How to Hedge an Option Against an Adversary: Black-Scholes Pricing is Minimax Optimal

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Abstract

We consider a popular problem in finance, option pricing, through the lens of an online learning game between Nature and an Investor. In the Black-Scholes option pricing model from 1973, the Investor can continuously hedge the risk of an option by trading the underlying asset, assuming that the asset's price fluctuates according to Geometric Brownian Motion (GBM). We consider a worst-case model, in which Nature chooses a sequence of price fluctuations under a cumulative quadratic volatility constraint, and the Investor can make a sequence of hedging decisions. Our main result is to show that the value of our proposed game, which is the "regret" of hedging strategy, converges to the Black-Scholes option price. We use significantly weaker assumptions than previous work—for instance, we allow large jumps in the asset price—and show that the Black-Scholes hedging strategy is near-optimal for the Investor even in this non-stochastic framework.

1 Introduction

An *option* is a financial contract that allows the purchase or sale of a given asset, such as a stock, bond, or commodity, for a predetermined price on a predetermined date. The contract is named as such because the transaction in question is optional for the purchaser of the contract. Options are bought and sold for any number of reasons, but in particular they allow firms and individuals with risk exposure to hedge against potential price fluctuations. Airlines, for example, have heavy fuel costs and hence are frequent buyers of oil options.

What ought we pay for the privilege of purchasing an asset at a fixed price on a future expiration date? The difficulty with this question, of course, is that while we know the asset's previous prices, we are uncertain as to its future price. In a seminal paper from 1973, Fischer Black and Myron Scholes introduced what is now known as the Black-Scholes Option Pricing Model, which led to a boom in options trading as well as a huge literature on the problem of derivative pricing [2]. Black and Scholes had a key insight that a firm which had sold/purchased an option could "hedge" against the future cost/return of the option by buying and selling the underlying asset as its price fluctuates. Their model is based on stochastic calculus and requires a critical assumption that the asset's price behaves according to a *Geometric Brownian Motion* (GBM) with known drift and volatility.

The GBM assumption in particular implies that (almost surely) an asset's price fluctuates continuously. The Black-Scholes model additionally requires that the firm be able to buy and sell continuously until the option's expiration date. Neither of these properties are true in practice: the stock market is only open eight hours per day, and stock prices are known to make significant jumps even

during regular trading. These and other empirical observations have led to much criticism of the Black-Scholes model.

An alternative model for option pricing was considered[1] by DeMarzo et al. [3], who posed the question: "Can we construct hedging strategies that are robust to *adversarially chosen* price fluctuations?" Essentially, the authors asked if we may consider hedging through the lens of *regret minimization in online learning*, an area that has proved fruitful, especially for obtaining guarantees robust to worst-case conditions. Within this minimax option pricing framework, DeMarzo et al. provided a particular algorithm resembling the Weighted Majority and Hedge algorithms [5, 6] with a nice bound.

Recently, Abernethy et al. [1] took the minimax option pricing framework a step further, analyzing the zero-sum game being played between an Investor, who is attempting to replicate the option payoff, and Nature, who is sequentially setting the price changes of the underlying asset. The Investor's goal is to "hedge" the payoff of the option as the price fluctuates, whereas Nature attempts to foil the Investor by choosing a challenging sequence of price fluctuations. The *value* of this game can be interpreted as the "minimax option price," since it is what the Investor should pay for the option against an adversarially chosen price path. The main result of Abernethy et al. was to show that the game value approaches the Black-Scholes option price as the Investor's trading frequency increases. Put another way, the minimax price tends to the option price under the GBM assumption. This lends significant further credibility to the Black-Scholes model, as it suggests that the GBM assumption may already be a "worst-case model" in a certain sense.

The previous result, while useful and informative, left two significant drawbacks. First, their techniques used minimax duality to compute the value of the game, but no particular hedging algorithm for the Investor is given. This is in contrast to the Black-Scholes framework (as well as to the De-Marzo et al.'s result [3]) in which a hedging strategy is given explicitly. Second, the result depended on a strong constraint on Nature's choice of price path: the multiplicative price variance is uniformly constrained, which forbids price jumps and other large fluctuations.

In this paper, we resolve these two drawbacks. We consider the problem of minimax option pricing with much weaker constraints: we restrict the *sum* over the length of the game of the squared price fluctuations to be no more than a constant c, and we allow arbitrary price jumps, up to a bound ζ. We show that the minimax option price is exactly the Black-Scholes price of the option, up to an additive term of $O(c\zeta^{1/4})$. Furthermore, we give an explicit hedging strategy: this upper bound is achieved when the Investor's strategy is essentially a version of the Black-Scholes hedging algorithm.

2   The Black-Scholes Formula

Let us now briefly review the Black-Scholes pricing formula and hedging strategy. The derivation requires some knowledge of continuous random walks and stochastic calculus—Brownian motion, Itô's Lemma, a second-order partial differential equation—and we shall only give a cursory treatment of the material. For further development we recommend a standard book on stochastic calculus, e.g. [8]. Let us imagine we have an underlying asset A whose price is fluctuating. We let $W(t)$ be a *Brownian motion*, also known as a *Weiner process*, with zero drift and unit variance; in particular, $W(0) = 0$ and $W(t) \sim N(0, t)$ for $t > 0$. We shall imagine that A's price path $G(t)$ is described by a *geometric Brownian motion* with drift μ and volatility σ, which we can describe via the definition of a Brownian motion: $G(t) \overset{d}{=} \exp\{(\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)\}$.

If an Investor purchases a *European call* option on some asset A (say, MSFT stock) with a *strike price* of $K > 0$ that matures at time T, then the Investor has the right to buy a share of A at price K at time T. Of course, if the market price of A at T is $G(T)$, then the Investor will only "exercise" the option if $G(T) > K$, since the Investor has no benefit of purchasing the asset at a price higher than the market price. Hence, the payoff of a European call option has a profit function of the form $\max\{0, G(T) - K\}$. Throughout the paper we shall use $g_{\text{EC}}(x) := \max\{0, x - K\}$ to refer to the payout of the European call when the price of asset A at time T is x (the parameter K is implicit).

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[1]Although it does not have quite the same flavor, a similar approach was explored in the book of Vovk and Shafer [7].

We assume the current time is t. The Black-Scholes derivation begins with a guess: assume that the "value" of the European call option can be described by a smooth function $\mathcal{V}(G(t), t)$, depending only on the current price of the asset $G(t)$ and the time to expiration $T - t$. We can immediately define a boundary condition on $\mathcal{V}$, since at the expiration time T the value of the option is $\mathcal{V}(G(T), 0) = g_{\text{EC}}(G(T))$.

So how do we arrive at a value for the option at another time point t? We assume the Investor has a *hedging strategy*, $\Delta(x, t)$ that determines the amount to invest when the current price is x and the time is t. Notice that if the asset's current price is $G(t)$ and the Investor purchases $\Delta(G(t), t)$ dollars of asset A at t, then the incremental amount of money made in an infinitesimal amount of time is $\Delta(G(t), t) \, dG/G(t)$, since $dG/G(t)$ is the instantaneous multiplicative price change at time t. Of course, if the earnings of the Investor are guaranteed to exactly cancel out the infinitesimal change in the value of the option $d\mathcal{V}(G(t), t)$, then the Investor is totally hedged with respect to the option payout for any sample of G for the remaining time to expiration. In other words, we hope to achieve $d\mathcal{V}(G, t) = \Delta(G, t) \, dG/G$. However, by Itô's Lemma [8] we have the following useful identity:

$$d\mathcal{V}(G, t) = \frac{\partial \mathcal{V}}{\partial x} dG + \frac{\partial \mathcal{V}}{\partial t} dt + \frac{1}{2} \sigma^2 G^2 \frac{\partial^2 \mathcal{V}}{\partial x^2} dt. \tag{1}$$

Black and Scholes proposed a generic hedging strategy, that the investor should invest

$$\Delta(x, t) = x \frac{\partial \mathcal{V}}{\partial x} \tag{2}$$

dollars in the asset A when the price of A is x at time t. As mentioned, the goal of the Investor is to hedge out risk so that it is always the case that $d\mathcal{V}(G, t) = \Delta(G, t) \, dG/G$. Combining this goal with Equations (1) and (2), we have

$$\frac{\partial \mathcal{V}}{\partial t} + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 \mathcal{V}}{\partial x^2} = 0. \tag{3}$$

Notice the latter is an entirely non-stochastic PDE, and indeed it can be solved explicitly:

$$\mathcal{V}(x, t) = \mathbb{E}_Y[g_{\text{EC}}(x \cdot \exp(Y))] \quad \text{where} \quad Y \sim \mathcal{N}(-\tfrac{1}{2}\sigma^2(T - t), \sigma^2(T - t)) \tag{4}$$

Remark: While we have described the derivation for the European call option, with payoff function g_{EC}, the analysis above does not rely on this specific choice of g. We refer the reader to a standard text on asset pricing for more on this [8].

3 The Minimax Hedging Game

We now describe a sequential decision protocol in which an Investor makes a sequence of trading decisions on some underlying asset, with the goal of hedging away the risk of some option (or other financial derivative) whose payout depends on the final price of the asset at the expiration time T. We assume the Investor is allowed to make a trading decision at each of n time periods, and before making this trade the investor observes how the price of the asset has changed since the previous period. Without loss of generality, we can assume that the current time is 0 and the trading periods occur at $\{T/n, \, 2T/n, \ldots, 1\}$, although this will not be necessary for our analysis.

The protocol is as follows.
1: Initial price of asset is $S = S_0$.
2: **for** $i = 1, 2, \ldots, n$ **do**
3:     Investor hedges, invests $\Delta_i \in \mathbb{R}$ dollars in asset.
4:     Nature selects a price fluctuation r_i and updates price $S \leftarrow S(1 + r_i)$.
5:     Investor receives (potentially negative) profit of $\Delta_i r_i$.
6: **end for**
7: Investor is charged the cost of the option, $g(S) = g\left(S_0 \cdot \prod_{i=1}^{n}(1 + r_i)\right)$.

Stepping back for a moment, we see that the Investor is essentially trying to minimize the following objective:

$$g\left(S_0 \cdot \prod_{i=1}^{n}(1 + r_i)\right) - \sum_{i=1}^{n} \Delta_i r_i.$$

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We can interpret the above expression as a form of *regret*: the Investor chose to execute a trading strategy, earning him $\sum_{i=1}^{n} \Delta_i r_i$, but in hindsight might have rather purchased the option instead, with a payout of $g\left(S_0 \cdot \prod_{i=1}^{n}(1 + r_i)\right)$. What is the best hedging strategy the Investor can execute to minimize the difference between the option payoff and the gains/losses from hedging? Indeed, how much regret may be suffered against a worst-case sequence of price fluctuations?

Constraining Nature. The cost of playing the above sequential game is clearly going to depend on how much we expect the price to fluctuate. In the original Black-Scholes formulation, the price volatility σ is a major parameter in the pricing function. In the work of Abernethy et al., a key assumption was that Nature may choose any $r_1, \ldots, r_n$ with the constraint that $\mathbb{E}[r_i^2 \mid r_1, \ldots, r_{i-1}] = O(1/n)$.[2] Roughly, this constraint means that in any ϵ-sized time interval, the price fluctuation variance shall be no more than ϵ. This constraint, however, does not allow for large price jumps during trading. In the present work, we impose a much weaker set of constraints, described as follows:[3]

- **TotVarConstraint**: The total price fluctuation is bounded by a constant c: $\sum_{i=1}^{n} r_i^2 \leq c$.
- **JumpConstraint**: Every price jump $|r_i|$ is no more than ζ, for some $\zeta > 0$ (which may depend on n).

The first constraint above says that Nature is bounded by how much, in total, the asset's price path can fluctuate. The latter says that at no given time can the asset's price jump more than a given value. It is worth noting that if $c \geq n\zeta^2$ then **TotVarConstraint** is superfluous, whereas **JumpConstraint** becomes superfluous if $c < \zeta^2$.

