CSCI 5822
Probabilistic Model of Human and Machine Learning

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Topics

Language modeling

Hierarchical processes

Pitman-Yor processes

Based on work of Teh (2006), “A hierarchical Bayesian language model based on Pitman-Yor processes”

Probabilistic model of language

n-gram model

P(word i | word$_{i-n+1}^{i-1}$)

Typically, trigram model (n=3)

Utility

e.g., speech, handwriting recognition

$$P(\text{sentence}) \approx \prod_{i=1}^{T} P(\text{word}_i \mid \text{word}_{i-n+1}^{i-1})$$

Challenge

If n too small, doesn't capture regularities of language

If n too large, insufficient data

Past approaches have used heuristics for choosing appropriate n, or by averaging predictions across a range of values of n, or by smoothing hacks of various sorts.

Hierarchical Pitman-Yor probabilistic model addresses this challenge via principled approach with explicit assumptions.

The Basic Story

Dirichlet process

- Distribution over distributions of a countably infinite number of alternatives
- E.g., Dirichlet process mixture model
  - Generative model of points in real-valued space
  - Partitioning of points into a countably infinite number of clusters

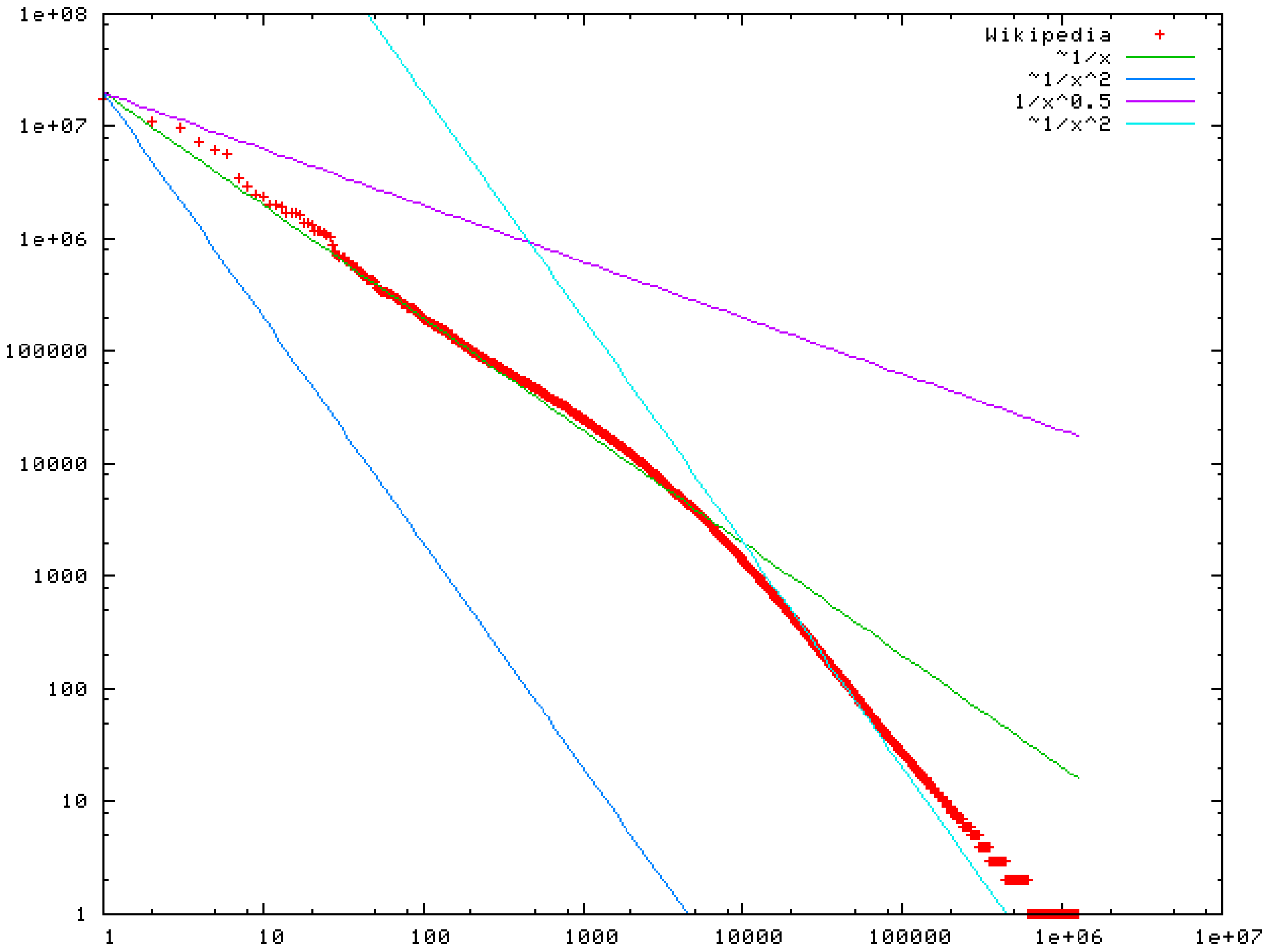
In principle, could use Dirichlet process to model language

