Minimax Option Pricing:
How Robust is Black-Scholes?

Rafael Frongillo

Department of Computer Science
University of California at Berkeley

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Joint work with Jake Abernethy and Andre Wibisono

Jake Abernethy

now at UPenn

Andre Wibisono

in 1225 this summer!

Financial Derivatives

A new financial instrument which is a function of old ones.

Class of derivatives we consider:

- Expiration date T (typically 1)
- Base stock/asset S
- Derivative pays out $g(S(T))$ at time T

  $S(t)$ is the value of S at time t

E.g. cos(gas price on Aug 1)

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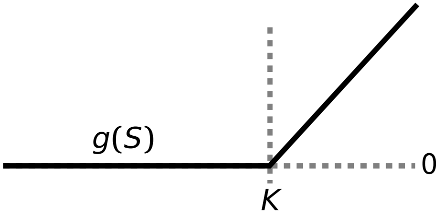
Options

Running example: European call option

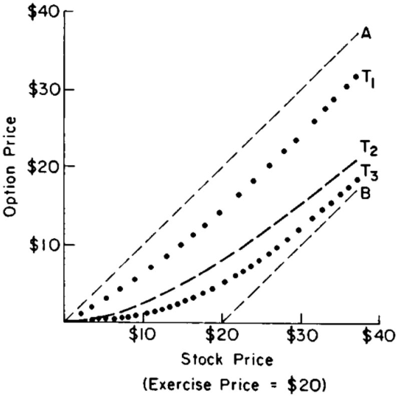


$g(S) = \max(0, S - K)$, where K is the *strike price*

Note: will use “option” and “derivative” interchangably

How to Price?

What is a derivative g worth?



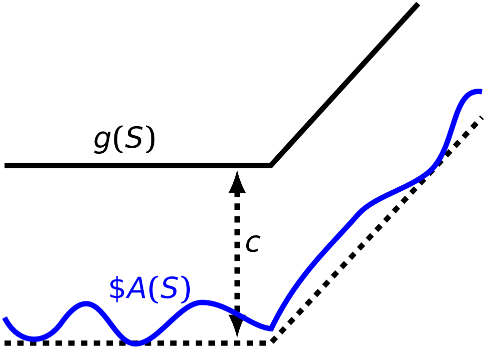
Black-Scholes Pricing

Fischer Black and Myron Scholes, 1973

- Intuition: price of derivative is cost of implementing it with existing instruments
- The algorithm which implements a derivative is a *replication strategy*
- The replication strategy has a fixed initial investment, which should be precisely the price of the derivative

Replication Strategies

Idea: As stock S fluctuates, use an algorithm A to “hedge” the option by buying and selling S



Result: guarantee the payoff of the option, minus a fixed cost c

Black-Scholes Assumptions

- No arbitrage opportunities
- 0% interest borrowing
- Can trade continuously
- No transaction fees, no dividend payments, etc
- Stock prices follow Geometric Brownian Motion (GBM)

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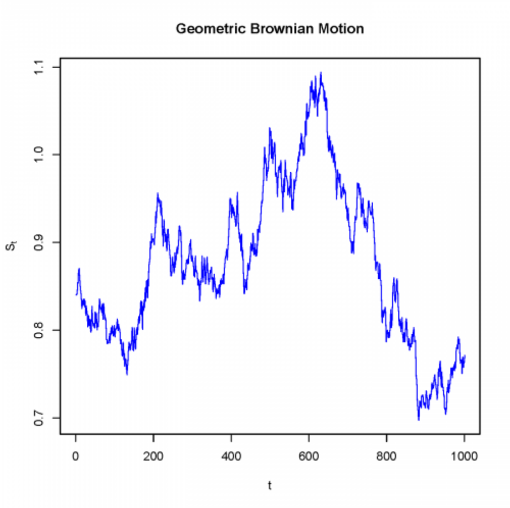

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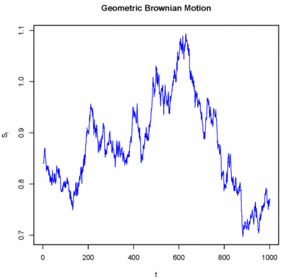
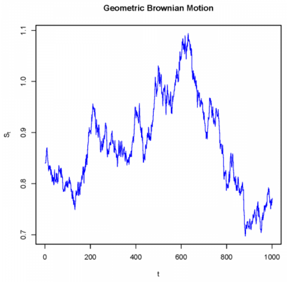


Let $W(t)$ be Brownian Motion with drift μ and volatility σ^2

- $W(0) = 0$
- $W(t) - W(s)$ and $W(u) - W(t)$ are indep. for $s < t < u$
- $W(t) - W(s) \sim N(\mu(t-s), \sigma^2(t-s))$

$G(t)$ is GBM $\iff$ $log(G(t))$ is Brownian Motion

Geometric Brownian Motion



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Delta-hedge Portfolio

Given option/derivative g:

- Let $V(S,t)$ be the value of the option at t
- Let $\frac{\partial V}{\partial S}$ be the replication portfolio

  Hold \$ $\frac{\partial V}{\partial S}(t)$ of stock @ time t

Now solve for V using the no-arbitrage condition:

- Stochastic PDE from Ito's Lemma:

$$\frac{\partial V}{\partial S} - \frac{1}{2} S^2 \frac{\partial^2 V}{\partial S^2} = 0$$

- Solution is:

$$V(S,t) = \mathbb{E}_{G \sim \mathrm{GBM}}[g(SG(T-t))]$$

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Price of option is therefore:

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Some surprises:

- Replication succeeds with probability 1!
- GBM above has drift 0 not μ!