The Minimax Option Price We are now in a position to define the *value* of the sequential option pricing game using a minimax formulation. That is, we shall ask how much the Investor loses when making optimal trading decisions against worst-case price fluctuations chosen by Nature. Let $V_\zeta^{(n)}(S; c, m)$ be the value of the game, measured by the investor's *loss*, when the asset's current price is $S \geq 0$, the **TotVarConstraint** is $c \geq 0$, the **JumpConstraint** is $\zeta > 0$, the total number of trading rounds are $n \in \mathbb{N}$, and there are $0 \leq m \leq n$ rounds remaining. We define recursively:

$$V_\zeta^{(n)}(S; c, m) = \inf_{\Delta \in \mathbb{R}} \sup_{r: |r| \leq \min\{\zeta, \sqrt{c}\}} -\Delta r + V_\zeta^{(n)}((1 + r)S; c - r^2, m - 1), \qquad (5)$$

with the base case $V_\zeta^{(n)}(S; c, 0) = g(S)$. Notice that the constraint under the supremum enforces both **TotVarConstraint** and **JumpConstraint**. For simplicity, we will write $V_\zeta^{(n)}(S; c) := V_\zeta^{(n)}(S; c, n)$. This is the value of the game that we are interested in analyzing.

Towards establishing an upper bound on the value (5), we shall discuss the question of how to choose the hedge parameter Δ on each round. We can refer to a "hedging strategy" in this game as a function of the tuple $(S, c, m, n, \zeta, g(\cdot))$ that returns hedge position. In our upper bound, in fact we need only consider hedging strategies $\Delta(S, c)$ that depend on S and c; there certainly will be a dependence on $g(\cdot)$ as well but we leave this implicit.

4 Asymptotic Results

The central focus of the present paper is the following question: "*For fixed c and S, what is the asymptotic behavior of the value $V_\zeta^{(n)}(S; c)$?*" and "*Is there a natural hedging strategy $\Delta(S, c)$ that (roughly) achieves this value?*" In other words, what is the minimax value of the option, as well as the optimal hedge, when we fix the variance budget c and the asset's current price S, but let the number of rounds tend to ∞? We now give answers to these questions, and devote the remainder of the paper to developing the results in detail.

We consider payoff functions $g \colon \mathbb{R}_0 \to \mathbb{R}_0$ satisfying three constraints:

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[2]The constraint in [1] was $\mathbb{E}[r_i^2 \mid r_1, \ldots, r_{i-1}] \leq \exp(c/n) - 1$, but this is roughly equivalent.

[3]We note that Abernethy et al. [1] also assumed that the multiplicative price jumps $|r_i|$ are bounded by $\hat{\zeta}_n = \Omega(\sqrt{(\log n)/n})$; this is a stronger assumption than what we impose on (ζ_n) in Theorem 1.

1. g is convex.

2. g is L-Lipschitz, i.e. $|g(x) - g(y)| \leq L|x - y|$.

3. g is *eventually linear*, i.e. there exists $K > 0$ such that $g(x)$ is a linear function for all $x \geq K$; in this case we also say g is K-*linear*.

We believe the first two conditions are strictly necessary to achieve the desired results. The K-linearity may not be necessary but makes our analysis possible. We note that the constraints above encompass the standard European call and put options.

Henceforth we shall let G be a *zero-drift* GBM with unit volatility. In particular, we have that $\log G(t) \sim \mathcal{N}(-\frac{1}{2}t, t)$. For $S, c \geq 0$, define the function

$$U(S, c) = \mathbb{E}_G[g(S \cdot G(c))],$$

and observe that $U(S, 0) = g(S)$. Our goal will be to show that U is asymptotically the minimax price of the option. Most importantly, this function $U(S, c)$ is *identical* to $\mathcal{V}(S, \frac{1}{\sigma^2}(T - c))$, the Black-Scholes value of the option in (4) when the GBM volatility parameter is σ in the Black-Scholes analysis. In particular, analogous to to (3), $U(S, c)$ satisfies a differential equation:

$$\frac{1}{2}S^2 \frac{\partial^2 U}{\partial S^2} - \frac{\partial U}{\partial c} = 0. \tag{6}$$

The following is our main result of this paper.

Theorem 1. *Let $S > 0$ be the initial asset price and let $c > 0$ be the variance budget. Assume we have a sequence $\{\zeta_n\}$ with $\lim_{n \to \infty} \zeta_n = 0$ and $\liminf_{n \to \infty} n\zeta_n^2 > c$. Then*

$$\lim_{n \to \infty} V_{\zeta_n}^{(n)}(S; c) = U(S, c).$$

This statement tells us that the minimax price of an option, when Nature has a total fluctuation budget of c, approaches the Black-Scholes price of the option when the time to expiration is c. This is particularly surprising since our minimax pricing framework made no assumptions as to the stochastic process generating the price path. This is the same conclusion as in [1], but we obtained our result with a significantly weaker assumption. Unlike [1], however, we do not show that the adversary's minimax optimal stochastic price path necessarily converges to a GBM. The convergence of Nature's price path to GBM in [1] was made possible by the uniform per-round variance constraint.

The previous theorem is the result of two main technical contributions. First, we prove a lower bound on the limiting value of $V_{\zeta_n}^{(n)}(S; c)$ by exhibiting a simple randomized strategy for Nature in the form of a stochastic price path, and appealing to the Lindeberg-Feller central limit theorem. Second, we prove an $O(c\zeta^{1/4})$ upper bound on the deviation between $V_{\zeta}^{(n)}(S; c)$ and $U(S, c)$. The upper bound is obtained by providing an explicit strategy for the Investor:

$$\Delta(S, c) = S \frac{\partial U(S, c)}{\partial S}$$

and carefully bounding the difference between the output using this strategy and the game value. In the course of doing so, because we are invoking Taylor's remainder theorem, we need to bound the first few derivatives of $U(S, c)$. Bounding these derivatives turns out to be the crux of the analysis; in particular, it uses the full force of the assumptions on g, including that g is eventually linear, to avoid the pathological cases when the derivative of g converges to its limiting value very slowly.

5 Lower Bound

In this section we prove that $U(S, c)$ is a lower bound to the game value $V_{\zeta_n}^{(n)}(S; c)$. We note that the result in this section does not use the assumptions in Theorem 1 that $\zeta_n \to 0$, nor that g is convex and eventually linear. In particular, the following result also applies when the sequence (ζ_n) is a constant $\zeta > 0$.

Theorem 2. *Let $g\colon \mathbb{R}_0 \to \mathbb{R}_0$ be an L-Lipschitz function, and let $\{\zeta_n\}$ be a sequence of positive numbers with $\liminf_{n\to\infty} n\zeta_n^2 > c$. Then for every $S, c > 0$,*

$$\liminf_{n\to\infty} V_{\zeta_n}^{(n)}(S; c) \geq U(S, c).$$

The proof of Theorem 2 is based on correctly "guessing" a randomized strategy for Nature. For each $n \in \mathbb{N}$, define the random variables $R_{1,n}, \ldots, R_{n,n} \sim \text{Uniform}\{\pm\sqrt{c/n}\}$ i.i.d. Note that $(R_{i,n})_{i=1}^n$ satisfies **TotVarConstraint** by construction. Moreover, the assumption $\liminf_{n\to\infty} n\zeta_n^2 > c$ implies $\zeta_n > \sqrt{c/n}$ for all sufficiently large n, so eventually $(R_{i,n})$ also satisfies **JumpConstraint**. We have the following convergence result for $(R_{i,n})$, which we prove in Appendix A.

Lemma 3. *Under the same setting as in Theorem 2, we have the convergence in distribution*

$$\prod_{i=1}^n (1 + R_{i,n}) \xrightarrow{\ d\ } G(c) \quad as \quad n \to \infty.$$

Moreover, we also have the convergence in expectation

$$\lim_{n\to\infty} \mathbb{E}\left[g\left(S \cdot \prod_{i=1}^n (1 + R_{i,n})\right)\right] = U(S, c). \tag{7}$$

With the help of Lemma 3, we are now ready to prove Theorem 2.

Proof of Theorem 2. Let n be sufficiently large such that $n\zeta_n^2 > c$. Let $R_{i,n} \sim \text{Uniform}\{\pm\sqrt{c/n}\}$ i.i.d., for $1 \leq i \leq n$. As noted above, $(R_{i,n})$ satisfies both **TotVarConstraint** and **JumpConstraint**. Then we have

$$
\begin{aligned}
V_{\zeta_n}^{(n)}(S; c) &= \inf_{\Delta_1} \sup_{r_1} \cdots \inf_{\Delta_n} \sup_{r_n} \ g\left(S \cdot \prod_{i=1}^n (1 + r_i)\right) - \sum_{i=1}^n \Delta_i r_i \\
&\geq \inf_{\Delta_1} \cdots \inf_{\Delta_n} \mathbb{E}\left[g\left(S \cdot \prod_{i=1}^n (1 + R_{i,n})\right) - \sum_{i=1}^n \Delta_i R_{i,n}\right] \\
&= \mathbb{E}\left[g\left(S \cdot \prod_{i=1}^n (1 + R_{i,n})\right)\right].
\end{aligned}
$$

The first line follows from unrolling the recursion in the definition (5); the second line from replacing the supremum over (r_i) with expectation over $(R_{i,n})$; and the third line from $\mathbb{E}[R_{i,n}] = 0$. Taking limit on both sides and using (7) from Lemma 3 give us the desired conclusion. $\qquad\square$

6 Upper Bound

In this section we prove that $U(S, c)$ is an upper bound to the limit of $V_\zeta^{(n)}(S; c)$.

Theorem 4. *Let $g\colon \mathbb{R}_0 \to \mathbb{R}_0$ be a convex, L-Lipschitz, K-linear function. Let $0 < \zeta \leq 1/16$. Then for any $S, c > 0$ and $n \in \mathbb{N}$, we have*

$$V_\zeta^{(n)}(S; c) \leq U(S, c) + \left(18c + \frac{8}{\sqrt{2\pi}}\right) LK\, \zeta^{1/4}.$$

We remark that the right-hand side of the above bound does not depend on the number of trading periods n. The key parameter is ζ, which determines the size of the largest price jump of the stock. However, we expect that as the trading frequency increases, the size of the largest price jump will shrink. Plugging a sequence $\{\zeta_n\}$ in place of ζ in Theorem 4 gives us the following corollary.

Corollary 1. *Let $g\colon \mathbb{R}_0 \to \mathbb{R}_0$ be a convex, L-Lipschitz, K-linear function. Let $\{\zeta_n\}$ be a sequence of positive numbers with $\zeta_n \to 0$. Then for $S, c > 0$,*

$$\limsup_{n\to\infty} V_{\zeta_n}^{(n)}(S; c) \leq U(S, c).$$

Note that the above upper bound relies on the convexity of g, for if g were concave, then we would have the reverse conclusion:

$$V_\zeta^{(n)}(S; c) \geq g(S) = g(S \cdot \mathbb{E}[G(c)]) \geq \mathbb{E}[g(S \cdot G(c))] = U(S, c).$$

Here the first inequality follows from setting all $r = 0$ in (5), and the second is by Jensen's inequality.