- Draws from DP would represent distribution over words (not clusters)
- In terms of CRP
  - Each table represents one word type
  - Each diner represents one word token
- Types vs. tokens
  - e.g., The_1 mean dog_1 ate the_2 small dog_2.

Zipf’s Law

Rank word types by frequency

Plot log frequency vs. log word index

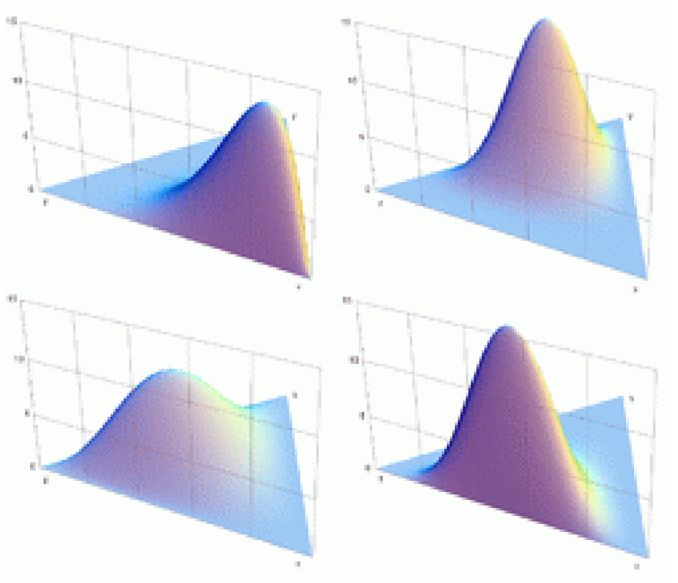


Dirichlet Distribution

Distribution over distributions over a finite set of alternatives

$$f(x_1,\dots,x_{K-1};\alpha_1,\dots,\alpha_K) = \frac{1}{\mathrm{B}(\alpha)}\prod_{i=1}^{K} x_i^{\alpha_i - 1}$$

$$\mathrm{B}(\alpha) = \frac{\prod_{i=1}^{K}\Gamma(\alpha_i)}{\Gamma\left(\sum_{i=1}^{K}\alpha_i\right)}, \qquad \alpha = (\alpha_1,\dots,\alpha_K).$$



E.g., uncertainty in distribution over a finite set of words

Dirichlet Process

Distribution over a distributions over a countably infinite set of alternatives

Dirichlet distribution

$$\begin{tikzcd}
\textbf{Dirichlet process} \\
\textbf{Dirichlet distribution} \arrow[u, "\text{infinite case}"']
\end{tikzcd}$$

$$\begin{tikzcd}
\text{Dirichlet process} \arrow[r, "\text{draw collapsed}", "\text{sample}"'] & \text{CRP} \\
\text{Dirichlet distribution} \arrow[u, "\text{infinite case}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\begin{array}{c}\text{stick breaking}\\ \text{process}\\ \beta_k \sim \text{Beta}(1,\alpha)\end{array} & \\
\begin{array}{c}\text{Dirichlet}\\ \text{process}\end{array} \arrow[u, "\text{defined via}"'] \arrow[r, "\text{draw collapsed}", "\text{sample}"'] & \text{CRP} \\
\begin{array}{c}\text{Dirichlet}\\ \text{distribution}\end{array} \arrow[u, "\text{infinite case}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\begin{array}{c}\textbf{stick breaking}\\ \textbf{process}\\ \beta_k \sim \text{Beta}(1,\alpha)\end{array} & \\
\begin{array}{c}\textbf{Dirichlet}\\ \textbf{process}\end{array} \arrow[u, "\text{defined via}"'] \arrow[r, "\text{draw collapsed}", "\text{sample}"'] & \textbf{CRP} \\
\underbrace{\begin{array}{c}\textbf{Dirichlet}\\ \textbf{distribution}\end{array}}_{\begin{array}{c}\text{simple}\\ \text{analytic form}\end{array}} \arrow[u, "\text{infinite case}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\begin{array}{c}\textbf{stick breaking}\\ \textbf{process}\\ \beta_k \sim \text{Beta}(1-d,\theta+dk)\end{array} & \\
\begin{array}{c}\textbf{Pitman-Yor}\\ \textbf{process}\end{array} \arrow[u, "\text{defined via}"'] \arrow[r, "\text{draw collapsed}", "\text{sample}"'] & \begin{array}{c}\textbf{fancy}\\ \textbf{CRP}\end{array} \\
\underbrace{\begin{array}{c}\textbf{Pitman-Yor}\\ \textbf{distribution}\end{array}}_{\begin{array}{c}\text{no known}\\ \text{analytic form}\end{array}} \arrow[u, "\text{infinite case}"'] &
\end{tikzcd}$$

