

Interpreting prediction markets: a stochastic approach

Rafael M. Frongillo

Nicolás Della Penna

Mark D. Reid

June 22, 2012

Abstract

We study the stochastic sequence of prices that results from traders with beliefs drawn from a given distribution sequentially interacting with an automated market maker. We relate this model to the classic market equilibrium analysis, as well as to the more recent line of research seeking to interpret prediction market prices. Along the way, we show a very natural stochastic market model which exactly corresponds to a stochastic mirror descent, and we leverage this new connection to make claims about the behavior of the market.

1. Introduction and literature review

The literature on the interpretation of prediction market prices (Manski, 2004; Wolfers & Zitzewitz, 2006) has had the goal of relating the equilibrium prices to the distribution of the beliefs of traders. More recent work (Othman & Sandholm, 2010) has looked at a stochastic model, and studied the behavior of simple agents sequentially interacting with the market. We continue this latter path of research, motivated by the observation that the equilibrium price may be a poor predictor of the behavior in a volatile prediction market. As such, we seek a more detailed understanding of the market than the equilibrium point – we would like to know what the “stationary distribution” of the price is, as time goes to infinity.

As is standard in the literature, we assume a fixed distribution over traders’ beliefs and wealth. Our model features an automated market maker which adjusts the prices *after* a trade, according to a fixed convex cost function C . Note that this differs slightly from the standard model, which adjusts

the price *continuously* during a trade.

We obtain two results. First, we prove that the stationary point of our stochastic process defined by the market maker and a belief distribution of traders converges to the Walrasian equilibrium of the market as traders wealth relative to market liquidity goes to zero. This result, stated in Theorem 1, is quite general as very few assumptions are made about the demand functions of the traders – as such, it can be seen as a generalisation of the stochastic result of (Othman & Sandholm, 2010) to cases where agents are not limited to linear demands. One noted exception, however, is that our price adjustment model simplifies the analysis slightly, whereas (Othman & Sandholm, 2010) consider continuous price adjustments.

Second, we show in Theorem 2 that when traders are Kelly bettors (i.e. have log-utilities), the resulting stochastic market process is equivalent to stochastic mirror descent; see e.g. (Duchi et al., 2010). This result adds to the growing literature which relates prediction markets, and automated market makers in general, to online learning; see e.g. (Abernethy et al., 2011), (Chen & Vaughan, 2010), and (Das & Magdon-Ismail, 2008).

This connection to mirror descent seems to suggest that the prices in a prediction market at any given time may be meaningless – the final point in stochastic mirror descent has poor convergence guarantees. However, standard results suggest that a prudent way to form a “consensus estimate” from a prediction market is to *average* the prices! The average price, assuming our market model is reasonable, is provably close to the stationary price. In Section 5 we give a natural example that exhibits this behavior.

Beyond this, however, Theorem 2 gives us insight into the relationship between the market liquidity and the convergence of prices; in particular it suggests that we should increase liquidity at a rate of \sqrt{t} if we wish the price to settle down at the right rate.

Appearing in *Proceedings of the 29th International Conference on Machine Learning*, Edinburgh, Scotland, UK, 2012. Copyright 2012 by the author(s)/owner(s).

Related work A important related question that has been examined is when are agents incentivized to reveal their information in the prediction markets setting. By construction a sequential proper scoring rule based market maker is always provides the right set of myopic incentives, (Chen et al., 2007) study the case when agents are not myopic but rather strategic. They show that when agents receive conditionally independent signals, it is optimal to sequentially adjust market prices to their posterior (given their signal and previous market prices) at an agents first opportunity when interacting with an automated market maker, in equilibrium. If the signals are conditionally dependent, then truthful reporting is not an equilibrium and agents incentives are to manipulate. Thus, our results apply when beliefs are conditionally independent, it is unclear how the strategic behaviour of agents would affect them if beliefs where conditionally dependent. Alternatively, they can be interpreted as being valid for myopic traders.

In (Beygelzimer et al., 2012) a model of inter-market dynamics with kelly bettors where each time period the market walrasian equilibrium is used (and the dynamics of how or why this is reached are not analysed) the outcome of the event causes to be wealth transferred. Our Theorem 2 can in a sense be seen as a related model with kelly bettors but focusing on the dynamics of prices relative to beliefs within a single market (i.e. intra-market dynamics).