6.1 Intuition

For brevity, we write the partial derivatives $U_c(S,c) = \partial U(S,c)/\partial c$, $U_S(S,c) = \partial U(S,c)/\partial S$, and $U_{S^2}(S,c) = \partial^2 U(S,c)/\partial S^2$. The proof of Theorem 4 proceeds by providing a "guess" for the Investor's action, which is a modification of the original Black-Scholes hedging strategy. Specifically, when the current price is S and the remaining budget is c, then the Investor invests

$$\Delta(S,c) := SU_S(S,c).$$

We now illustrate how this strategy gives rise to a bound on $V_\zeta^{(n)}(S;c)$ as stated in Theorem 4. First suppose for some $m \geq 1$ we know that $V_\zeta^{(n)}(S;c, m-1)$ is a rough approximation to $U(S,c)$. Note that a Taylor approximation of the function $r_m \mapsto U(S + Sr_m, c - r_m^2)$ around $U(S,c)$ gives us

$$U(S + Sr_m, c - r_m^2) = U(S,c) + r_m SU_S(S,c) - r_m^2 U_c(S,c) + \frac{1}{2}r_m^2 S^2 U_{S^2}(S,c) + O(r_m^3)$$

$$= U(S,c) + r_m SU_S(S,c) + O(r_m^3),$$

where the last line follows from the Black-Scholes equation (6). Now by setting $\Delta = SU_S(S,c)$ in the definition (5), and using the assumption and the Taylor approximation above, we obtain

$$V_\zeta^{(n)}(S;c,m) = \inf_{\Delta \in \mathbb{R}} \sup_{|r_m| \leq \min\{\zeta, \sqrt{c}\}} -\Delta r_m + V_\zeta^{(n)}(S + Sr_m; c - r_m^2, m-1)$$

$$\leq \sup_{r_m} -r_m SU_S(S,c) + V_\zeta^{(n)}(S + Sr_m; c - r_m^2, m-1)$$

$$= \sup_{r_m} -r_m SU_S(S,c) + U(S + Sr_m, c - r_m^2) + \text{(approx terms)}$$

$$= U(S,c) + O(r_m^3) + \text{(approx terms)}.$$

In other words, on each round of the game we add an $O(r_m^3)$ term to the approximation error. By the time we reach $V_\zeta^{(n)}(S;c,n)$ we will have an error term that is roughly on the order of $\sum_{m=1}^n |r_m|^3$. Since $\sum_{m=1}^n r_m^2 \leq c$ and $|r_m| \leq \zeta$ by assumption, we get $\sum_{m=1}^n |r_m|^3 \leq \zeta c$.

The details are more intricate because the error $O(r_m^3)$ from the Taylor approximation also depends on S and c. Trading off the dependencies of c and ζ leads us to the bound stated in Theorem 4.

6.2 Proof (Sketch) of Theorem 4

In this section we describe an outline of the proof of Theorem 4. Throughout, we assume g is a convex, L-Lipschitz, K-linear function, and $0 < \zeta \leq 1/16$. The proofs of Lemma 5 and Lemma 7 are provided in Appendix B, and Lemma 6 is proved in Appendix C.

For $S, c > 0$ and $|r| \leq \sqrt{c}$, we define the *(single-round) error term* of the Taylor approximation,

$$\epsilon_r(S,c) := U(S + Sr, c - r^2) - U(S,c) - rSU_S(S,c). \tag{8}$$

We also define a sequence $\{\alpha^{(n)}(S,c,m)\}_{m=0}^n$ to keep track of the cumulative errors. We define this sequence by setting $\alpha^{(n)}(S,c,0) = 0$, and for $1 \leq m \leq n$,

$$\alpha^{(n)}(S,c,m) := \sup_{|r| \leq \min\{\zeta, \sqrt{c}\}} \epsilon_r(S,c) + \alpha^{(n)}(S + Sr, c - r^2, m-1). \tag{9}$$

For simplicity, we write $\alpha^{(n)}(S,c) \equiv \alpha^{(n)}(S,c,n)$. Then we have the following result, which formalizes the notion from the preceding section that $V_\zeta^{(n)}(S;c,m)$ is an approximation to $U(S,c)$.

Lemma 5. *For $S, c > 0$, $n \in \mathbb{N}$, and $0 \leq m \leq n$, we have*

$$V_\zeta^{(n)}(S;c,m) \leq U(S,c) + \alpha^{(n)}(S,c,m). \tag{10}$$

It now remains to bound $\alpha^{(n)}(S,c)$ from above. A key step in doing so is to show the following bounds on ϵ_r. This is where the assumptions that g be L-Lipschitz and K-linear are important.

Lemma 6. *For $S, c > 0$, and $|r| \leq \min\{1/16, \sqrt{c}/8\}$, we have*

$$\epsilon_r(S, c) \leq 16LK \left(\max\{c^{-3/2}, c^{-1/2}\} |r|^3 + \max\{c^{-2}, c^{-1/2}\} r^4 \right). \tag{11}$$

Moreover, for $S > 0$, $0 < c \leq 1/4$, and $|r| \leq \sqrt{c}$, we also have

$$\epsilon_r(S, c) \leq \frac{4LK}{\sqrt{2\pi}} \cdot \frac{r^2}{\sqrt{c}}. \tag{12}$$

Using Lemma 6, we have the following bound on $\alpha^{(n)}(S, c)$.

Lemma 7. *For $S, c > 0$, $n \in \mathbb{N}$, and $0 < \zeta \leq 1/16$, we have*

$$\alpha^{(n)}(S, c) \leq \left(18c + \frac{8}{\sqrt{2\pi}} \right) LK \, \zeta^{1/4}.$$

Proof (sketch). By unrolling the inductive definition (9), we can write $\alpha^{(n)}(S, c)$ as the supremum of $f(r_1, \ldots, r_n)$, where

$$f(r_1, \ldots, r_n) = \sum_{m=1}^{n} \epsilon_{r_m} \left(S \prod_{i=1}^{m-1}(1 + r_i), \, c - \sum_{i=1}^{m-1} r_i^2 \right).$$

Let $(r_1, \ldots, r_n)$ be such that $|r_m| \leq \zeta$ and $\sum_{m=1}^{n} r_m^2 \leq c$. We will show that $f(r_1, \ldots, r_n) \leq (18c + 8/\sqrt{2\pi}) LK \, \zeta^{1/4}$. Let $0 \leq n_* \leq n$ be the largest index such that $\sum_{m=1}^{n_*} r_m^2 \leq c - \sqrt{\zeta}$. We split the analysis into two parts.

1. **For $1 \leq m \leq \min\{n, \, n_* + 1\}$:** By (11) from Lemma 6 and a little calculation, we have

$$\epsilon_{r_m} \left(S \prod_{i=1}^{m-1}(1 + r_i), \, c - \sum_{i=1}^{m-1} r_i^2 \right) \leq 18LK \, \zeta^{1/4} \, r_m^2.$$

Summing over $1 \leq m \leq \min\{n, \, n_* + 1\}$ then gives us

$$\sum_{m=1}^{\min\{n,\, n_*+1\}} \epsilon_{r_m} \left(S \prod_{i=1}^{m-1}(1+r_i), c - \sum_{i=1}^{m-1} r_i^2 \right) \leq 18LK\zeta^{1/4} \sum_{m=1}^{\min\{n,\, n_*+1\}} r_m^2 \leq 18LK\zeta^{1/4} c.$$

2. **For $n_* + 2 \leq m \leq n$ (if $n_* \leq n - 2$):** By (12) from Lemma 6, we have

$$\epsilon_{r_m} \left(S \prod_{i=1}^{m-1}(1 + r_i), \, c - \sum_{i=1}^{m-1} r_i^2 \right) \leq \frac{4LK}{\sqrt{2\pi}} \cdot \frac{r_m^2}{\sqrt{\sum_{i=m}^{n} r_i^2}}.$$

Therefore,

$$\sum_{m=n_*+2}^{n} \epsilon_{r_m} \left(S \prod_{i=1}^{m-1}(1 + r_i), \, c - \sum_{i=1}^{m-1} r_i^2 \right) \leq \frac{4LK}{\sqrt{2\pi}} \sum_{m=n_*+2}^{n} \frac{r_m^2}{\sqrt{\sum_{i=m}^{n} r_i^2}} \leq \frac{8LK}{\sqrt{2\pi}} \zeta^{1/4},$$

where the last inequality follows from Lemma 8 in Appendix B.

Combining the two cases above gives us the desired conclusion. $\qquad\square$

Proof of Theorem 4. Theorem 4 follows immediately from Lemma 5 and Lemma 7. $\qquad\square$

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SUPPLEMENTARY MATERIAL TO
How to Hedge an Option Against an Adversary: Black-Scholes Pricing is Minimax Optimal

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A Proof of Lemma 3

For each $1 \leq i \leq n$, the random variable $\log(1 + R_{i,n})$ has mean and variance, respectively,

$$\mu_n = \frac{1}{2} \log \left(1 - \frac{c}{n} \right) \quad \text{and} \quad \sigma_n^2 = \frac{1}{4} \log^2 \left(\frac{\sqrt{n} + \sqrt{c}}{\sqrt{n} - \sqrt{c}} \right).$$

We now define

$$X_{i,n} := \frac{\log(1 + R_{i,n}) - \mu_n}{\sigma_n \sqrt{n}}, \tag{13}$$

so $X_{1,n}, \ldots, X_{n,n}$ are i.i.d. random variables with $\mathbb{E}[X_{i,n}] = 0$ and $\sum_{i=1}^{n} \mathbb{E}[X_{i,n}^2] = 1$. Recalling that $R_{i,n} \in \{\pm\sqrt{c/n}\}$, we see that the two possible values for $X_{i,n}$ both approach 0 as $n \to \infty$. This means for any $\epsilon > 0$ we can find a sufficiently large n such that $|X_{i,n}| < \epsilon$ for all $1 \leq i \leq n$. In particular, this implies the Lindeberg condition for the triangular array $(X_{i,n}, 1 \leq i \leq n)$: for all $\epsilon > 0$,

$$\lim_{n \to \infty} \sum_{i=1}^{n} \mathbb{E}[X_{i,n}^2 \, \mathbf{1}\{|X_{i,n}| > \epsilon\}] = 0.$$

Thus, by the Lindeberg central limit theorem [4, Theorem 3.4.5], we have the convergence in distribution $\sum_{i=1}^{n} X_{i,n} \xrightarrow{d} Z$, where $Z \sim \mathcal{N}(0, 1)$ is a standard Gaussian random variable.

Clearly $\mu_n \to 0$ as $n \to \infty$. Furthermore, one can easily verify that by the L'Hôpital's rule,

$$\lim_{n \to \infty} \sigma_n \sqrt{n} = \sqrt{c} \quad \text{and} \quad \lim_{n \to \infty} n\mu_n = -\frac{c}{2}.$$

Therefore, from the convergence $\sum_{i=1}^{n} X_{i,n} \xrightarrow{d} Z$ and recalling the definition (13) of $X_{i,n}$, we also obtain

$$\sum_{i=1}^{n} \log(1 + R_{i,n}) = \sum_{i=1}^{n} \left(\mu_n + \sigma_n \sqrt{n} X_{i,n} \right) = n\mu_n + \sigma_n \sqrt{n} \sum_{i=1}^{n} X_{i,n} \xrightarrow{d} \frac{-c}{2} + \sqrt{c}Z.$$

In particular, by the continuous mapping theorem,

$$\prod_{i=1}^{n} (1 + R_{i,n}) \xrightarrow{d} \exp \left(-\frac{c}{2} + \sqrt{c}Z \right) \overset{d}{=} G(c).$$

We now want to show that we also have convergence in expectation when g is an L-Lipschitz function, namely, that $\mathbb{E}[g(S \cdot \prod_{i=1}^{n}(1 + R_{i,n}))] \to \mathbb{E}[g(S \cdot G(c))]$. Without loss of generality (by replacing $g(x)$ by $\hat{g}(x) = g(S \cdot x) - g(0)$) we may assume $S = 1$ and $g(0) = 0$. For simplicity, let $S_n = \prod_{i=1}^{n}(1 + R_{i,n})$. For each $M > 0$ define the continuous bounded function $g_M(x) = \min\{g(x), M\}$. The convergence in distribution $S_n \xrightarrow{d} G(c)$ gives us

$$\lim_{n \to \infty} \mathbb{E}[g_M(S_n))] = \mathbb{E}[g_M(G(c))] \quad \text{for all } M > 0. \tag{14}$$

Since $g_M \uparrow g$ pointwise, by the monotone convergence theorem we also have

$$\lim_{M \to \infty} \mathbb{E}[g_M(G(c))] = \mathbb{E}[g(G(c))]. \tag{15}$$

Now observe that $\mathbb{E}[S_n] = 1$ and $\mathbb{E}[S_n^2] = (1 + c/n)^n \leq \exp(c)$. Since $g(0) = 0$ and g is L-Lipschitz, we have $g(x) \leq Lx$ for all $x \geq 0$. In particular, $\mathbb{E}[g(S_n)^2] \leq L^2\mathbb{E}[S_n^2] \leq L^2 \exp(c)$. Moreover, by Markov's inequality,

$$\mathbb{P}(g(S_n) > M) \leq \mathbb{P}\left(S_n > \frac{M}{L}\right) \leq \frac{\mathbb{E}[S_n]}{M/L} = \frac{L}{M}.$$

Therefore, for each n and for all $M > 0$, by Cauchy-Schwarz inequality,

$$\begin{aligned}
\left|\mathbb{E}[g(S_n)] - \mathbb{E}[g_M(S_n)]\right| &= \mathbb{E}[(g(S_n) - M) \cdot \mathbf{1}\{g(S_n) > M\}] \\
&\leq \mathbb{E}[g(S_n) \cdot \mathbf{1}\{g(S_n) > M\}] \\
&\leq \mathbb{E}[g(S_n)^2]^{1/2}\, \mathbb{P}(g(S_n) > M)^{1/2} \\
&\leq \left(L^3 \exp(c)/M\right)^{1/2}.
\end{aligned}$$

Since the final bound does not involve n, this shows that $\lim_{M \to \infty} \mathbb{E}[g_M(S_n)] \to \mathbb{E}[g(S_n)]$ uniformly in n. This allows us to interchange the order of the limit operations below, which, together with (14) and (15), give us our desired result:

$$\lim_{n \to \infty} \mathbb{E}[g(S_n)] = \lim_{n \to \infty} \lim_{M \to \infty} \mathbb{E}[g_M(S_n)] = \lim_{M \to \infty} \lim_{n \to \infty} \mathbb{E}[g_M(S_n)] = \lim_{M \to \infty} \mathbb{E}[g_M(G(c))] = \mathbb{E}[g(G(c))].$$

This completes the proof of Lemma 3.