Pitman-Yor

Pitman-Yor distribution

Generalization of Dirichlet distribution

No known analytic form for density of PY for finite vocabulary.

Pitman-Yor Process

$G \sim PY(d, \alpha, G_0)$

d: discount parameter, $0 <= d < 1$

α: concentration parameter, $\alpha > -d$

Large α = more tables occupied

Large d = population at tables is more

DP is special case of PY Process

$PY(0, \alpha, G_0) = DP(\alpha, G_0)$

see stick breaking construction

Understanding G_0

With Gaussian Mixture Model, domain of G_0 is real-valued vector

$G_0(\theta)$: θ is vector $(\mu\ \Sigma)$ that specifies the mean μ and covariance Σ of a Gaussian bump.

$G_0(\theta)$ represents prior distribution over Gaussian parameters.

With natural language, think of domain of G_0 as all letter strings (potential words)

$G_0(w)$ is the prior probability of “word w”

Drawing Sample Of Words Directly From PY Process Prior

Chinese restaurant process

person 1 comes into restaurant and sits at table 1

person c+1 comes into restaurant and sits at one of the populated tables, k, $k \in \{1, ..., t\}$, with probability ~ c_k (# people at table k)

... or sits at a new table (t+1) with probability ~ α

Modified CRP

person 1 comes into restaurant and sits at table 1

person c+1 comes into restaurant and sits at one of the populated tables, k, $k \in \{1, ..., t\}$, with probability ~ $c_k - d$

... or sits at a new table (t+1) with probability ~ $\alpha + d\,t$

Pitman Yor

Why the restrictions $0 <= d < 1$ and $\alpha > -d$?

Generalized version of CRP

person 1 comes into restaurant and sits at table 1

person c+1 comes into restaurant and sits at one of the populated tables, k, $k \in \{1, ..., t\}$, with probability $\sim c_k - d$

... or sits at a new table (t+1) with probability $\sim \alpha + d\,t$

Pitman Yor

Like the CRP, notion that a specific “meal” is served at each table

- $\theta_k \sim G_0$ is meal for table k
- $\phi_c = \theta_k$ is meal instance served to individual c sitting at table k
- With DP mixture model, “meal” served was parameters of a gaussian bump
- With DP/PY language model, “meal” served is a word

Types vs. tokens

- e.g., The$_1$ mean dog$_1$ ate the$_2$ small dog$_2$.

Algorithm for drawing word c+1 according to Pitman-Yor

```
if $c = 0$              /* special case for first word */
   $t \leftarrow 1$              /* t: number of tables */
   $\theta_t \sim G_0$          /* meal associated with table */
   $c_t \leftarrow 1$             /* $c_t$: count of customers at table t */
   $\phi_c \leftarrow \theta_t$          /* $\phi_c$: meal associated with customer c */

else
   choose $k \in \{1, ..., t\}$ with probability $\sim c_k - d$,
                $k = t+1$ with probability $\sim \alpha + d\ t$

   if $k = t + 1$
      $t \leftarrow t + 1$
      $\theta_t \sim G_0$
      $c_t \leftarrow 1$
   else
      $c_k \leftarrow c_k + 1$
   endif

   $\phi_c \leftarrow \theta_k$

endif
```

