An Algorithm for Learning Switched Linear Dynamics from Data.

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NEURAL INFORMATION PROCESSING SYSTEMS

Problem Formulation

Switched System Identification

Inputs: Full-state observations of a system (with noise):

$$(\mathbf{x}(t), \mathbf{x}(t+1)),\ t = 0, \ldots, N-1,$$

Number of modes m,
Error tolerances $\epsilon, \tau > 0$,

Output: Find m $d \times d$ matrices $A_1, \ldots, A_m$ s.t.

$$(\forall\, t)\ (\exists\, j)\ ||\mathbf{x}(t+1) - A_j \mathbf{x}(t)||_\infty \leq \epsilon||\mathbf{x}(t)|| + \tau$$

Relative Error ($\epsilon||\mathbf{x}(t)||$), Absolute Error ($\tau$)

Applications

1. Cyber-Physical Systems, Robotics, and Control.
2. Machine Learning: k-linear regression problem – fit $k \geq 2$ "straight lines" to data.

Theorem (Lauer and Bloch [4, Theorem 5.1]). The problem above is NP-hard.

Mixed Integer LP Formulation (MILP): Exponential time in number of data points N.

Contributions

1. Reformulation of problem with a *gap*.
2. More efficient algorithm:
   - Linear in number of data points N,
   - Exponential in number of modes m,
   - Exponential in the dimensionality of state space d.
3. Empirical evaluation and comparison against related techniques including MILP.

Reformulation with a Gap

Original Problem

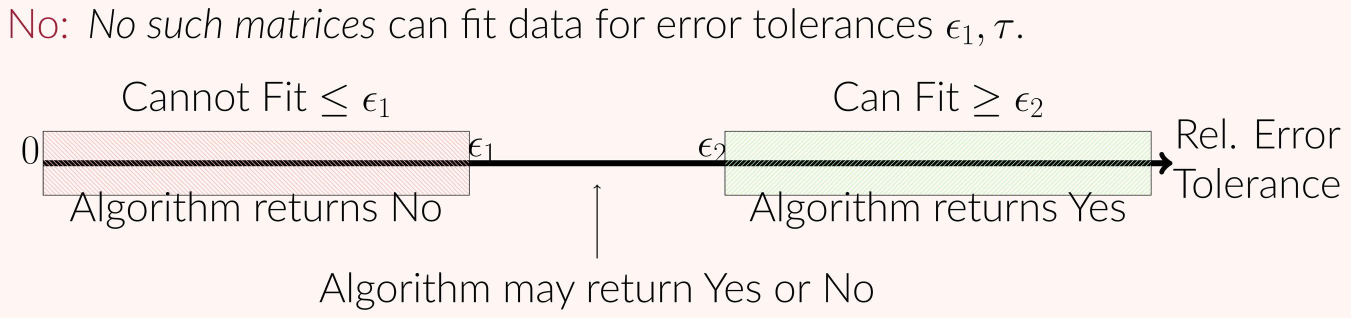
There are two possible outcomes:

Yes: Successfully found m matrices satisfying error tolerances ϵ, τ.
No: *No such matrices* can fit the given data.

Idea # 1: Reformulate problem with a *gap*.

Input *two* relative error tolerances $\epsilon_1 < \epsilon_2$.

Yes: Successfully found m matrices satisfying error tolerances ϵ_2, τ.
No: *No such matrices* can fit data for error tolerances ϵ_1, τ.



