Gravity-driven instabilities of thin viscoelastic films on an inverted plane

NSF Capstone Laboratory Project

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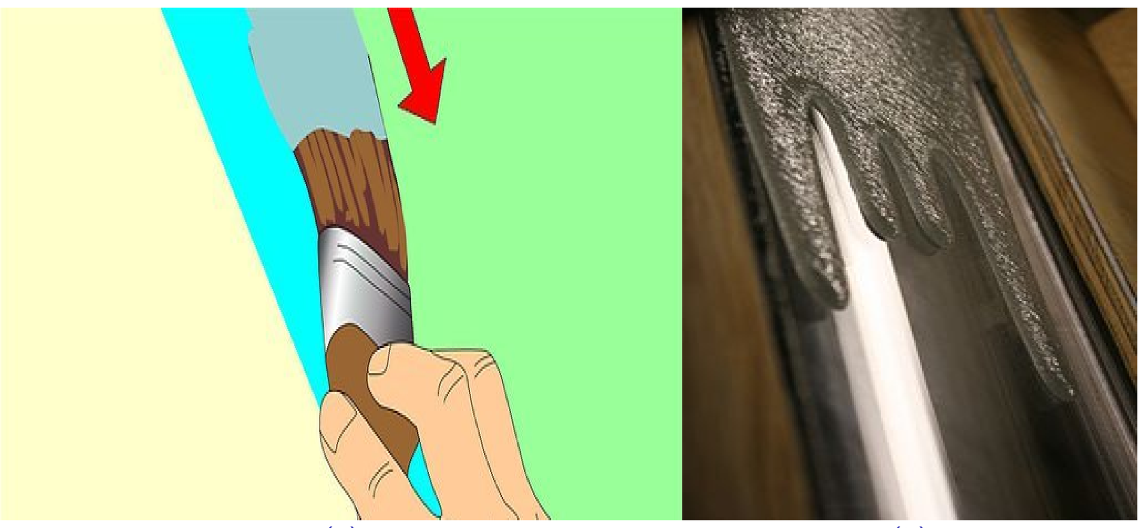
Introduction

This work was carried out during the spring semester of the 2016 − 2017 school year as part of an applied mathematics Capstone group project at NJIT. The theoretical and numerical study concerned the gravity-driven evolution of the interface of a thin viscoelastic film laying on an inverted plane. The governing equation was obtained as a long-wave approximation of the Navier-Stokes equations, including the gravitational body force, and the Jeffreys model for viscoelastic stresses. The Linear Stability Analysis was performed to compare theoretical predictions of the early stage of the dynamics, with the numerical results obtained. The competing effects of the physical parameters involved on the length and time scales of the instabilities were analyzed, in the linear and nonlinear regimes.



(a) (b)

Figure 1: Examples of viscoelastic fluids on inclined planes: (a) paint; (b) slurry.