B   Proofs of Lemma 5 and Lemma 7

B.1   Proof of Lemma 5

Lemma 5 essentially follows from the definition of $\alpha^{(n)}$.

Proof of Lemma 5.   We proceed by induction on m. For the base case $m = 0$, we use Jensen's inequality and the fact that $\mathbb{E}[G(c)] = 1$:

$$V_\zeta^{(n)}(S; c, 0) = g(S) = g(S \cdot \mathbb{E}[G(c)]) \leq \mathbb{E}[g(S \cdot G(c))] = U(S, c).$$

Now assume the statement (10) holds for $m - 1$. Then for m,

$$\begin{aligned}
V_\zeta^{(n)}(S; c, m) &= \inf_{\Delta \in \mathbb{R}} \sup_{|r| \leq \min\{\zeta, \sqrt{c}\}} -\Delta r + V_\zeta^{(n)}(S + Sr; c - r^2, m - 1) \\
&\leq \inf_{\Delta \in \mathbb{R}} \sup_{|r| \leq \min\{\zeta, \sqrt{c}\}} -\Delta r + U(S + Sr, c - r^2) + \alpha^{(n)}(S + Sr, c - r^2, m - 1) \\
&\leq \sup_{|r| \leq \min\{\zeta, \sqrt{c}\}} -rSU_S(S, c) + U(S + Sr, c - r^2) + \alpha^{(n)}(S + Sr, c - r^2, m - 1) \\
&= \sup_{|r| \leq \min\{\zeta, \sqrt{c}\}} U(S, c) + \epsilon_r(S, c) + \alpha^{(n)}(S + Sr, c - r^2, m - 1) \\
&= U(S, c) + \alpha^{(n)}(S, c, m).
\end{aligned}$$

The first line is from the definition (5); the second line is using the inductive hypothesis that (10) holds for $m - 1$; the third line is from substituting the choice $\Delta = SU_S(S, c)$; the fourth line is from the definition of ϵ_r; and the last line is from the definition of $\alpha^{(n)}(S, c, m)$. $\qquad \square$

B.2 Proof of Lemma 7

For completeness, we provide a more detailed proof of Lemma 7.

Proof of Lemma 7. Unrolling the inductive definition (9), we can write

$$\alpha^{(n)}(S, c) = \sup_{\substack{r_1, \ldots, r_n \\ |r_m| \leq \zeta, \ \sum_{m=1}^n r_m^2 \leq c}} f(r_1, \ldots, r_n),$$

where f is the function

$$f(r_1, \ldots, r_n) = \sum_{m=1}^n \epsilon_{r_m}\left(S \prod_{i=1}^{m-1}(1 + r_i), \ c - \sum_{i=1}^{m-1} r_i^2\right).$$

Let $(r_1, \ldots, r_n)$ be such that $|r_m| \leq \zeta$ and $\sum_{m=1}^n r_m^2 \leq c$. We will show that $f(r_1, \ldots, r_n) \leq (18c + 8/\sqrt{2\pi}) LK \zeta^{1/4}$.

Assume for now that $\zeta \leq c^2$. Let $0 \leq n_* \leq n$ be the largest index such that

$$\sum_{m=1}^{n_*} r_m^2 \leq c - \sqrt{\zeta}.$$

We split the analysis into two parts.

For $1 \leq m \leq \min\{n, \ n_* + 1\}$: We want to apply the bound in Lemma 6, so let us verify that the conditions in Lemma 6 are satisfied. Clearly $|r_m| \leq \zeta \leq 1/16$. Moreover, since $c - \sum_{i=1}^{m-1} r_i^2 \geq c - \sum_{i=1}^{n_*} r_i^2 \geq \sqrt{\zeta}$ and $\zeta \leq 1/16$, we also have

$$|r_m| \leq \zeta \leq \frac{\zeta^{1/4}}{8} \leq \frac{\sqrt{c - \sum_{i=1}^{m-1} r_i^2}}{8}.$$

Therefore, by (11) from Lemma 6,

$$\epsilon_{r_m}\left(S\prod_{i=1}^{m-1}(1+r_i), \ c - \sum_{i=1}^{m-1} r_i^2\right) \leq 16LK\left(\max\left\{\left(c - \sum_{i=1}^{m-1} r_i^2\right)^{-3/2}, \left(c - \sum_{i=1}^{m-1} r_i^2\right)^{-1/2}\right\} |r_m|^3\right.$$

$$\left. + \max\left\{\left(c - \sum_{i=1}^{m-1} r_i^2\right)^{-2}, \left(c - \sum_{i=1}^{m-1} r_i^2\right)^{-1/2}\right\} r_m^4\right)$$

$$\leq 16LK\left(\max\{\zeta^{-3/4}, \zeta^{-1/4}\} |r_m|^3 + \max\{\zeta^{-1}, \zeta^{-1/4}\} r_m^4\right)$$

$$= 16LK\left(\zeta^{-3/4} |r_m|^3 + \zeta^{-1} r_m^4\right) \qquad \text{(since } \zeta < 1\text{)}$$

$$\leq 16LK\left(\zeta^{1/4} r_m^2 + \zeta r_m^2\right) \qquad \text{(since } |r_m| \leq \zeta\text{)}$$

$$\leq 16LK\left(\zeta^{1/4} r_m^2 + \zeta^{1/4} \frac{1}{16^{3/4}} r_m^2\right) \qquad \text{(since } \zeta \leq 1/16\text{)}$$

$$= 18LK \zeta^{1/4} r_m^2.$$

Summing over $1 \leq m \leq \min\{n, \ n_* + 1\}$ gives us

$$\sum_{m=1}^{\min\{n, \ n_*+1\}} \epsilon_{r_m}\left(S\prod_{i=1}^{m-1}(1+r_i), c - \sum_{i=1}^{m-1} r_i^2\right) \leq 18LK\zeta^{1/4} \sum_{m=1}^{\min\{n, \ n_*+1\}} r_m^2 \leq 18LK\zeta^{1/4} c. \quad (16)$$

For $n_* + 2 \leq m \leq n$, if $n_* \leq n - 2$: Without loss of generality we may assume $r_n \neq 0$, for if $r_n = 0$, then the term depending on r_n does not affect $f(r_1, \ldots, r_n)$ since

$$\epsilon_{r_n}\left(S\prod_{i=1}^{n-1}(1+r_i), \ c - \sum_{i=1}^{n-1} r_i^2\right) = 0,$$

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so we can remove r_n and only consider $n_* + 2 \le m \le n - 1$. From the definition of n_* we see that $\sum_{m=1}^{n_*+1} r_m^2 > c - \sqrt{\zeta}$, and since $\sum_{m=1}^{n} r_m^2 \le c$, this implies

$$\sum_{m=n_*+2}^{n} r_m^2 \le c - \sum_{m=1}^{n_*+1} r_m^2 < c - (c - \sqrt{\zeta}) = \sqrt{\zeta}. \tag{17}$$

Note also that for each $n_* + 2 \le m \le n$,

$$0 < r_n^2 \le \sum_{i=m}^{n} r_i^2 \le c - \sum_{i=1}^{m-1} r_i^2 \le c - \sum_{i=1}^{n_*+1} r_i^2 \le \sqrt{\zeta} \le \frac{1}{4},$$

so by (12) from Lemma 6,

$$\epsilon_{r_m}\left(S\prod_{i=1}^{m-1}(1+r_i),\ c - \sum_{i=1}^{m-1} r_i^2\right) \le \frac{4LK}{\sqrt{2\pi}} \cdot \frac{r_m^2}{\sqrt{c - \sum_{i=1}^{m-1} r_i^2}} \le \frac{4LK}{\sqrt{2\pi}} \cdot \frac{r_m^2}{\sqrt{\sum_{i=m}^{n} r_i^2}}.$$

Therefore, by applying Lemma 8 below to $x_i = r_{n_*+1+i}^2$, we see that

$$\begin{aligned}
\sum_{m=n_*+2}^{n} \epsilon_{r_m}\left(S\prod_{i=1}^{m-1}(1+r_i),\ c - \sum_{i=1}^{m-1} r_i^2\right) &\le \frac{4LK}{\sqrt{2\pi}} \sum_{m=n_*+2}^{n} \frac{r_m^2}{\sqrt{\sum_{i=m}^{n} r_i^2}} \\
&\le \frac{8LK}{\sqrt{2\pi}}\left(\sum_{m=n_*+2}^{n} r_m^2\right)^{1/2} \le \frac{8LK}{\sqrt{2\pi}} \zeta^{1/4},
\end{aligned} \tag{18}$$

where the last inequality follows from (17). Combining (16) and (18) gives us the desired conclusion.

Now if $\zeta > c^2$, then the argument in the second case above (for $n_* + 2 \le m \le n$) still holds with n_* set to be -1, so we still get the same conclusion. $\qquad \square$

It now remains to prove the following result, which we use at the end of the proof of Lemma 7.