2. Models

For simplicity, we state our model for the binary outcome prediction market, though in principle these techniques could be easily extended to more complicated settings. Thus, we need only track the price π of “contract 1”, as the price of contract 0 will be $1 - \pi$. We will make use of an automated market maker – following (Abernethy et al., 2011), we will equip our market maker with a convex function C such that $\nabla C(\mathbb{R}) \subseteq [0, 1]$. For brevity we will write $\varphi := \nabla C$.

In full generality, our model is the following. The market maker posts the current price π_t , and at each time $t = 1 \dots T$, a trader is chosen with belief $p \sim \mathcal{P}$ and wealth $W \sim \mathcal{W}$ and some demand function $d(W, \pi_t, p) \in \mathbb{R}$ which gives the number of contracts of the positive event that the trader purchases. The price is then updated to $\pi_{t+1} = \varphi(\varphi^{-1}(\pi_t) + d(W, \pi_t, p))$.

We will make the following assumptions:

1. Bounded wealth: $\exists W_{\max} \Pr_{\mathcal{W}}[W > W_{\max}] = 0$
2. Decreasing demands: $d(W, \pi + \epsilon, p) \leq d(W, \pi, p)$
3. No debt: $-W/(1 - \pi) \leq d(W, \pi, p) \leq W/\pi$
4. Stationary only at belief: $\frac{\partial d}{\partial W} = 0 \iff \pi = p$

Note that according the standard automated market model, where the price varies continuously as contracts are purchased, the total money required for purchase $d = W/\pi$ will in general be higher than W . In this sense, our model is a *fixed-price* approximation to the standard model, where the price is fixed during orders, and is only updated afterwards. This is crucial for our analysis in Section 4, but could in principle be relaxed in Section 3.

3. Stationarity and equilibrium

We will show that in the fixed-price model, the stationary point of the stochastic process approaches the Walrasian equilibrium point as the wealth of the traders approaches 0 (relative to the liquidity of the market). So that the equilibria in question are unique, we will assume that the distributions \mathcal{P} and \mathcal{W} have simply-connected support.

First, let us analyze the equilibrium point π^* , which is simply the price at which the market clears. Formally, this is the following condition:

$$\int_0^1 \int_0^{W_{\max}} d(W, \pi^*, p) d\mathcal{P}(p) d\mathcal{W}(W) = 0. \quad (1)$$

The stationary point of our stochastic process, on the other hand, is the price π^s for which the expected price fluctuation is 0. Formally, we have

$$\mathbb{E}_{(W,p) \sim (\mathcal{P}, \mathcal{W})} [\Delta(\pi^s, d(W, \pi^s, p))] = 0, \quad (2)$$

where $\Delta(\pi, d) := \varphi(\varphi^{-1}(\pi) + d) - \pi$ is the price fluctuation.

We now consider the limit of our stochastic process as the ratio of the wealth to the market liquidity approaches 0. As we have fixed the liquidity, we will take wealth to 0, but one could equivalently think of this limit as increasing the liquidity. As our wealth is stochastic, we will simply scale each wealth value by a constant $\alpha \rightarrow 0$, so instead of changing the distribution \mathcal{W} we can simply modify our demands to $d_\alpha(W, \pi, p) := d(\alpha W, \pi, p)$. Now we define π_α^* and π_α^s to be the equilibrium and stationary points of our process with distributions \mathcal{P} and \mathcal{W} and demands d_α , and let π_0^* and π_0^s be the respective limits as $\alpha \rightarrow 0$.

Theorem 1. For demands d satisfying conditions 1 through 4, and all convex C , $\pi_0^s = \pi_0^*$.

Proof. We define ‘‘excess demand’’ functions for our two cases:

$$\begin{aligned} Z_\alpha^*(\pi) &:= \frac{1}{\alpha} \mathbb{E}[d_\alpha(W, \pi, p)] \\ Z_\alpha^s(\pi) &:= \frac{1}{\alpha} \mathbb{E}[\Delta(\pi, d_\alpha(W, \pi, p))], \end{aligned}$$

where expectations are taken over \mathcal{P}, \mathcal{W} here and throughout. Note that for all α , π_α^s is the unique zero of Z_α^s , and likewise for π_α^* and Z_α^* , since the expectation is the same integral as in (1).