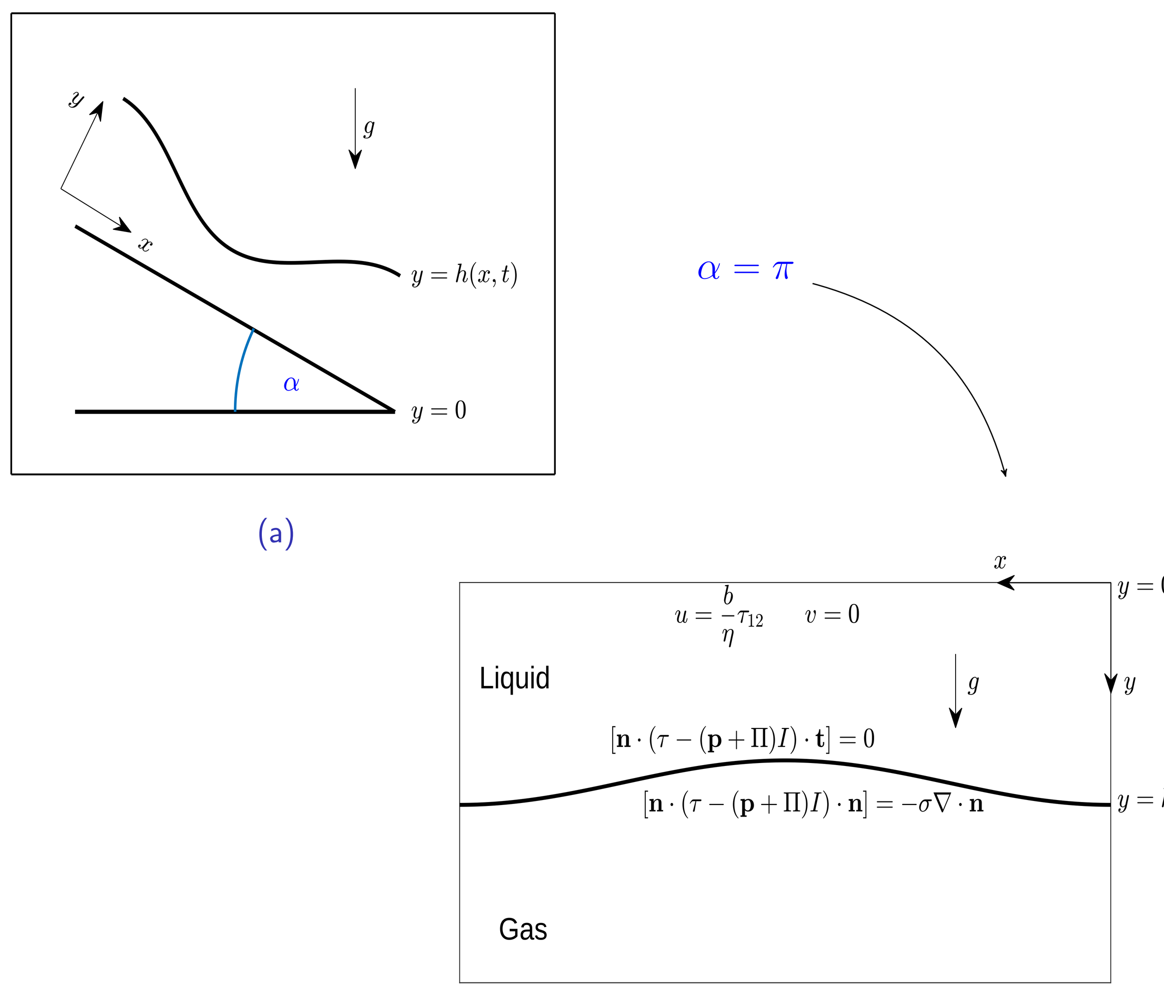
Governing Equations

The equation governing the dynamics of the fluid interface of viscoelastic liquids on an inclined plane, is derived as a long-wave approximation of the conservation laws. The liquid is considered incompressible, with mass density ρ. We consider a plane inclined of an angle α with the positive x-axis (see fig. 2). The equations of conservation of momentum and mass, respectively, are

$$\rho\left(\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla(p+\Pi) + \nabla\cdot\boldsymbol{\tau} + \mathbf{F}\,, \tag{1a}$$

$$\nabla\cdot\mathbf{u} = 0\,, \tag{1b}$$

where $\mathbf{u} = (u(x,y,t), v(x,y,t))$ is the velocity field in the Cartesian xy-plane, $\nabla = (\partial_x, \partial_y)$, $\boldsymbol{\tau}$ is the stress tensor, p is the pressure, Π is the disjoining pressure induced by the van der Waals solid-liquid interaction force, and $\mathbf{F} = (\rho g \sin\alpha, -\rho g\cos\alpha)$, with $g > 0$, is the gravitational acceleration (positive for the reference system considered, see fig. 2).



(a)

(b)

Figure 2: Setup and reference coordinate system.