Lemma 8. *For $x_1, \ldots, x_k \ge 0$ with $x_k > 0$, we have*

$$\sum_{i=1}^{k} \frac{x_i}{\sqrt{x_i + x_{i+1} + \cdots + x_k}} \le 2\left(\sum_{i=1}^{k} x_i\right)^{1/2}.$$

Proof. Let $\mathcal{L}_k$ denote the objective function that we wish to bound,

$$\mathcal{L}_k(x_1, \ldots, x_k) = \sum_{i=1}^{k} \frac{x_i}{\sqrt{x_i + x_{i+1} + \cdots + x_k}},$$

and note that for any $t > 0$,

$$\mathcal{L}_k(tx_1, \ldots, tx_k) = \sqrt{t}\, \mathcal{L}_k(x_1, \ldots, x_k), \tag{19}$$

For each $k \in \mathbb{N}$, let Δ_k denote the unit simplex in $\mathbb{R}^k$ with $x_k > 0$,

$$\Delta_k = \left\{(x_1, \ldots, x_k) :\ x_1, \ldots, x_{k-1} \ge 0,\ x_k > 0,\ \sum_{i=1}^{k} x_i = 1\right\},$$

and let η_k denote the supremum of the function $\mathcal{L}_k$ over $x \in \Delta_k$. Given $x = (x_1, \ldots, x_k) \in \Delta_k$, define $y = (y_1, \ldots, y_{k-1})$ by $y_i = x_{i+1}/(1 - x_1)$, so $y \in \Delta_{k-1}$. Then we can write

$$\begin{aligned}
\mathcal{L}_k(x_1, \ldots, x_k) &= \frac{x_1}{\sqrt{x_1 + \cdots + x_k}} + \mathcal{L}_{k-1}(x_2, \ldots, x_k) \\
&= x_1 + \sqrt{1 - x_1}\, \mathcal{L}_{k-1}(y_1, \ldots, y_{k-1}) \\
&\le x_1 + \sqrt{1 - x_1}\, \eta_{k-1},
\end{aligned}$$

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where the second equality is from (19) and the last inequality is from the definition of η_{k-1}. The function $x_1 \mapsto x_1 + \sqrt{1 - x_1}\, \eta_{k-1}$ is concave and maximized at $x_1^* = 1 - \eta_{k-1}^2/4$, giving us

$$\mathcal{L}_k(x_1, \ldots, x_k) \leq x_1^* + \sqrt{1 - x_1^*}\, \eta_{k-1} = 1 - \frac{\eta_{k-1}^2}{4} + \sqrt{\frac{\eta_{k-1}^2}{4}}\, \eta_{k-1} = 1 + \frac{\eta_{k-1}^2}{4}.$$

Taking the supremum over $x \in \Delta_k$ gives us the recursion

$$\eta_k \leq 1 + \frac{\eta_{k-1}^2}{4},$$

which, along with the base case $\eta_1 = 1$, easily implies $\eta_k \leq 2$ for all $k \in \mathbb{N}$. Now given $x_1, \ldots, x_k \geq 0$ with $x_k > 0$, let $x' = (tx_1, \ldots, tx_k)$ with $t = 1/(x_1 + \cdots + x_k)$, so $x' \in \Delta_k$. Then using (19) and the bound $\eta_k \leq 2$, we get

$$\mathcal{L}_k(x_1, \ldots, x_k) = \frac{1}{\sqrt{t}}\, \mathcal{L}_k(tx_1, \ldots, tx_k) \leq \eta_k \left(\sum_{i=1}^k x_i \right)^{1/2} \leq 2 \left(\sum_{i=1}^k x_i \right)^{1/2},$$

as desired. $\qquad\square$

C Proof of Lemma 6

In this section we provide a proof of Lemma 6. Throughout the rest of this paper, we use the following notation for the higher-order partial derivatives of U,

$$U_{S^a c^b}(S, c) = \frac{\partial^{a+b} U(S, c)}{\partial S^a \partial c^b}, \quad a, b \in \mathbb{N}_0.$$

We will use the following bounds on U_{S^2}, U_{S^3}, and U_{S^4}, which we prove in Appendix D. These bounds are where we use the crucial assumptions that the payoff function g is convex, L-Lipschitz, and K-linear.

Lemma 9. *Let $g \colon \mathbb{R}_0 \to \mathbb{R}_0$ be a convex, L-Lipschitz, K-linear function. Then for all $S, c > 0$,*

$$|U_{S^2}(S, c)| \leq \frac{2LK}{\sqrt{2\pi}} \cdot \frac{1}{S^2 \sqrt{c}} \tag{20}$$

$$|U_{S^3}(S, c)| \leq 7LK \cdot \frac{\max\{c^{-3/2},\, c^{-1/2}\}}{S^3}, \tag{21}$$

$$|U_{S^4}(S, c)| \leq 28LK \cdot \frac{\max\{c^{-2},\, c^{-1/2}\}}{S^4}. \tag{22}$$

We will also use the following property of the function U.

Lemma 10. *The function $U(S, c)$ is convex in S and non-decreasing in c.*

Proof. For each fixed $c \geq 0$ and for each realization of the random variable $G(c) > 0$, the function $S \mapsto g(S \cdot G(c))$ is convex. Therefore, $U(S, c)$ is convex in S, being a nonnegative linear combination of convex functions. In particular, this implies $U_{S^2}(S, c) \geq 0$. So by the Black-Scholes equation (6), we also have $U_c(S, c) = \frac{1}{2}S^2 U_{S^2}(S, c) \geq 0$. $\qquad\square$

We are now ready to prove Lemma 6. For clarity, we divide the proof into two parts: we first prove the bound (12), then prove the bound (11).

Proof of (12) **in Lemma 6.** Recall that $U(S, c)$ is non-decreasing in c by Lemma 10. Then by the Taylor remainder theorem, we can write

$$\begin{aligned}
\epsilon_r(S, c) &= U(S + Sr, c - r^2) - U(S, c) - rSU_S(S, c) \\
&\leq U(S + Sr, c) - U(S, c) - rSU_S(S, c) \\
&= \frac{1}{2}r^2 S^2 U_{S^2}(S + S\xi, c)
\end{aligned}$$

where ξ is some value between 0 and r. Since $|\xi| \le |r| \le \sqrt{c} \le 1/2$, we have $(1 + \xi)^2 \ge 1/4$. Moreover, from (20) in Lemma 9, we have

$$\left|(1 + \xi)^2 S^2 U_{S^2}(S + S\xi, c)\right| \le \frac{2LK}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{c}}.$$

Combining the bounds above gives us

$$\epsilon_r(S, c) \le \frac{1}{2} \frac{r^2}{(1 + \xi)^2} \left|(1 + \xi)^2 S^2 U_{S^2}(S + S\xi, c)\right| \le \frac{4LK}{\sqrt{2\pi}} \cdot \frac{r^2}{\sqrt{c}},$$

as desired. $\qquad\square$

Proof of (11) **in Lemma 6.** Fix $S, c > 0$, and consider the function

$$f(r) = U(S + Sr, c - r^2), \quad |r| \le \sqrt{c}.$$

By repeatedly applying the Black-Scholes differential equation (6), we can easily verify that $f(0) = U(S, c)$, $f'(0) = SU_S(S, c)$, and

$$f''(r) = p_2(r)\, r\, S^2 U_{S^2}(S + Sr, c - r^2)\; +\; p_3(r)\,(1 + r)^2 r\, S^3 U_{S^3}(S + Sr, c - r^2) \\ +\; (1 + r)^4 r^2\, S^4 U_{S^4}(S + Sr, c - r^2), \tag{23}$$

where p_2, p_3 are the polynomials $p_2(r) = 2r^3 + 4r^2 - 3r - 6$ and $p_3(r) = 4r^2 + 4r - 2$.

Noting that we can write

$$\epsilon_r(S, c) = f(r) - f(0) - f'(0)r,$$

another application of Taylor's remainder theorem allows us to write

$$\epsilon_r(S, c) = \frac{1}{2} f''(\xi) r^2$$

for some ξ lying between 0 and r. It is easy to verify that we have

$$\left|\frac{p_2(\xi)}{(1 + \xi)^2}\right| \le 7, \qquad \left|\frac{p_3(\xi)}{(1 + \xi)}\right| \le 3 \qquad \text{for all } |\xi| \le |r| \le \frac{1}{16}.$$

Moreover, since $\xi^2 \le r^2 \le c/64$, we have $c - \xi^2 \ge \frac{63}{64}c$. Then from the bound (20) in Lemma 9, we have

$$\left|(1 + \xi)^2 S^2 U_{S^2}(S + S\xi, c - \xi^2)\right| \le \frac{2LK}{\sqrt{2\pi}} \cdot \frac{1}{(c - \xi^2)^{1/2}} \le \frac{2LK}{\sqrt{2\pi}} \cdot \frac{1}{(\frac{63}{64}c)^{1/2}} \le LK\, c^{-1/2}.$$

We also get from the bound (21) in Lemma 9,

$$\begin{aligned}
\left|(1 + \xi)^3 S^3 U_{S^3}(S + S\xi, c - \xi^2)\right| &\le 7LK\; \max\{(c - \xi^2)^{-3/2}, (c - \xi^2)^{-1/2}\} \\
&\le 7LK\; \max\left\{\left(\frac{63}{64}c\right)^{-3/2}, \left(\frac{63}{64}c\right)^{-1/2}\right\} \\
&\le 7LK\; \left(\frac{64}{63}\right)^{3/2} \max\{c^{-3/2}, c^{-1/2}\} \\
&\le 8LK\; \max\{c^{-3/2}, c^{-1/2}\}.
\end{aligned}$$

Similarly, the bound (22) in Lemma 9 gives us

$$\left|(1 + \xi)^4 S^4 U_{S^4}(S + S\xi, c - \xi^2)\right| \le 29LK\; \max\{c^{-2}, c^{-1/2}\}.$$

Applying the bounds above to (23) gives us

$$\begin{aligned}
|f''(\xi)| &\le \left|\frac{p_2(\xi)}{(1 + \xi)^2}\right| \cdot |\xi| \cdot \left|(1 + \xi)^2 S^2 U_{S^2}(S + S\xi, c - \xi^2)\right| \\
&\quad + \left|\frac{p_3(\xi)}{(1 + \xi)}\right| \cdot |\xi| \cdot \left|(1 + \xi)^3 S^3 U_{S^3}(S + S\xi, c - \xi^2)\right| \\
&\quad + \xi^2 \cdot \left|(1 + \xi)^4 S^4 U_{S^4}(S + S\xi, c - \xi^2)\right| \\
&\le 7LK\, |r|\, c^{-1/2} + 24LK\, |r|\, \max\{c^{-3/2}, c^{-1/2}\} + 29LK\, r^2\, \max\{c^{-2}, c^{-1/2}\} \\
&\le 31LK\, |r|\, \max\{c^{-3/2}, c^{-1/2}\} + 29LK\, r^2\, \max\{c^{-2}, c^{-1/2}\}.
\end{aligned}$$

Therefore, we obtain

$$|\epsilon_r(S,c)| = \frac{1}{2}|f''(\xi)| \cdot r^2 \le 16LK \left(|r|^3 \max\{c^{-3/2}, c^{-1/2}\} + r^4 \max\{c^{-2}, c^{-1/2}\}\right),$$

as desired. $\qquad\square$

D Proof of the Bounds on the Derivatives (Lemma 9)

In this section we prove the bounds on the higher-order derivatives $U_{S^a}(S,c)$, $a \ge 0$. Proving the bounds in Lemma 9 is more difficult than the analysis that we have done so far, and uses the full force of the assumptions that the payoff function g is convex, L-Lipschitz, and K-linear.

The outline of the proof is as follows. By writing $U(S,c)$ as a convolution, we can write its derivatives $U_{S^a}(S,c)$ as an expectation of $g(S \cdot G(c))$ modulated by certain polynomials (Appendix D.1). The K-linearity of g allows us to approximate g by the European-option payoff function g_{EC} that we encountered in Section 2, so we first prove Lemma 9 for the specific case when the payoff function is g_{EC} (Appendix D.2). We extend the bound on $U_{S^2}(S,c)$ to the general case by dominating the function inside the expectation by another carefully constructed function (Appendix D.3). Finally, we use the approximation of g by g_{EC} to prove the bounds on the higher-order derivatives U_{S^3} and U_{S^4} (Appendix D.4). In particular, Lemma 15 proves the bound (20), and Lemma 18 proves the bounds (21) and (22).

Throughout the rest of this appendix, $Z \sim \mathcal{N}(0,1)$ denotes a standard Gaussian random variable, and Φ and ϕ denote the cumulative distribution function and the probability density function, respectively, of the standard Gaussian distribution. The symbol $*$ denotes the convolution operator on $\mathbb{R}$. We also use the fact that $G(c) \overset{d}{=} \exp(-\frac{1}{2}c + \sqrt{c}Z)$. Recall that the convexity of g implies differentiability almost everywhere, so we can work with its derivative g', which is necessarily increasing (since g is convex) and satisfies $|g'(x)| \le L$ (since g is L-Lipschitz).

Finally, in the proofs below we use the following easy property, which we state without proof.

Lemma 11. *For $f \colon \mathbb{R} \to \mathbb{R}$, $Z \sim \mathcal{N}(0,1)$, and $c \ge 0$, we have*

$$\mathbb{E}[f(Z) \exp(\sqrt{c}Z)] = \exp\left(\frac{c}{2}\right) \mathbb{E}[f(Z + \sqrt{c})],$$

provided all the expectations above exist.