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Beyond Black-Scholes

Problems with Black-Scholes

- Continuous-time trading
- Assumes GBM!

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Prices respond to decisions of other traders!

Why not *adversarial* prices? [DeMarzo, Kremer, Mansour '08]

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An Option-Pricing Game

$$\inf_{A \in \mathcal{A}} \sup_{X \in \mathcal{X}} \mathbb{E}\left[g(X(1)) - \sum_{m=1}^{n} T_m \Delta_m \right]$$

- An n-round game between Investor and Nature
- Discrete-time trades at $t = m/n$, $m \in [n]$

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- Earnings of Investor
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Value of the Game

$$\inf_{A \in \mathcal{A}} \sup_{X \in \mathcal{X}} \mathbb{E}\Big[\; g(X(1)) - \sum_{m=1}^{n} T_m \; \Delta_m \; \Big]$$

Value of the game $\geq$ option price!

Upper bound because of the worst-case assumptions

Interested continuous trading limit as $n \rightarrow \infty$

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Constraining Nature

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What price paths $\mathcal{X}$ can Nature choose from?

We require: $\mathbb{E}[T_m^2 | T_{m-1}] \leq \frac{c}{n}$ c is the "volatility"

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By Sion's Minimax Theorem, we can swap $\inf$ and $\sup$!

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Step II: Martingale

$$\sup_{X \in \mathcal{X}} \inf_{A \in \mathcal{A}} \mathbb{E}\left[g(X(1)) - \sum_{m=1}^{n} T_m \Delta_m \right]$$

Now $\{T_m\}$ must be a martingale sequence

- Assume not: $\mathbb{E}[T_m | T_{m-1}] \neq 0$
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But now the algorithm is completely irrelevant!

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Martingale sequence with conditional variance c/n

Applying a martingale CLT: *Lindeberg–Feller Theorem*

Theorem

As $n \to \infty$, $X_n^* \xrightarrow{d} GBM$

Corollary

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What Just Happened?

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Our constraint on Nature:

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[DeMarzo, Kremer, Mansour '08] use a *cumulative* constraint:

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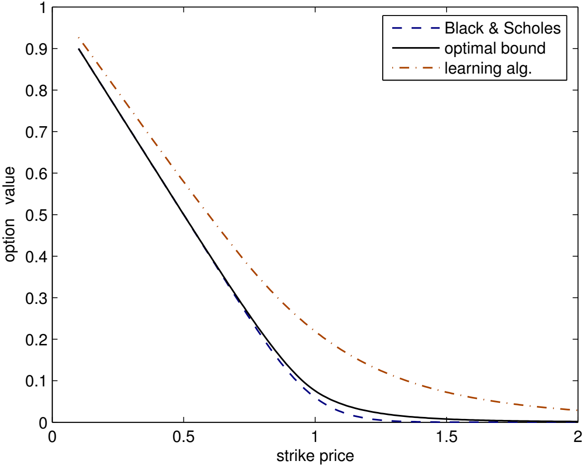
Still Obtain Black-Scholes Price?

From [DeMarzo, Kremer, Mansour '08]:



Some Speculation

We believe:

- $X_n^* \not\longrightarrow$ GBM
- $X_n^*(1) \longrightarrow \text{GBM}(1)$

Hence, we would still obtain the Black-Scholes price!

Proof ideas:

- $\text{support}(T_m) = 2$ in dual game
- Optimal Δ_m balances these two points
- Then Δ_m is a discrete derivative of V
- This V approaches Black-Scholes V, and Δ_m approaches the delta-hedge portfolio!

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New Analysis

Consider the value function for this game:

$$V_n(S,n) := g(S)$$

$$V_n(S,m) := \inf_{\Delta\in\mathbb{R}} \sup_{t\in[-z,z]} \Delta t + V_n\Big(S(1+t), m-1\Big)$$

And let $\Delta = \Delta(S,m)$ be the optimal investment for Investor

Lemma

If $\Delta = \Delta(S,m)$, then Nature's $\sup_t$ is achieved by at least two points $t_1, -t_2$ with $t_1, t_2 > 0$

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$$V_n(S,m) = \Delta(S,m)\,t_1 + V_n\Big(S(1+t_1), m-1\Big)$$
$$= -\Delta(S,m)\,t_2 + V_n\Big(S(1-t_2), m-1\Big)$$

Solving for Δ:

$$\Delta(S,m) = \frac{V_n\Big(S(1-t_2), m-1\Big) - V_n\Big(S(1+t_1), m-1\Big)}{t_1 + t_2}$$

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A discrete derivative... reminiscent of the delta-hedge portfolio!

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Martingale??

Plugging Δ back in:

$$V_n(S,m) = \frac{t_1}{t_1+t_2} V_n\Big(S(1-t_2), m-1\Big) + \frac{t_2}{t_1+t_2} V_n\Big(S(1+t_1), m-1\Big)$$

Introduce a random variable $T = \begin{cases} t_1 & \text{w.p. } \frac{t_2}{t_1+t_2} \\ -t_2 & \text{w.p. } \frac{t_1}{t_1+t_2} \end{cases}$ *Note* $\mathbb{E}[T] = 0$

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Applying this at every round:

$$V_n(S,0) = \mathbb{E}\left[V_n\left(S \cdot \prod_{m=1}^{n}(1+T_m), n\right)\right]$$
$$= \mathbb{E}\left[g\left(S \cdot \prod_{m=1}^{n}(1+T_m)\right)\right]$$

Conjectures

1. $V_n(S,n) \longrightarrow V_{\text{B-S}}(S,1)$
2. $\Delta(S,m) \longrightarrow \frac{\partial}{\partial S} V_{\text{B-S}}(S, \frac{m}{n})$

thank you