Governing Equations (cont'd)

The Jeffreys constitutive model for viscoelastic stress follows

$$\boldsymbol{\tau} + \lambda_1\partial_t\boldsymbol{\tau} = \eta(\dot{\gamma} + \lambda_2\partial_t\dot{\gamma})\,, \tag{2}$$

where $\dot{\gamma}$ is the strain rate tensor, e.g. $\dot{\gamma}_{ij} = \partial u_j/\partial x_i + \partial u_i/\partial x_j$, η is the shear viscosity coefficient, λ_1 and λ_2, the *relaxation time* and the *retardation time*, respectively, such that $\lambda_2 = \lambda_1\eta_s/\eta_s + \eta_p$ ($\Rightarrow \lambda_1 \geq \lambda_2$). Here η_s and η_p are the viscosity coefficients of the Newtonian solvent and the polymeric solute, respectively, such that $\eta = \eta_s + \eta_p$.

We define L the characteristic length scale of the film, H the characteristic height scale, such that $H/L = \varepsilon \ll 1$ is the small parameter considered for the asymptotic approximation of the system (1). We obtain the thin film equation

$$\begin{aligned}(1+\lambda_2\partial_t)h_t + (\lambda_2-\lambda_1)\frac{\partial}{\partial x}\left[\left(\frac{h^2}{2}\mathbf{Q} - h\mathbf{R}\right)h_t\right] + \\ + \frac{\partial}{\partial x}\Bigg[(1+\lambda_1\partial_t)\frac{h^3}{3}(h_{xxx} + \Pi'(h)h_x - \mathcal{C}h_x + \mathcal{S}) + \\ +(1+\lambda_2\partial_t)bh^2(h_{xxx} + \Pi'(h)h_x - \mathcal{C}h_x + \mathcal{S})\Bigg] = 0\,,\end{aligned} \tag{3}$$

where b is the slip coefficient with the substrate, and $\mathbf{Q}$ and $\mathbf{R}$ satisfy the ODEs:

$$(1+\lambda_2\partial_t)\mathbf{Q} = \frac{\partial p}{\partial x} - \mathcal{S} \equiv \frac{\partial}{\partial x}(-h_{xx} - \Pi'(h) + \mathcal{C}h_x) - \mathcal{S}\,, \tag{4a}$$

$$(1+\lambda_2\partial_t)\mathbf{R} = h\left(\frac{\partial p}{\partial x} - \mathcal{S}\right) \equiv h\left[\frac{\partial}{\partial x}(-h_{xx} - \Pi'(h) + \mathcal{C}h) - \mathcal{S}\right]\,, \tag{4b}$$

and where we have used $\mathcal{S} = \mathcal{B}\sin\alpha$ $\mathcal{C} = \mathcal{B}\cos\alpha$, with $\mathcal{B} = \rho g L^2\varepsilon^3/U\eta = O(1)$ is the Bond number, and U a characteristic scale for the velocity.

Linear Stability Analysis

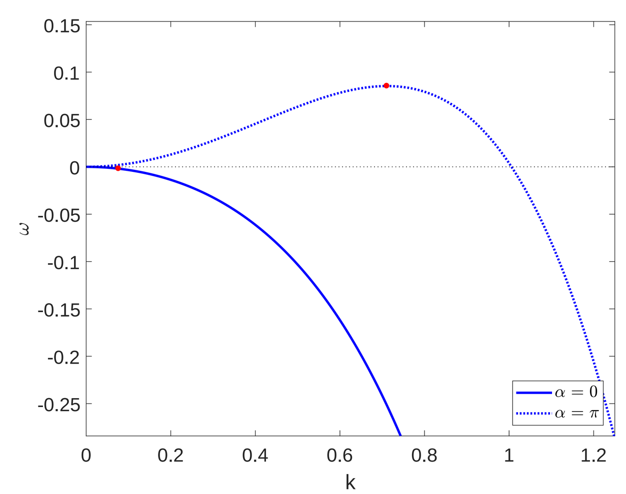
To study the film's response to a perturbation, we consider $h(x,t) = h_0 + \delta h_0 e^{ikx+\omega t}$, where h_0 is the flat reference thickness, δ a small amplitude, k the wave number $k = 2\pi/\lambda$, and ω the growth rate. We obtain the following dispersion relation

$$\begin{aligned}\lambda_2\omega^2 + \left[1 + (k^4 - (\Pi'(h_0) - \mathcal{C})k^2 + i\mathcal{S}k)\left(\lambda_1\frac{h_0^3}{3} + \lambda_2 bh_0^2\right)\right]\omega \\ + (k^4 - (\Pi'(h_0) - \mathcal{C})k^2 + i\mathcal{S}k)\left(\frac{h_0^3}{3} + bh_0^2\right) = 0\,.\end{aligned} \tag{5}$$