Pitman Yor Produces Power Law Distribution

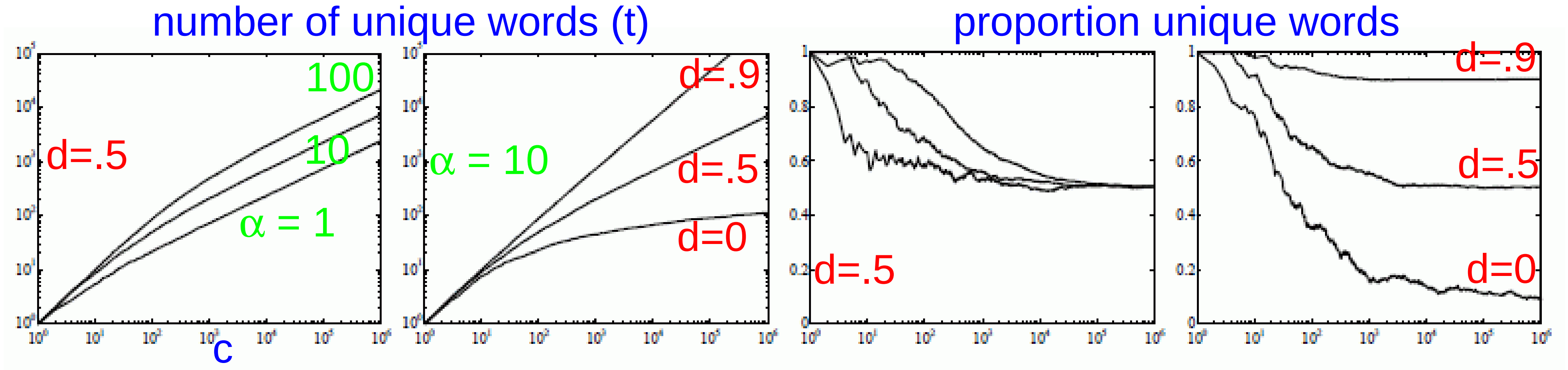


Figure 1: First panel: number of unique words as a function of the number of words drawn on a log-log scale, with $d = .5$ and $\alpha = 1$ (bottom), 10 (middle) and 100 (top). Second panel: same, with $\alpha = 10$ and $d = 0$ (bottom), .5 (middle) and .9 (top). Third panel: proportion of words appearing only once, as a function of the number of words drawn, with $d = .5$ and $\alpha = 1$ (bottom), 10 (middle), 100 (top). Last panel: same, with $\alpha = 10$ and $d = 0$ (bottom), .5 (middle) and .9 (top).

d is asymptotic proportion of words appearing only once

note d=0 ⇔ asymptotically no single-token words

Number of unique words as a function of c

- $d > 0$: $O(\alpha c^d)$
- $d = 0$: $O(\alpha \log c)$

Hierarchical Pitman-Yor (or Dirichlet) Process

Suppose you want to model where people hang out in a town (hot spots)

Not known in advance how many locations need to be modeled

Some spots are generally popular, others not so much.

But individuals also have preferences that deviate from the population preference.

E.g., bars are popular, but not for individuals who don’t drink.

Need to model distribution over hot spots at level of both population and individual.

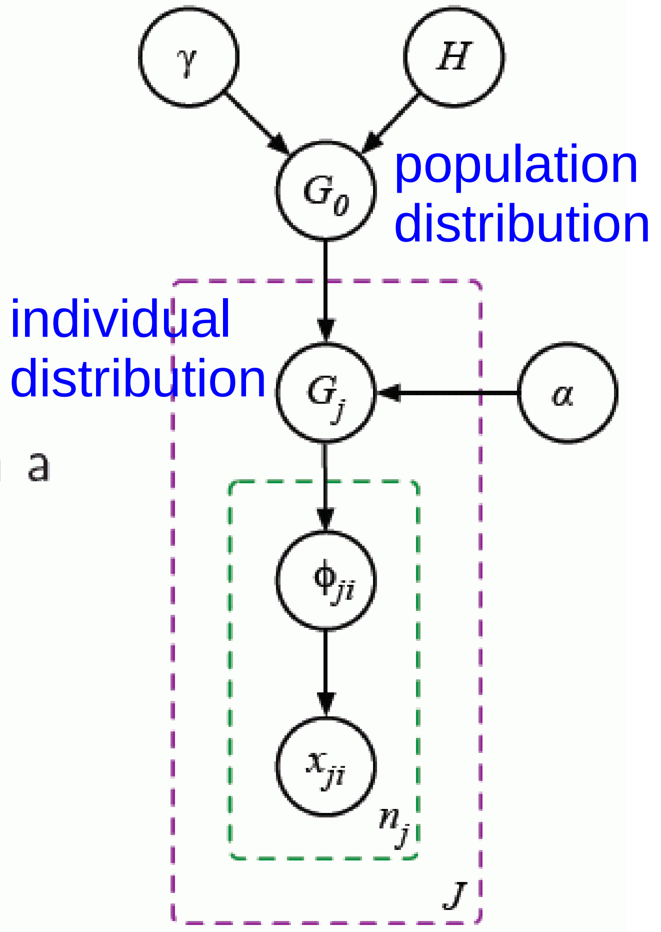
Hierarchical Pitman-Yor (or Dirichlet) Process