Main Result

Algorithm with time complexity

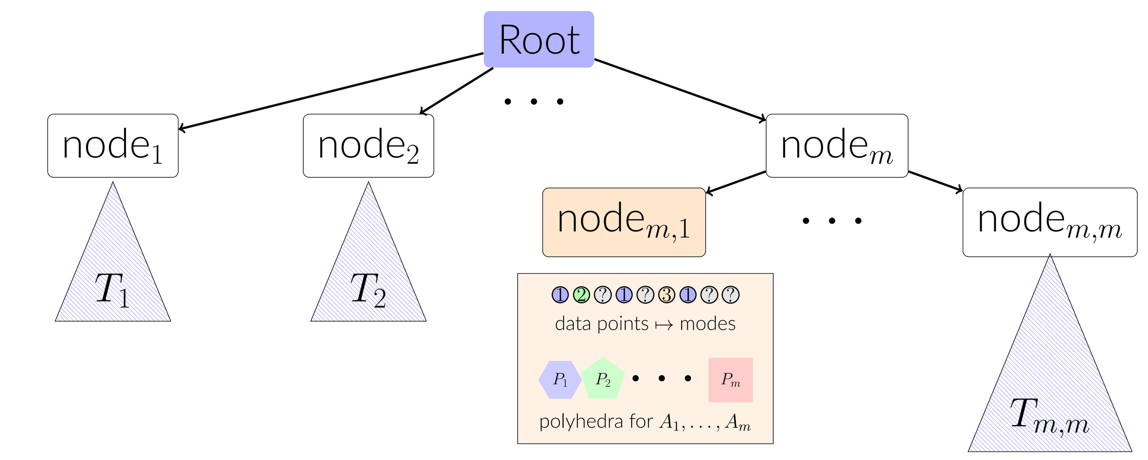
$$O\left(m^{C\, m\, d^3\, |\log(d\gamma/(\epsilon_2-\epsilon_1))|} \times N \times \mathrm{poly}(m, d)\right)$$

Exp. m, # dim., $\frac{1}{\epsilon_2-\epsilon_1}$ — Lin. over # data — Solving LPs

1. Combination of simple ideas.
2. Easy to implement and works well in practice.

Overall Algorithm

Organize constraints using a *tree* data structure.



Each node carries the following information.

- Data points that have been assigned to modes.
- Unassigned data points.
- Polyhedra $P_1, \ldots, P_m$ representing constraints for $A_1, \ldots, A_m$ resp.

Initial Tree

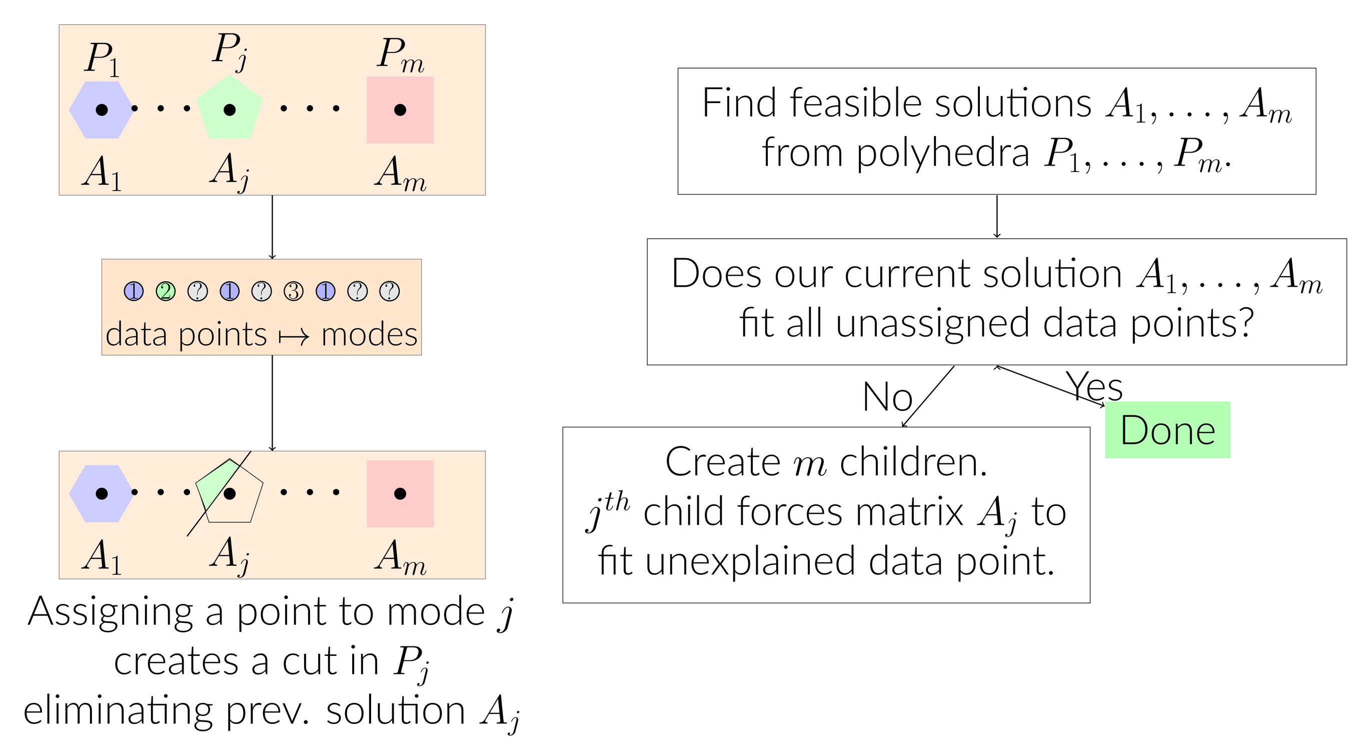
- Single root node with all data points unassigned.
- Polyhedra $P_1, \ldots, P_m$ are initialized to compact sets.