Considering only real roots

$$\begin{aligned}Re\{\omega_{1,2}\} = \frac{-\left[1 + (k^4 - k^2(\Pi'(h_0) - \mathcal{C}))\left(\lambda_1\frac{h_0^3}{3} + \lambda_2 bh_0^2\right)\right]}{2\lambda_2} \pm \\ \pm\frac{\sqrt{\left[1 + (k^4 - k^2(\Pi'(h_0) - \mathcal{C}))\left(\lambda_1\frac{h_0^3}{3} + \lambda_2 bh_0^2\right)\right]^2 - 4\lambda_2(k^4 - k^2(\Pi'(h_0) - \mathcal{C}))\left(\frac{h_0^3}{3} + bh_0^2\right)}}{2\lambda_2}\,.\end{aligned} \tag{6}$$

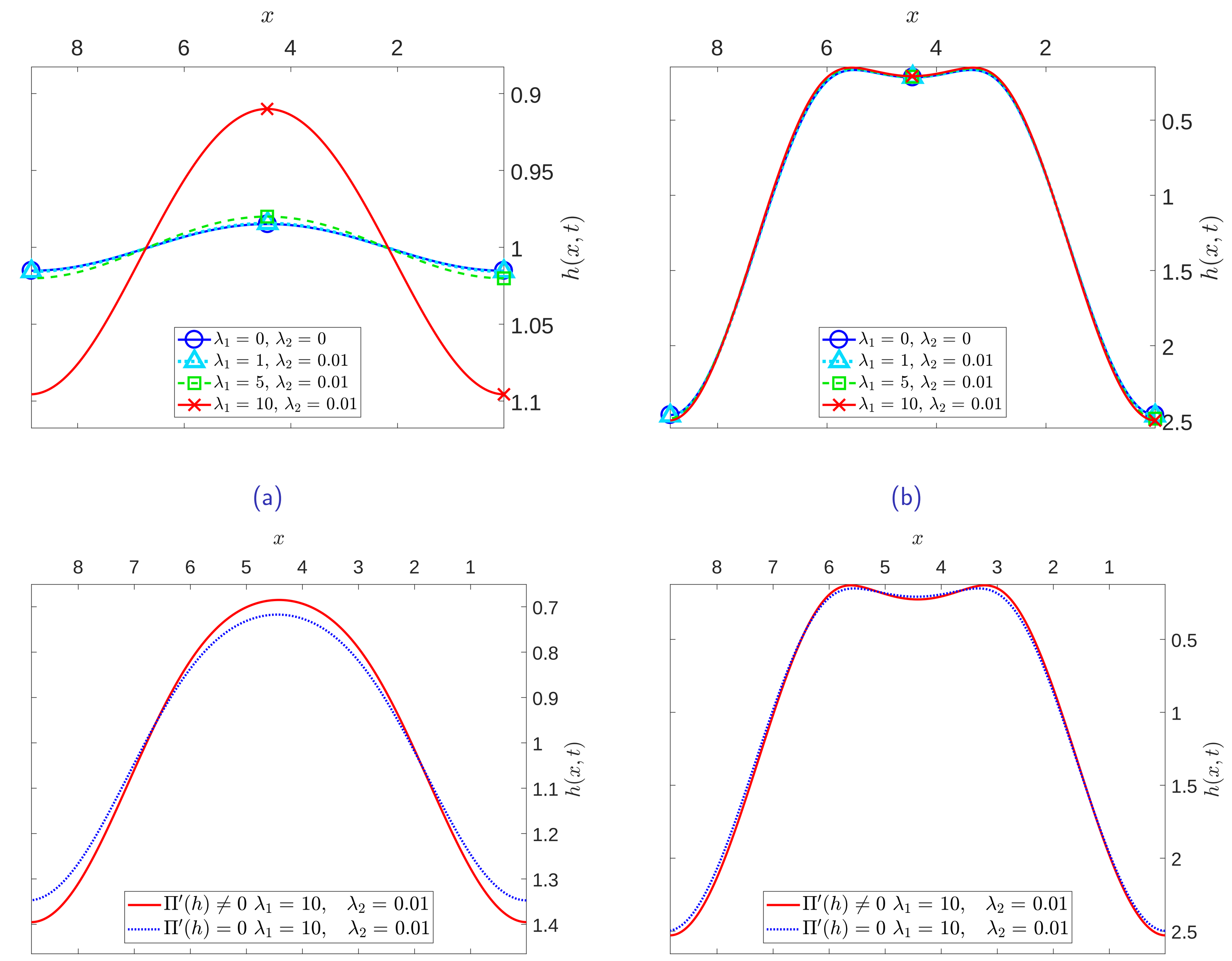
For which the critical wave number satisfies $k_c^2 = \Pi'(h_0) - \mathcal{C}$, and the wavenumber of maximum growth is $k_m = k_c/\sqrt{2}$.