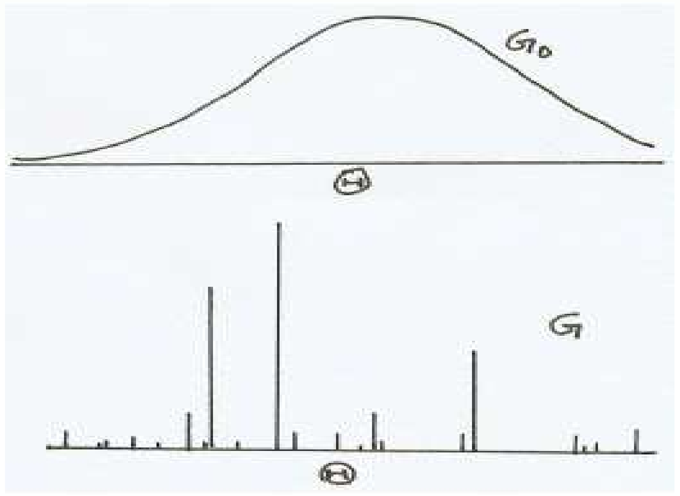
In an HDP there is a common DP:

$$G_0 | H, \gamma \sim \text{DP}(\cdot | H, \gamma)$$

Which forms the base measure for a draw from a DP within each group

$$G_j | G_0, \alpha \sim \text{DP}(\cdot | G_0, \alpha)$$





From Two-Level Hot-Spot Model To Multilevel Word Model: Capturing Levels Of Generality

P(hot spot)

P(hot spot | individual)

P(word$_i$)

P(word$_i$|word$_{i-1}$)

P(word$_i$|word$_{i-1}$,word$_{i-2}$)

Intuition

Use the more general distribution as the basis for the more context-specific distribution

P(word$_i$ | word$_{i-1}$) is specialized version of P(word$_i$)

P(word$_i$ | word$_{i-1}$,word$_{i-2}$) is specialized version of P(word$_i$ | word$_{i-1}$)

Formally

$$\underbrace{G_{\mathbf{u}}}_{\text{context (preceding n–1 words)}} \sim \mathrm{PY}(d_{|\mathbf{u}|}, \alpha_{|\mathbf{u}|}, \underbrace{G_{\pi(\mathbf{u})}}_{\text{suffix of u consisting of all but the earliest word}})$$

states supreme

United states supreme

Defines G$_u$ recursively, anchored with

$$G_\emptyset \sim \mathrm{PY}(d_0, \alpha_0, G_0)$$

Sampling With Hierarchical CRP

Imagine a different restaurant for every possible context u

- e.g., “supreme”, “states supreme”, “united states supreme”, “united states of”

... each with its own set of tables and own population of diners

The meals served at the popular tables at restaurant u will tend to be same as meals at popular tables at restaurant π(u)

- Consider restaurants (contexts) **u** and π(**u**)
  - e.g., **u** = “the united” and π(**u**) = “united”
- and some past (in generative model) assignments of tables based on
  - ‘united states’
    ‘united we stand’
    ‘united under god’
    ‘the united states’

Notation

$c_{\mathbf{u}.k}$

- number of diners in restaurant **u** assigned to table *k*
- number of tokens of the word indexed by *k* appearing in context **u**
- But we also need to represent the meal served at a table ...