Note that by conditions 2 and 3, we must have for all $\pi \leq p$,

$$\lim_{\alpha \rightarrow 0} d_\alpha(W) = \lim_{\alpha \rightarrow 0} d(\alpha W) \leq \lim_{\alpha \rightarrow 0} \alpha W / \pi = 0. \quad (3)$$

Moreover, this limit is uniform in both p and W , since $d_\alpha(W, \pi, p) \leq \alpha W_{\max} / \pi$. The same argument holds for $\pi > p$. Now let $s = \varphi^{-1}(\pi)$, and consider the pointwise limit of the fluctuations:

$$\begin{aligned} \lim_{\alpha \rightarrow 0} \frac{\Delta(\pi, d_\alpha(W, \pi, p))}{d_\alpha(W, \pi, p)} &= \lim_{d \rightarrow 0} \frac{\Delta(\pi, d)}{d} \\ &= \lim_{d \rightarrow 0} \frac{\varphi(\varphi^{-1}(\pi) + d) - \pi}{d} \\ &= \lim_{d \rightarrow 0} \frac{\varphi(s + d) - \varphi(s)}{d} \\ &= \nabla \varphi(s) = \nabla^2 C(s). \end{aligned}$$

Relatedly, observe that $d_\alpha(0, \pi, p) = 0$ by condition 3, and thus

$$\begin{aligned} \lim_{\alpha \rightarrow 0} \frac{d_\alpha(W, \pi, p)}{\alpha} &= \lim_{\alpha \rightarrow 0} -W \frac{d(0, \pi, p) - d(\alpha W, \pi, p)}{W\alpha} \\ &= -W \frac{\partial}{\partial \bar{W}} d(0, \pi, p). \end{aligned}$$

We are now ready to analyze the limit excess demands Z_0^* and Z_0^s :

$$\begin{aligned} Z_0^s(\pi) &:= \lim_{\alpha \rightarrow 0} \frac{1}{\alpha} \mathbb{E}[\Delta(\pi, d_\alpha(W, \pi, p))] \\ &= \lim_{\alpha \rightarrow 0} \mathbb{E} \left[\frac{\Delta(\pi, d_\alpha(W, \pi, p))}{d_\alpha(W, \pi, p)} \frac{d_\alpha(W, \pi, p)}{\alpha} \right] \\ &= \mathbb{E} \left[\lim_{\alpha \rightarrow 0} \frac{\Delta(\pi, d_\alpha(W, \pi, p))}{d_\alpha(W, \pi, p)} \frac{d_\alpha(W, \pi, p)}{\alpha} \right] \\ &= \mathbb{E} \left[\nabla^2 C(s) \left(-W \frac{\partial}{\partial \bar{W}} d(0, \pi, p) \right) \right] \\ &= -\bar{W} \cdot \nabla^2 C(s) \mathbb{E} \left[\frac{\partial}{\partial \bar{W}} d(0, \pi, p) \right], \end{aligned}$$

where we used the limit uniformity to exchange the limit and expectation. More directly, we also have

$$Z_0^*(\pi) = -\bar{W} \mathbb{E} \left[\frac{\partial}{\partial \bar{W}} d(0, \pi, p) \right].$$

Thus, as $\nabla^2 C \geq 0$ by convexity of C , the zeros of Z_0^* and Z_0^s must be the same, namely $\pi_0^* = \pi_0^s$. \square

4. Kelly model as mirror descent

In the fixed-price model with Kelly betters, our update is the following

$$\pi_{t+1} = \varphi \left(\varphi^{-1}(\pi_t) + \frac{W}{\pi} \frac{p - \pi}{1 - \pi} \right), \quad (4)$$

where W and p are drawn (independently) from \mathcal{P} and \mathcal{W} . We will show that this is equivalent to a stochastic mirror descent of the form

$$x_{t+1} = \operatorname{argmin}_{x \in \mathbb{R}} \{ \eta x \cdot \nabla F(x_t; \xi) + D_R(x, x_t) \}, \quad (5)$$

where at each step $\xi \sim \Xi$ are i.i.d. and R is some strictly convex function. We will refer to an algorithm of the form (5) a *stochastic mirror descent* of $f(x) := \mathbb{E}_{\xi \sim \Xi} [F(x; \xi)]$.