D.1 Formulae for the Derivatives

In this section we show that the partial derivative $U_{S^a}(S,c)$ can be expressed as an expectation of a polynomial modulated by the payoff function g. We define the family of polynomials $p^{[a]}(x,y)$, $a \ge 0$, as follows:

$$\begin{aligned}
p^{[0]}(x,y) &= 1 \\
p^{[a+1]}(x,y) &= (x - ay)\, p^{[a]}(x,y) - p_x^{[a]}(x,y) \quad \text{for } a \ge 1,
\end{aligned} \tag{24}$$

where $p_x^{[a]}(x,y) = \partial p^{[a]}(x,y)/\partial x$.

The following is the main result in this section; note that we only assume that g is Lipschitz.

Lemma 12. *Let $g \colon \mathbb{R}_0 \to \mathbb{R}_0$ be an L-Lipschitz function. For $a \ge 0$ and $S, c > 0$,*

$$U_{S^a}(S,c) = \frac{1}{S^a c^{a/2}} \mathbb{E}\left[p^{[a]}(Z, \sqrt{c}) \cdot g\left(S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}Z\right)\right)\right],$$

where $Z \sim \mathcal{N}(0,1)$.

In proving Lemma 12 we will need the following result, which allows us to differentiate the convolution.

Lemma 13. *Fix $c > 0$. Let $g \colon \mathbb{R}_0 \to \mathbb{R}_0$ be an L-Lipschitz function, and let $\tilde{g}(x) = g(\exp(x))$. Let $\omega \colon \mathbb{R} \to \mathbb{R}$ be given by $\omega(x) = p(x)\, \phi(x/\sqrt{c})$, where $p(x)$ is a polynomial in x with coefficients involving c. Finally, let $f \colon \mathbb{R} \to \mathbb{R}$ be given by $f(r) = (\tilde{g} * \omega)(r)$. Then the derivative $f'(r) = df(r)/dr$ can be written as the derivative of the convolution, $f'(r) = (\tilde{g} * \omega')(r)$.*

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Proof. Fix $r \in \mathbb{R}$. For $h \neq 0$, consider the quantity $\rho_h = \frac{1}{h}(f(r+h) - f(r))$, and note that $f'(r) = \lim_{h \to 0} \rho_h$. Recalling the definition of f as a convolution and using the mean-value theorem, we can write ρ_h as

$$\rho_h = \int_{-\infty}^{\infty} \tilde{g}(x) \left(\frac{\omega(r-x+h) - \omega(r-x)}{h} \right) dx = \int_{-\infty}^{\infty} g(x)\, \omega'(r-x+\xi_h)\, dx,$$

for some ξ_h between 0 and h. Let

$$\rho_0 := \int_{-\infty}^{\infty} \tilde{g}(x)\, \omega'(r-x)\, dx = (\tilde{g} * \omega')(r).$$

Then by another application of the mean-value theorem, we can write

$$\begin{aligned}
\Delta_h := \rho_h - \rho_0 &= \xi_h \int_{-\infty}^{\infty} \tilde{g}(x) \left(\frac{\omega'(r-x+\xi_h) - \omega'(r-x)}{\xi_h} \right) dx \\
&= \xi_h \int_{-\infty}^{\infty} \tilde{g}(x)\, \omega''(r-x+\xi_h^{(2)})\, dx,
\end{aligned} \tag{25}$$

for some $\xi_h^{(2)}$ lying between 0 and ξ_h. One can easily verify that the second derivative of ω is given by

$$\omega''(x) = \frac{q(x)}{c^2}\, \phi\left(\frac{x}{\sqrt{c}} \right),$$

where $q(x)$ is the polynomial $q(x) = (x^2 - c)p(x) - 2cxp'(x) + c^2 p''(x)$. Since g is L-Lipschitz, for each $x \in \mathbb{R}$ we have

$$0 \leq \tilde{g}(x) = g(\exp(x)) \leq g(0) + |g(\exp(x)) - g(0)| \leq g(0) + L\exp(x)$$

This gives us the estimate

$$\left| \int_{-\infty}^{\infty} \tilde{g}(x)\, \omega''(r-x+\xi_h^{(2)})\, dx \right|$$

$$\leq \frac{1}{c^2} \int_{-\infty}^{\infty} \left(g(0) + L\exp(x) \right) \cdot |q(r-x+\xi_h^{(2)})| \cdot \phi\left(\frac{r-x+\xi_h^{(2)}}{\sqrt{c}} \right) dx$$

$$= \frac{1}{c^{3/2}} \int_{-\infty}^{\infty} \left(g(0) + L\exp(r + \xi_h^{(2)} - \sqrt{c}y) \right) \cdot |q(\sqrt{c}y)| \cdot \phi(y)\, dy < \infty,$$

where in the computation above we have used the substitution $y = (r - x + \xi_h^{(2)})/\sqrt{c}$. The last expression above shows that the integral is finite, since we are integrating exponential and polynomial functions against the Gaussian density. Plugging this bound to (25) and recalling that $|\xi_h| \leq |h|$, we obtain

$$|\Delta_h| \leq |h| \cdot \left| \int_{-\infty}^{\infty} \tilde{g}(x)\, \omega''(r-x+\xi_h^{(2)})\, dx \right| \to 0 \quad \text{as} \quad h \to 0.$$

Since $\Delta_h = \rho_h - \rho_0$, this implies our desired conclusion,

$$f'(r) = \lim_{h \to 0} \rho_h = \rho_0 = (\tilde{g} * \omega')(r).$$

$\square$

We are now ready to prove Lemma 12.

Proof of Lemma 12. We proceed by induction on a. The base case $a = 0$ follows from the definition of U. Assume the statement holds for some $a \geq 0$; we prove it also holds for $a + 1$. Our strategy is to express U_{S^a} as a convolution, use Lemma 13 to differentiate the convolution, and write the result back as an expectation.

Fix $S, c > 0$ for the rest of this proof. Let $\tilde{g}(x) = g(\exp(x))$ and $\phi_c(x) = \phi(x/\sqrt{c})$. From the inductive hypothesis and the fact that $-Z \overset{d}{=} Z$, we have

$$
\begin{aligned}
U_{S^a}(S, c) &= \frac{1}{S^a c^{a/2}} \, \mathbb{E}\left[p^{[a]}(-Z, \sqrt{c}) \cdot \tilde{g}\left(\log S - \frac{c}{2} - \sqrt{c}Z \right) \right] \\
&= \frac{1}{S^a c^{a/2}} \int_{-\infty}^{\infty} p^{[a]}(-x, \sqrt{c}) \cdot \tilde{g}\left(\log S - \frac{c}{2} - \sqrt{c}x \right) \cdot \phi(x) \, dx \\
&= \frac{1}{S^a c^{(a+1)/2}} \int_{-\infty}^{\infty} p^{[a]}\left(-\frac{y}{\sqrt{c}}, \sqrt{c} \right) \cdot \tilde{g}\left(\log S - \frac{c}{2} - y \right) \cdot \phi_c(y) \, dy \\
&= \frac{1}{S^a c^{(a+1)/2}} \int_{-\infty}^{\infty} \tilde{g}\left(\log S - \frac{c}{2} - y \right) \cdot \omega(y) \, dy \\
&= \frac{1}{S^a c^{(a+1)/2}} (\tilde{g} * \omega)\left(\log S - \frac{c}{2} \right),
\end{aligned}
$$

where in the computation above we have used the substitution $y = \sqrt{c}x$, and we have defined the function

$$
\omega(y) = p^{[a]}\left(-\frac{y}{\sqrt{c}}, \sqrt{c} \right) \cdot \phi\left(\frac{y}{\sqrt{c}} \right).
$$

In particular, ω has derivative

$$
\omega'(y) = -\frac{1}{\sqrt{c}} \left(p_x^{[a]}\left(-\frac{y}{\sqrt{c}}, \sqrt{c} \right) + \frac{y}{\sqrt{c}} p^{[a]}\left(-\frac{y}{\sqrt{c}}, \sqrt{c} \right) \right) \phi\left(\frac{y}{\sqrt{c}} \right).
$$

Differentiating U_{S^a} with respect to S and using the result of Lemma 13 give us

$$
\begin{aligned}
U_{S^{a+1}}(S, c) &= -\frac{a}{S^{a+1} c^{(a+1)/2}} (\tilde{g} * \omega)\left(\log S - \frac{c}{2} \right) + \frac{1}{S^{a+1} c^{(a+1)/2}} (\tilde{g} * \omega')\left(\log S - \frac{c}{2} \right) \\
&= \frac{1}{S^{a+1} c^{(a+1)/2}} \int_{-\infty}^{\infty} \tilde{g}\left(\log S - \frac{c}{2} - y \right) (\omega'(y) - a\omega(y)) \, dy \\
&= \frac{1}{S^{a+1} c^{a/2}} \int_{-\infty}^{\infty} \tilde{g}\left(\log S - \frac{c}{2} - \sqrt{c}x \right) (\omega'(\sqrt{c}x) - a\omega(\sqrt{c}x)) \, dx \\
&= \frac{1}{S^{a+1} c^{a/2}} \int_{-\infty}^{\infty} \tilde{g}\left(\log S - \frac{c}{2} - \sqrt{c}x \right) \frac{((-x - a\sqrt{c})p^{[a]}(-x, \sqrt{c}) - p_x^{[a]}(-x, \sqrt{c}))}{\sqrt{c}} \phi(x) \, dx \\
&= \frac{1}{S^{a+1} c^{(a+1)/2}} \int_{-\infty}^{\infty} \tilde{g}\left(\log S - \frac{c}{2} - \sqrt{c}x \right) p^{[a+1]}(-x, \sqrt{c}) \phi(x) \, dx \\
&= \frac{1}{S^{a+1} c^{(a+1)/2}} \, \mathbb{E}\left[p^{[a+1]}(-Z, \sqrt{c}) \cdot \tilde{g}\left(\log S - \frac{c}{2} - \sqrt{c}Z \right) \right] \\
&= \frac{1}{S^{a+1} c^{(a+1)/2}} \, \mathbb{E}\left[p^{[a+1]}(Z, \sqrt{c}) \cdot g\left(S \cdot \left(-\frac{c}{2} + \sqrt{c}Z \right) \right) \right],
\end{aligned}
$$

as desired. In the computation above we have again used the substitution $x = y/\sqrt{c}$ and the fact that $-Z \overset{d}{=} Z$. This completes the induction step and the proof of the lemma. □

As an example, the first few polynomials $p^{[a]}(x, y)$ are

$$
\begin{aligned}
p^{[0]}(x, y) &= 1 \\
p^{[1]}(x, y) &= x \\
p^{[2]}(x, y) &= x^2 - yx - 1 \\
p^{[3]}(x, y) &= x^3 - 3yx^2 + (2y^2 - 3)x + 3y,
\end{aligned}
$$

giving us the formulae

$$U(S,c) = \mathbb{E}\left[g\left(S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}Z\right)\right)\right]$$

$$U_S(S,c) = \frac{1}{S\sqrt{c}}\mathbb{E}\left[Z \cdot g\left(S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}Z\right)\right)\right]$$

$$U_{S^2}(S,c) = \frac{1}{S^2 c}\mathbb{E}\left[(Z^2 - \sqrt{c}Z - 1) \cdot g\left(S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}Z\right)\right)\right]$$

$$U_{S^3}(S,c) = \frac{1}{S^3 c^{3/2}}\mathbb{E}\left[(Z^3 - 3\sqrt{c}Z^2 + (2c-3)Z + 3\sqrt{c}) \cdot g\left(S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}Z\right)\right)\right].$$

We also have the following easy corollaries.