Key Steps of the Algorithm:

1. Choose a previously unexplored leaf.
2. Expand the leaf (see below):
   - Discover matrices $A_1, \ldots, A_m$ or
   - Add m new children.

Exploring a Tree Leaf

Leaf with unassigned data points U, polyhedra $P_1, \ldots, P_m$.



Time Complexity

Idea # 2: Choose maximum volume ellipsoid (MVE) center of polyhedra.
⟶ Volume shrinks by at least $\alpha < 1$ [1, § 4.3].

Idea # 3: If a leaf has solutions, then its volume $\geq$ a fixed lower bound.
⟶ The gap formulation ($\epsilon_2 - \epsilon_1 > 0$) is essential for this.

Ideas #1 + # 2 + # 3 : Upper bound on the maximum depth of the tree.
⟶ Bound on time complexity of the algorithm.

Implementation

Implemented in the Python programming language.

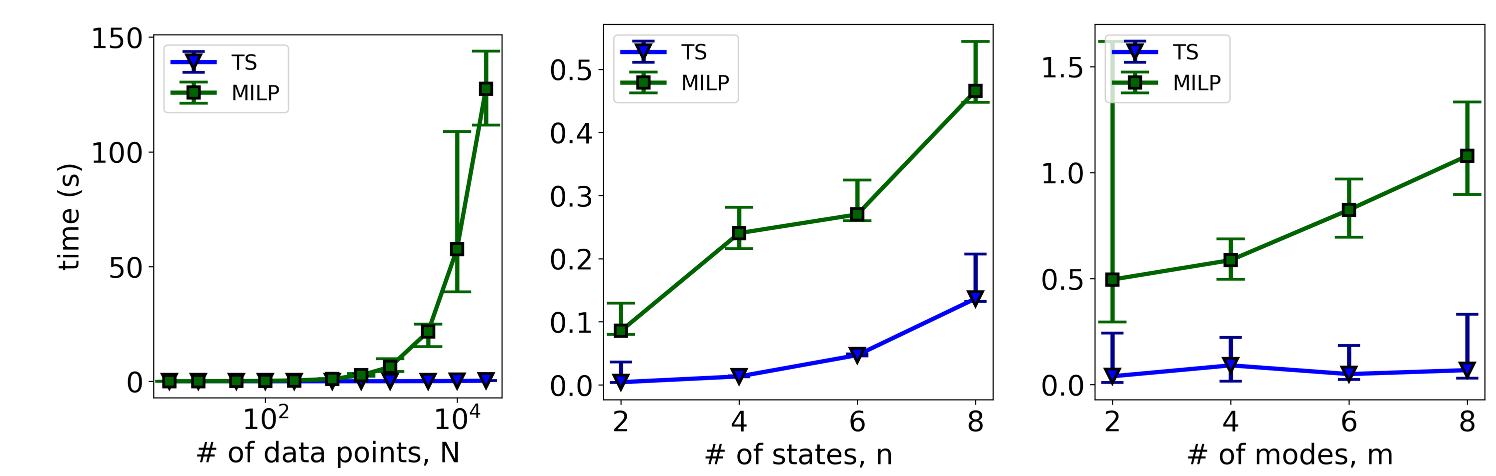
- Gurobi LP solver (free academic license).
- Use Chebyshev center instead of MVE center.