Theorem 2. The stochastic update for fixed-price Kelly betters (4) is exactly a stochastic mirror descent of $f(\pi) = \bar{W} \cdot \text{KL}(\bar{p}, \pi)$, where \bar{p} and \bar{W} are the means of \mathcal{P} and \mathcal{W} , respectively.

Proof. By standard arguments, the mirror descent update (5) can be rewritten as

$$x_{t+1} = \nabla R^*(\nabla R(x_t) - \nabla F(x_t; \xi)),$$

where R^* is the conjugate dual of R . Take $R = C^*$, let $\xi = (p, W) \sim (\mathcal{P}, \mathcal{W})$, and take $F(x; (p, W)) = W \cdot (\text{KL}(p, x) + H(p))$. Then

$$\begin{aligned} \nabla F(x; (p, W)) &= W \left(\frac{-p}{x} + \frac{1-p}{1-x} \right) \\ &= -\frac{W}{x} \frac{p-x}{1-x}. \end{aligned}$$

Furthermore, as $\nabla R^* = \nabla C = \varphi$, we have $\varphi^{-1} = (\nabla R^*)^{-1} = \nabla R$ by duality, and thus our update becomes

$$\begin{aligned} x_{t+1} &= \nabla R^*(\nabla R(x_t) - \nabla F(x_t; \xi)), \\ &= \varphi \left(\varphi^{-1}(x_t) - \frac{W}{x} \frac{p-x}{1-x} \right), \end{aligned}$$

which exactly matches the Kelly update (4). Finally, the function this mirror descent is minimizing is

$$\begin{aligned} f(x) &= \mathbb{E}[F(x; \xi)] \\ &= \mathbb{E}[Wp \log x + W(1-p) \log(1-x)] \\ &= \bar{W} \cdot (\text{KL}(\bar{p}, x) + H(\bar{p})), \end{aligned}$$

which of course is equivalent to $\bar{W} \cdot \text{KL}(\bar{p}, x)$ as the entropy term does not depend on x . \square

Theorem 2 not only identifies a fascinating connection between machine learning and our stochastic prediction market model, but it also allows us to use powerful existing techniques to make broad conclusions about the behavior of our model. Consider the following result:

Proposition 1 ((Duchi et al., 2010)). *If $\|\nabla F(\pi; p)\|^2 \leq G^2$ for all p, π , and R is σ -strongly convex, then with probability $1 - \delta$,*

$$f(\bar{\pi}_T) \leq \min_{\pi} f(\pi) + \left(\frac{D^2}{\eta T} + \frac{G^2 \eta}{2\sigma} \right) \left(1 + 4\sqrt{\log \frac{1}{\delta}} \right).$$

In our context, Proposition 1 says that the *average* of the prices will be a very good estimate of the minimizer of f , which as suggested by happens to be the underlying mean belief \bar{p} of the traders! Moreover, as the Kelly demands are linear in both p and W , it is easy to see from Theorem 1 that \bar{p} is also the stationary point and the Walrasian equilibrium point (the latter was also shown by (Wolfers & Zitzewitz, 2006)). On the other hand, as we demonstrate next, it is not hard to come up with an example where the *instantaneous* price π_t is quite far from the equilibrium at any given time period.

Before moving to our example, we make one final point. The above relationship between our stochastic market model and mirror descent sheds light on an important question: how might an automated market maker adjust the liquidity so that the market actually converges to the mean of the traders' beliefs? The learning parameter η can be thought of as the inverse of the liquidity, and as such, Proposition 1 suggests that increasing the liquidity as \sqrt{t} will cause the mean price to converge to the mean belief.

5. Example: biased coin

Consider a classic Bayesian setting where a coin has unknown bias $\Pr[\text{heads}] = q$, and traders have a prior $\beta(\alpha, \alpha)$ over q (i.e., traders are α -confident that the coin is fair). Now suppose each trader independently observes n flips from the coin, and updates her belief; upon seeing k heads, a trader would have posterior $\beta(\alpha + k, \alpha + n - k)$.