Corollary 2. *For $a \geq 1$, $\mathbb{E}[p^{[a]}(Z, \sqrt{c})] = 0$. For $a \geq 2$, we also have $\mathbb{E}[p^{[a]}(Z + \sqrt{c}, \sqrt{c})] = 0$.*

Proof. First assume $a \geq 1$, and take g to be the constant function $g(x) = 1$. In this case $U(S,c) = 1$ and $U_{S^a}(S,c) = 0$, so by the result of Lemma 12, $\mathbb{E}[p^{[a]}(Z, \sqrt{c})] = S^a c^{a/2} U_{S^a}(S,c) = 0$. Next, assume $a \geq 2$, and take g to be the linear function $g(x) = x$. In this case $U(S,c) = \mathbb{E}[S \cdot G(c)] = S$, so $U_{S^a}(S,c) = 0$. Then using the results of Lemma 11 and Lemma 12,

$$\mathbb{E}\left[p^{[a]}(Z + \sqrt{c}, \sqrt{c})\right] = \exp\left(-\frac{c}{2}\right)\mathbb{E}\left[p^{[a]}(Z, \sqrt{c})\exp(\sqrt{c}Z)\right] = S^{a-1}c^{a/2}U_{S^a}(S,c) = 0.$$

$\square$

D.2 Calculations for the European-Option Payoff Function

In this section, we bound the derivatives $U_{S^a}(S,c)$ for the special case when g is the payoff function of the European call function, $g(x) = \max\{0, x - K\}$, where $K > 0$ is a constant. Note that the bounds on U_{S^3} and U_{S^4} are slightly stronger than the stated bounds (21) and (22), because in this case we are able to compute the derivatives exactly.

Lemma 14. *Let $g(x) = \max\{0, x - K\}$. Then for all $S, c > 0$,*

$$|U_{S^2}(S,c)| \leq \frac{K}{\sqrt{2\pi}} \cdot \frac{1}{S^2\sqrt{c}}$$

$$|U_{S^3}(S,c)| \leq \frac{K}{\sqrt{2\pi}} \cdot \frac{(2\sqrt{c}+1)}{S^3 c}$$

$$|U_{S^4}(S,c)| \leq \frac{K}{\sqrt{2\pi}} \cdot \frac{(6c + 5\sqrt{c} + 2)}{S^4 c^{3/2}}$$

Proof. We first compute the Black-Scholes value $U(S,c)$. Define

$$\alpha \equiv \alpha(S,c) = -\frac{1}{\sqrt{c}}\log\frac{S}{K} + \frac{\sqrt{c}}{2},$$

and observe that $S \cdot \exp(-c/2 + \sqrt{c}Z) \geq K$ if and only if $Z \geq \alpha$. Then using the result of Lemma 11, we have

$$U(S,c) = \mathbb{E}\left[\left(S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}Z\right) - K\right) \cdot \mathbf{1}\{Z \geq \alpha\}\right]$$

$$= S \cdot \exp\left(-\frac{c}{2}\right)\mathbb{E}\left[\exp(\sqrt{c}Z) \cdot \mathbf{1}\{Z \geq \alpha\}\right] - K\,\mathbb{P}(Z \geq \alpha)$$

$$= S\,\mathbb{P}(Z \geq \alpha - \sqrt{c}) - K\,\mathbb{P}(Z \geq \alpha)$$

$$= S\,\Phi(-\alpha + \sqrt{c}) - K\,\Phi(-\alpha).$$

Differentiating the formula above with respect to c and applying the Black-Scholes differential equation (6), we get

$$U_{S^2}(S,c) = \frac{2}{S^2}U_c(S,c) = \frac{1}{S^2 c}\left[S\alpha\phi\left(-\alpha + \sqrt{c}\right) + K(-\alpha + \sqrt{c})\phi(\alpha)\right] = \frac{K}{S^2\sqrt{c}}\phi(\alpha),$$

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where the last equality follows from the relation $S\phi(-\alpha+\sqrt{c}) = K\phi(\alpha)$. In particular, we have the bound $0 \le U_{S^2}(S,c) \le K/(S^2\sqrt{2\pi c})$. A direct calculation reveals that the higher order derivatives of U are given by

$$U_{S^3}(S,c) = \frac{K}{S^3 c} \left(\alpha - 2\sqrt{c}\right) \phi(\alpha) \quad \text{and} \quad U_{S^4}(S,c) = \frac{K}{S^4 c^{3/2}} \left(\alpha^2 - 5\sqrt{c}\alpha + 6c - 1\right) \phi(\alpha).$$

It is not difficult to see that we have $|\alpha \exp(-\alpha^2/2)| \le 1$ and $|\alpha^2 \exp(-\alpha^2/2)| \le 1$. Applying these bounds to the formulae above gives us the desired conclusion. $\square$

D.3 Bounding the Second Derivative $U_{S^2}(S,c)$

We now bound the second-order derivative $U_{S^2}(S,c)$ in the general case.

Lemma 15. *Let $g\colon \mathbb{R}_0 \to \mathbb{R}_0$ be a convex, L-Lipschitz, K-linear function. Then for all $S,c > 0$,*

$$0 \le U_{S^2}(S,c) \le \frac{2LK}{\sqrt{2\pi}} \cdot \frac{1}{S^2\sqrt{c}}.$$

Proof. Recall that $U(S,c)$ is convex in S (Lemma 10), so $U_{S^2}(S,c) \ge 0$. If g is a linear function, say $g(x) = \gamma x$ for some $0 \le \gamma \le L$, then $U(S,c) = \mathbb{E}[\gamma S \cdot G(c)] = \gamma S$. In this case $U_{S^2}(S,c) = 0$, and we are done.

Now assume g is not a linear function. Since g is non-negative, L-Lipschitz, and K-linear, we can find $0 \le \gamma \le L$ such that $g'(x) = \gamma$ for $x \ge K$. Moreover, since g is convex and not a linear function, we also have that $\gamma > g'(0)$. Define the function $\tilde{g}\colon \mathbb{R}_0 \to \mathbb{R}_0$ by

$$\tilde{g}(x) = \frac{g(x) - g(0) - xg'(0)}{\gamma - g'(0)}, \tag{26}$$

and note that $\tilde{g}$ is an increasing, 1-Lipschitz convex function with $\tilde{g}(0) = \tilde{g}'(0) = 0, 0 \le \tilde{g}'(x) \le 1$, and $\tilde{g}'(x) = 1$ for $x \ge K$.

Consider the quantity $V(S,c) = \mathbb{E}[\tilde{g}(S \cdot G(c))]$, and note that we can write

$$V(S,c) = \frac{\mathbb{E}\left[g(S \cdot G(c)) - g(0) - g'(0) \cdot S \cdot G(c)\right]}{\gamma - g'(0)} = \frac{U(S,c) - g(0) - g'(0) \cdot S}{\gamma - g'(0)}.$$

Taking second derivative with respect to S on both sides and using the fact that $0 \le \gamma - g'(0) \le 2L$, we obtain

$$0 \le U_{S^2}(S,c) = (\gamma - g'(0)) \cdot V_{S^2}(S,c) \le 2L \cdot V_{S^2}(S,c).$$

We already know that $V_{S^2}(S,c) \ge 0$ since $\tilde{g}$ is convex, so we only need to show that

$$V_{S^2}(S,c) \le \frac{K}{\sqrt{2\pi}} \cdot \frac{1}{S^2\sqrt{c}}.$$

For $0 < S \le K$, using the formula from Lemma 12 and the result of Lemma 16 below, we obtain

$$V_{S^2}(S,c) = \frac{1}{S^2 c}\mathbb{E}\left[(Z^2 - \sqrt{c}Z - 1) \cdot \tilde{g}\left(S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}Z\right)\right)\right] \le \frac{1}{S^2 c} \cdot \frac{S\sqrt{c}}{\sqrt{2\pi}} \le \frac{K}{\sqrt{2\pi}} \cdot \frac{1}{S^2\sqrt{c}},$$

and for $S \ge K$, we use the result of Lemma 17 to obtain

$$V_{S^2}(S,c) = \frac{1}{S^2 c}\mathbb{E}\left[(Z^2 - \sqrt{c}Z - 1) \cdot \tilde{g}\left(S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}Z\right)\right)\right] \le \frac{1}{S^2 c} \cdot \frac{K\sqrt{c}}{\sqrt{2\pi}} = \frac{K}{\sqrt{2\pi}} \cdot \frac{1}{S^2\sqrt{c}}.$$

This completes the proof of the lemma. $\square$

It remains to prove the following two results, which we use in the proof of Lemma 15 above with $\tilde{g}$ in place of g. Note that the first result below does not use the assumption that g is eventually linear.

Lemma 16. *Let $g\colon \mathbb{R}_0 \to \mathbb{R}_0$ be an increasing, nonnegative, convex, 1-Lipschitz function. Then for all $S,c > 0$,*

$$\mathbb{E}\left[(Z^2 - \sqrt{c}Z - 1) \cdot g\left(S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}Z\right)\right)\right] \le \frac{S\sqrt{c}}{\sqrt{2\pi}}.$$

Proof. Fix $S, c > 0$, and define the following quantities:

$$t_1 = \frac{\sqrt{c} - \sqrt{c+4}}{2} \qquad\qquad t_2 = \frac{\sqrt{c} + \sqrt{c+4}}{2}$$

$$\lambda_1 = S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}\, t_1\right) \qquad\qquad \lambda_2 = S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}\, t_2\right)$$

$$g_1 = g(\lambda_1) \qquad\qquad g_2 = g(\lambda_2)$$

$$t_* = \frac{1}{\sqrt{c}} \log\left(\exp(\sqrt{c}t_2) - \frac{1}{S} \cdot \exp\left(\frac{c}{2}\right) \cdot (g_2 - g_1)\right).$$

Furthermore, define the function $h\colon \mathbb{R} \to \mathbb{R}_0$ by

$$h(x) = g_1 + \left(g_2 - g_1 - \lambda_2 + S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}\, x\right)\right) \cdot \mathbf{1}\{x \geq t_*\}.$$

We will show that

$$\mathbb{E}\left[(Z^2 - \sqrt{c}Z - 1) \cdot g\left(S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}Z\right)\right)\right] \leq \mathbb{E}\left[(Z^2 - \sqrt{c}Z - 1) \cdot h(Z)\right], \qquad (27)$$

and furthermore, we can evaluate the latter expectation explicitly:

$$\mathbb{E}\left[(Z^2 - \sqrt{c}Z - 1) \cdot h(Z)\right] = S\sqrt{c}\, \phi(t_* - \sqrt{c}) \leq \frac{S\sqrt{c}}{\sqrt{2\pi}}.$$

We begin by noting that t_1 and t_2 are the two roots of the polynomial $x^2 - \sqrt{c}x - 1$. Since g is increasing and 1-Lipschitz,

$$g_2 - g_1 = g(\lambda_2) - g(\lambda_1) \leq \lambda_2 - \lambda_1 = S \cdot \exp\left(-\frac{c}{2}\right)\left(\exp(\sqrt{c}\, t_2) - \exp(\sqrt{c}\, t_1)\right).$$

Therefore, from the definition of t_*, we see that

$$\exp(\sqrt{c}\, t_2) - \exp(\sqrt{c}\, t_*) = \frac{1}{S} \cdot \exp\left(\frac{c}{2}\right) \cdot (g_2 - g_1) \leq \exp(\sqrt{c}\, t_2) - \exp(\sqrt{c}\, t_1),$$

so $t_1 \leq t_* \leq t_2$. Furthermore, by construction,

$$S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}\, t_*\right) = S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}\, t_2\right) - (g_2 - g_1) = \lambda_2 - g_2 + g_1,$$

so $h(t_*) = g_1$. This means h is a continuous convex function of x (although we will not actually use this property). We will now show that pointwise,

$$\phi(x) \cdot (x^2 - \sqrt{c}x - 1) \cdot g\left(S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}x\right)\right) \leq \phi(x) \cdot (x^2 - \sqrt{c}x - 1) \cdot h(x). \qquad (28)$$

We consider four cases:

- Suppose $x \leq t_1$, so $x^2 - \sqrt{c}x - 1 \geq 0$. Since g is increasing and nonnegative,

$$0 \leq g\left(S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}x\right)\right) \leq g\left(S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}\, t_1\right)\right) = g_1 = h(x).$$

- Suppose $t_1 \leq x \leq t_*$, so $x^2 - \sqrt{c}x - 1 \leq 0$. Since g is increasing,

$$g\left(S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}x\right)\right) \geq g\left(S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}\, t_1\right)\right) = g_1 = h(x) \geq 0.$$