Comparison against two methods:

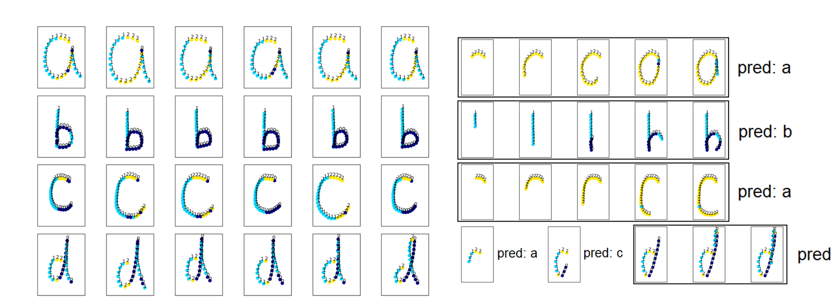
MILP Solver: Comparison with MILP.
- Implemented using Gurobi: state-of-the-art solver [2].
- Worst-case exponential in the number of data points

Clustering-Based: Fast method but inexact [3].

Microbenchmark Comparisons

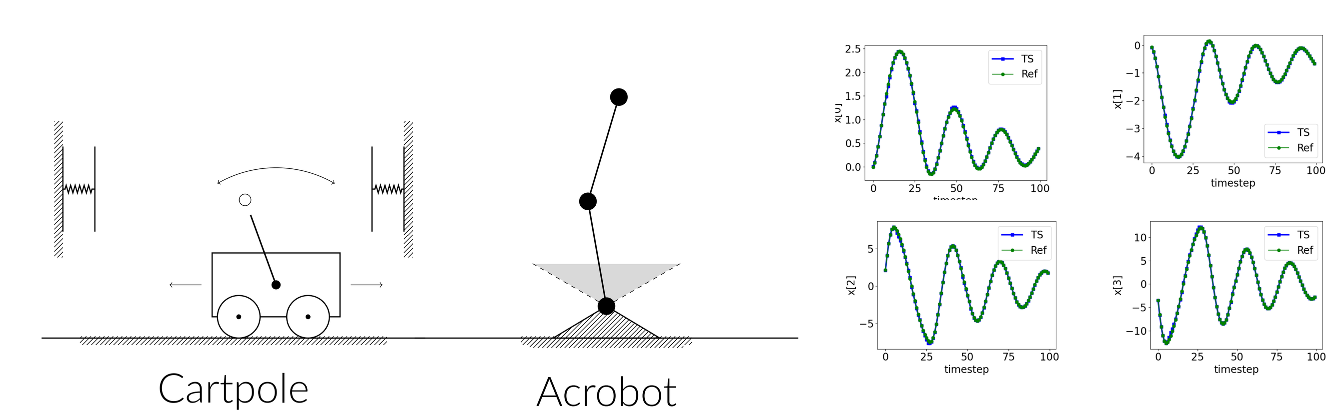


Handwritten Character Modeling



- Handwritten alphabet on a tablet.
- Combine our approach with automata learning (EDSM).
- Successfully learn model to predict written character.

Mechanical Systems with Contact Forces



References

[1] Stephen Boyd and Lieven Vandenberghe.
Localization and cutting-plane methods.
From Stanford EE 364b lecture notes, 2007.

[2] Gurobi Optimization, LLC.
Gurobi Optimizer Reference Manual, 2022.

[3] Fabien Lauer.
Estimating the probability of success of a simple algorithm for switched linear regression.
Nonlinear Analysis: Hybrid Systems, 8:31–47, 2013.

[4] Fabien Lauer and Gérard Bloch.
Hybrid system identification: theory and algorithms for learning switching models.
Springer, 2019.

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