Numerical Results and Discussion

We numerically solve eq. (3) using Newton's method of the non-linear terms, and a finite difference semi-implicit Crank-Nicolson scheme for the second order in time and fourth order in space PDE. The two ODEs (4) are solved with forward Euler method. We employ a fixed grid for the spacial discretization, and an adaptive time step for computational advantage and numerical stability. We recast the governing equation to isolate the time derivatives to apply the iterative scheme

$$\begin{aligned}\lambda_2 h_{tt} + \left\{1 + (\lambda_2-\lambda_1)\left[\frac{\partial}{\partial x}\left(\frac{h^2}{2}\mathbf{Q} - h\mathbf{R}\right)\right]\right\}h_t + \frac{\partial}{\partial t}\left(\frac{\partial h}{\partial x}\right)(\lambda_2-\lambda_1)\left(\frac{h^2}{2}\mathbf{Q} - h\mathbf{R}\right) + \\ + \lambda_1\frac{\partial}{\partial t}\left[\frac{\partial}{\partial x}\left(\frac{h^3}{3}(h_{xxx} + \Pi'(h)h_x - \mathcal{C}h_x + \mathcal{S})\right)\right] + \frac{\partial}{\partial x}\left[\left(\frac{h^3}{3} + bh^2\right)(h_{xxx} + \Pi'(h)h_x - \mathcal{C}h_x + \mathcal{S})\right] + \\ \lambda_2\frac{\partial}{\partial t}\left[\frac{\partial}{\partial x}\left(bh^2(h_{xxx} + \Pi'(h)h_x - \mathcal{C}h_x + \mathcal{S})\right)\right] = 0\,.\end{aligned} \tag{7}$$



(a) (b)

(c) (d)

Figure 4: Evolution of different viscoelastic films, in the absence of the van der Waals potential with the substrate, for different values of the retardation time: in (a) at time $t = 5$, in (b) at time $t = 100$. Evolution in the presence/absence of van der Waals potential with the substrate for a viscoelastic film, in (c) at time $t = 10$, in (d) at time $t = 100$.

Conclusions and Future Work

We numerically solved the non-linear PDE governing the interface of viscoelastic liquids on an inverted plane. Our numerical results agree with the theoretical predictions in the linear regime. In the non-linear phase, they show that elastic effects enhance the dewetting of a viscoelastic film on an inverted plane, in agreement with previous findings for flat non-inverted planes [1]. The investigation of the interfacial dynamics of viscoelastic films on inclined planes with arbitrary angles is considered for future work.

References

[1] Barra V., Afkhami S., Kondic L., *Interfacial dynamics of thin viscoelastic films and drops*, J. Non-Newt. Fluid Mech. **237**, 26 − 38, (2016)

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