When presented with a prediction market with contracts for a single toss of the coin, where contract 0 pays \$1 for tails and contract 1 pays \$1 for

heads, a trader would purchase contracts as if according to the mean of their posterior. Hence, the belief distribution \mathcal{P} of the market assigns weight $\mathcal{P}(p) = \binom{n}{k} q^k (1-q)^{n-k}$ to belief $p = (\alpha + k)/(2\alpha + n)$, yielding a biased mean belief of $(\alpha + nq)/(2\alpha + n)$.

We show a typical simulation of this market in

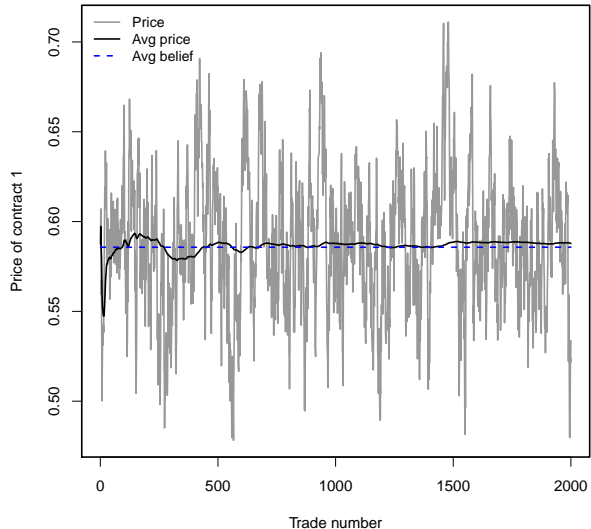


Figure 1. Price movement for Kelly bettors with binomial($q = 0.6$, $n = 6$, $\alpha = 0.5$) beliefs in the LMSR market with liquidity parameter $b = 10$.

Figure 1, where traders behave as Kelly bettors in the fixed-price LMSR. Clearly, after almost every trade, the market price is quite far from the equilibrium/stationary point, and hence the classical supply and demand analysis of this market yields a poor description of the actual behavior, and in particular, of the predictive quality of the price at any given time. However, the mean price is consistently close to the mean belief of the traders, which in turn is quite close to the true parameter q .

6. Conclusion and future work

There appears to be substantial possibility to generalise theorem 1 to more general classes of cost functions and relaxing the fixed-price assumption. The equivalence to mirror decent established in theorem 2 may lead to a better understanding of the optimal manner in which a automated prediction market ought to increase liquidity so as to maximise efficiency.

References

- Abernethy, J., Chen, Y., and Vaughan, J.W. An optimization-based framework for automated market-making. In *Proceedings of the 11th ACM conference on Electronic Commerce (EC'11)*, 2011.
- Beygelzimer, A., Langford, J., and Pennock, D. Learning performance of prediction markets with kelly bettors. In *ICML*, 2012.
- Chen, Y. and Vaughan, J.W. *A new understanding of prediction markets via no-regret learning*, pp. 189198. 2010.
- Chen, Y., Reeves, D.M., Pennock, D.M., Hanson, R.D., Fortnow, L., and Gonen, R. Bluffing and strategic reticence in prediction markets. In *Proceedings of the 3rd international conference on Internet and network economics*, pp. 70–81. Springer-Verlag, 2007.
- Das, S. and Magdon-Ismail, M. Adapting to a market shock: Optimal sequential market-making. *Advances in Neural Information Processing Systems (NIPS)*, pp. 361368, 2008.
- Duchi, J., Shalev-Shwartz, S., Singer, Y., and Tewari, A. Composite objective mirror descent. *COLT*, 2010.
- Manski, C.F. Interpreting the predictions of prediction markets. Technical report, National Bureau of Economic Research, 2004.
- Othman, A. and Sandholm, T. When do markets with simple agents fail? In *Proceedings of the 9th International Conference on Autonomous Agents and Multiagent Systems: volume 1-Volume 1*, pp. 865–872. International Foundation for Autonomous Agents and Multiagent Systems, 2010.
- Wolfers, J. and Zitzewitz, E. Interpreting prediction market prices as probabilities. Technical report, National Bureau of Economic Research, 2006.