- Suppose $t_* \leq x \leq t_2$, so $x^2 - \sqrt{c}x - 1 \leq 0$. Since g is increasing and 1-Lipschitz,

$$g\left(S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}x\right)\right)$$
$$\geq g\left(S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}\, t_2\right)\right) + S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}x\right) - S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}\, t_2\right)$$
$$= g_2 - \lambda_2 + S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}x\right) = h(x) \geq 0.$$

- Suppose $x \geq t_2$, so $x^2 - \sqrt{c}\,x - 1 \geq 0$. Since g is increasing and 1-Lipschitz,

$$g\left(S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}\,x\right)\right)$$
$$\leq g\left(S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}\,t_2\right)\right) + S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}\,x\right) - S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}\,t_2\right)$$
$$= g_2 - \lambda_2 + S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}\,x\right) = h(x).$$

Integrating (28) over $x \in \mathbb{R}$ gives us the desired inequality (27). Let us now evaluate the expectation on the right hand side of (27). A simple computation using the properties of $Z \sim \mathcal{N}(0,1)$ gives us

$$
\begin{aligned}
\mathbb{E}\left[(Z^2 - \sqrt{c}Z - 1) \cdot h(Z)\right] &= g_1\,\mathbb{E}\left[(Z^2 - \sqrt{c}Z - 1)\right] \\
&\quad + (g_2 - g_1 - \lambda_2) \cdot \mathbb{E}\left[(Z^2 - \sqrt{c}Z - 1) \cdot \mathbf{1}\{Z \geq t_*\}\right] \\
&\quad + S \cdot \exp\left(-\frac{c}{2}\right) \cdot \mathbb{E}\left[(Z^2 - \sqrt{c}Z - 1)\,\exp(\sqrt{c}Z) \cdot \mathbf{1}\{Z \geq t_*\}\right] \\
&= (g_2 - g_1 - \lambda_2) \cdot (t_* - \sqrt{c})\,\phi(t_*) + St_*\,\phi(t_* - \sqrt{c}) \\
&= -S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}\,t_*\right) \cdot (t_* - \sqrt{c})\,\phi(t_*) + St_*\,\phi(t_* - \sqrt{c}) \\
&= -S(t_* - \sqrt{c})\,\phi(t_* - \sqrt{c}) + St_*\,\phi(t_* - \sqrt{c}) \\
&= S\sqrt{c}\,\phi(t_* - \sqrt{c}),
\end{aligned}
$$

as desired. $\qquad\square$

The following result is similar to Lemma 16, except that this result assumes the K-linearity of g and achieves a stronger result.

Lemma 17. *Let $g \colon \mathbb{R}_0 \to \mathbb{R}_0$ be an increasing, nonnegative, convex, 1-Lipschitz function with the property that $g'(x) = 1$ for $x \geq K$. Then for all $S \geq K$ and $c > 0$,*

$$\mathbb{E}\left[(Z^2 - \sqrt{c}Z - 1) \cdot g\left(S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}Z\right)\right)\right] \leq \frac{K\sqrt{c}}{\sqrt{2\pi}}.$$

Proof. This proof is similar in nature to the proof of Lemma 16, and we omit some of the details.

Case 1: Suppose $S \geq K \exp(\sqrt{c(c+4)}/2)$. Recall the European-option payoff function $g_{\mathrm{EC}}(x) = \max\{0, x - K\}$ from Section 2, and note that the K-linearity of g implies $g(x) = g(K) + g_{\mathrm{EC}}(x)$ for $x \geq K$. Using the fact that g is increasing and K-linear, we can show that for all $x \in \mathbb{R}$ we have

$$(x^2 - \sqrt{c}\,x - 1) \cdot g\left(S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}\,x\right)\right) \leq (x^2 - \sqrt{c}\,x - 1) \cdot \left\{g(K) + g_{\mathrm{EC}}\left(S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}\,x\right)\right)\right\}.$$

Integrating both sides above with $Z \sim \mathcal{N}(0,1)$ in place of x and using the result of Lemma 14, we obtain

$$
\begin{aligned}
\mathbb{E}\Big[(Z^2 &- \sqrt{c}Z - 1) \cdot g\left(S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}Z\right)\right)\Big] \\
&\leq \mathbb{E}\left[(Z^2 - \sqrt{c}Z - 1) \cdot \left\{g(K) + g_{\mathrm{EC}}\left(S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}Z\right)\right)\right\}\right] \\
&= \mathbb{E}\left[(Z^2 - \sqrt{c}Z - 1) \cdot g_{\mathrm{EC}}\left(S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}Z\right)\right)\right] \leq \frac{K\sqrt{c}}{\sqrt{2\pi}}.
\end{aligned}
$$

Case 2: Suppose $K \leq S \leq K \exp(\sqrt{c(c+4)}/2)$. Define the following quantities:

$$t_0 = \frac{\sqrt{c}}{2} - \frac{1}{\sqrt{c}} \log \frac{S}{K - g(K) + g_1}, \qquad \lambda_1 = S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}\left(\frac{\sqrt{c} - \sqrt{c+4}}{2}\right)\right),$$

and $g_1 = g(\lambda_1)$. Consider the function $h_2 \colon \mathbb{R} \to \mathbb{R}_0$ given by

$$h_2(x) = g_1 + \left(g(K) - g_1 - K + S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}\,x\right)\right) \cdot \mathbf{1}\{x \geq t_0\}.$$

Using the fact that g is increasing, 1-Lipschitz, and K-linear, we can show that for all $x \in \mathbb{R}$,

$$(x^2 - \sqrt{c}x - 1) \cdot g\left(S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}x\right)\right) \le (x^2 - \sqrt{c}x - 1) \cdot h_2(x).$$

Integrating both sides above with $Z \sim \mathcal{N}(0,1)$ in place of x, we get

$$\mathbb{E}\left[(Z^2 - \sqrt{c}Z - 1) \cdot g\left(S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}Z\right)\right)\right] \le \mathbb{E}\left[(Z^2 - \sqrt{c}Z - 1) \cdot h_2(Z)\right].$$

Following the same calculation as in the proof of Lemma 16, we can evaluate the latter expectation to be

$$\mathbb{E}\left[(Z^2 - \sqrt{c}Z - 1) \cdot h_2(Z)\right] = (K - g(K) + g_1)\sqrt{c}\,\phi(t_0) \le \frac{K\sqrt{c}}{\sqrt{2\pi}},$$

where the last inequality follows from the relation $0 \le g(K) - g_1 \le K - \lambda_1$, since g is increasing and 1-Lipschitz. $\qquad\square$

D.4 Bounding the Higher-Order Derivatives

We now turn to bounding the higher-order derivatives $U_{S^3}(S,c)$ and $U_{S^4}(S,c)$. Our strategy is to approximate the eventually linear payoff function g by the European-option payoff g_{EC} and applying the bounds for g_{EC} developed in Lemma 14.

Lemma 18. *Let $g\colon \mathbb{R}_0 \to \mathbb{R}_0$ be a convex, L-Lipschitz, K-linear function. Then for all $S, c > 0$,*

$$|U_{S^3}(S,c)| \le 7LK \cdot \frac{\max\{c^{-3/2},\, c^{-1/2}\}}{S^3},$$

$$|U_{S^4}(S,c)| \le 28LK \cdot \frac{\max\{c^{-2},\, c^{-1/2}\}}{S^4}.$$

Proof. Since g is L-Lipschitz and K-linear, we can find $0 \le \gamma \le L$ such that $g(x) = g(K) + \gamma(x - K)$ for $x \ge K$. We decompose g into two parts,

$$g(x) = \gamma g_{\text{EC}}(x) + g^*(x),$$

where $g_{\text{EC}}(x) = \max\{0, x - K\}$ is the European-option payoff function, and $g^*\colon \mathbb{R}_0 \to \mathbb{R}_0$ is given by $g^*(x) = g(x)$ for $0 \le x \le K$, and $g^*(x) = g(K)$ otherwise.

Then the Black-Scholes value $U(S,c)$ also decomposes,

$$U(S,c) = \mathbb{E}[g(S \cdot G(c))] = \gamma\,\mathbb{E}[g_{\text{EC}}(S \cdot G(c))] + \mathbb{E}[g^*(S \cdot G(c))] \equiv \gamma\,U^{\text{EC}}(S,c) + U^*(S,c),$$

and similarly for the derivatives,

$$U_{S^a}(S,c) = \gamma\,U^{\text{EC}}_{S^a}(S,c) + U^*_{S^a}(S,c), \qquad a \ge 0. \tag{29}$$

For the function g_{EC}, Lemma 14 tells us that for all $S, c > 0$,

$$|U^{\text{EC}}_{S^3}(S,c)| \le \frac{3K}{\sqrt{2\pi}} \cdot \frac{\max\{c^{1/2},\, 1\}}{S^3 c}, \qquad |U^{\text{EC}}_{S^4}(S,c)| \le \frac{13K}{\sqrt{2\pi}} \cdot \frac{\max\{c,\, 1\}}{S^4 c^{3/2}}. \tag{30}$$

Now for the second function g^*, we use Lemma 12 to write

$$U^*_{S^a}(S,c) = \frac{1}{S^a c^{a/2}}\,\mathbb{E}\left[p^{[a]}(Z, \sqrt{c}) \cdot g^*\left(S \cdot \exp\left(-\frac{c}{2} + \sqrt{c}Z\right)\right)\right]. \tag{31}$$

Since $\mathbb{E}[p^{[a]}(Z, \sqrt{c})] = 0$ for $a \ge 1$ (Corollary 2), we may assume that $g(0) = 0$, so $g^*(0) = 0$ as well. Since g is L-Lipschitz, this implies

$$\sup_{x \in \mathbb{R}}\left|g^*(x)\right| = \max_{0 \le x \le K}|g(x)| \le \max_{0 \le x \le K} Lx = LK.$$

Therefore, by applying triangle inequality and Cauchy-Schwarz inequality to (31), we get for $a \ge 1$,

$$|U^*_{S^a}(S,c)| \le \frac{1}{S^a c^{a/2}}\,\mathbb{E}\left[\left|p^{[a]}(Z, \sqrt{c})\right| \cdot LK\right] \le \frac{LK}{S^a c^{a/2}}\,\mathbb{E}\left[\left(p^{[a]}(Z, \sqrt{c})\right)^2\right]^{1/2}. \tag{32}$$

For $a = 3, 4$, we use the recursion (24) to compute the polynomials $p^{[a]}(Z, \sqrt{c})$, and we evaluate the expectation $\mathbb{E}[(p^{[a]}(Z, \sqrt{c}))^2]$. Plugging in this expectation to (32) with $a = 3$ gives us

$$|U_{S^3}^*(S, c)| \leq \frac{LK}{S^3 c^{3/2}} \cdot \left(4c^2 + 18c + 6\right)^{1/2} \leq \sqrt{28} \cdot LK \cdot \frac{\max\{c, 1\}}{S^3 c^{3/2}}. \tag{33}$$

Therefore, by combining the bound above with the first inequality in (30) and using the decomposition (29), we get the first part of our lemma,

$$|U_{S^3}(S, c)| \leq \frac{3}{\sqrt{2\pi}} \cdot LK \cdot \frac{\max\{c^{1/2}, 1\}}{S^3 c} + \sqrt{28} \cdot LK \cdot \frac{\max\{c, 1\}}{S^3 c^{3/2}} \leq 7LK \cdot \frac{\max\{c, 1\}}{S^3 c^{3/2}}.$$

A similar computation with $a = 4$ yields the second part of the lemma,

$$|U_{S^4}(S, c)| \leq \frac{13}{\sqrt{2\pi}} \cdot LK \cdot \frac{\max\{c, 1\}}{S^4 c^{3/2}} + \sqrt{518} \cdot LK \cdot \frac{\max\{c^{3/2}, 1\}}{S^4 c^2} \leq 28LK \cdot \frac{\max\{c^{3/2}, 1\}}{S^3 c^2}.$$

$\square$