$c_{\mathbf{u}w.}$

- number of diners in restaurant **u** assigned to a table with meal *w*
- number of tokens of word w appearing in context **u**
- NOTE: unlike DPMM, this count is an *observation*

$c_{\mathbf{u}wk}$

- number of diners in restaurant **u** assigned to table *k* which has meal *w*
- in context **u**, number of tokens of word *w*, when word *w* is indexed by *k*

$t_{\mathbf{u}\cdot\cdot}$

- formerly just t: number of occupied tables in restaurant **u**
- number of word types appearing in context **u**

$t_{\mathbf{u}w\cdot}$

- 1 if restaurant **u** serves meal *w,* 0 otherwise
- 1 if word *w* appears in context **u**

$t_{\mathbf{u}wk}$

- 1 if restaurant **u** has meal *w* assigned to table *k,* 0 otherwise

Redundant indexing of c and t by both *k* and *w* necessary to allow for reference either by

table (for generative model) or

meal (for probability computation)

Algorithm for drawing word w in context u

```
Function $w$ = DrawWord($\mathbf{u}$)

if $\mathbf{u} = 0$
   $w \sim G_0$

else
   choose $k \in \{1, \ldots, t_{\mathbf{u}\cdot\cdot}\}$ with probability $\sim c_{\mathbf{u}\cdot k} - d_{|\mathbf{u}|}$,
            $k = t_{\mathbf{u}\cdot\cdot}+1$ with probability $\sim \alpha_{|\mathbf{u}|} + d_{|\mathbf{u}|}\, t_{\mathbf{u}\cdot\cdot}$

   if $k = t_{\mathbf{u}\cdot\cdot} + 1$
      $\theta_{\mathbf{u}k} \leftarrow$ DrawWord($\pi(\mathbf{u})$)        <- new table drawn or restaurant is
      $w \leftarrow \theta_{\mathbf{u}k}$                                     empty
      $c_{\mathbf{u}wk} \leftarrow 1$
      $t_{\mathbf{u}wk} \leftarrow 1$                                    COOL RECURSION!
   else
      $w \leftarrow \theta_{\mathbf{u}k}$
      $c_{\mathbf{u}wk} \leftarrow c_{\mathbf{u}wk} + 1$
endif
```

Inference

Given training data, $\mathcal{D}$, compute predictive posterior over words w for a given context, u:

$$p(w|\mathrm{u},\mathcal{D}) = \int p(w|\mathrm{u},\mathcal{S},\Theta)p(\mathcal{S},\Theta|\mathcal{D})\,d(\mathcal{S},\Theta)$$

$\Theta = \{\alpha_m, d_m: 0 <= m <= n-1\}$

$\mathcal{S}$ = seating arrangement

Because of structured relationship among the contexts, this probability depends on more than the empirical count.

e.g., p(“of” | “united states”) will be influenced by p(“of” | “states”), p(“of”), p(“of” | “altered states”), etc.

Inference

Given training data, $\mathcal{D}$, compute predictive posterior over words w for a given context, u:

$$p(w|\mathbf{u},\mathcal{D}) = \int \boxed{p(w|\mathbf{u},\mathcal{S},\Theta)}\, p(\mathcal{S},\Theta|\mathcal{D})\, d(\mathcal{S},\Theta)$$

$\Theta = \{\alpha_m, d_m : 0 <= m <= n-1\}$

$\mathcal{S}$ = seating arrangement

Compute P(w|u,$\mathcal{S}$,Θ) with...

```
Function WordProb($\mathbf{u}$,$w$):
Returns the probability that the next word after context $\mathbf{u}$ will be $w$.
If $\mathbf{u} = 0$, return $G_0(w)$. Else return
    $\frac{c_{\mathbf{u}w\cdot} - d_{|\mathbf{u}|} t_{\mathbf{u}w\cdot}}{\alpha_{|\mathbf{u}|} + c_{\mathbf{u}\cdot\cdot}} + \frac{\alpha_{|\mathbf{u}|} + d_{|\mathbf{u}|} t_{\mathbf{u}\cdot\cdot}}{\alpha_{|\mathbf{u}|} + c_{\mathbf{u}\cdot\cdot}}$WordProb($\pi(\mathbf{u})$,$w$).
```

Inference

Given training data, $\mathcal{D}$, compute predictive posterior over words w for a given context, u:

$$p(w|\mathbf{u},\mathcal{D}) = \int p(w|\mathbf{u},\mathcal{S},\Theta)\boxed{p(\mathcal{S},\Theta|\mathcal{D})}\,d(\mathcal{S},\Theta)$$

$\Theta = \{\alpha_m, d_m : 0 <= m <= n-1\}$

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Compute P(w|u,$\mathcal{S}$,Θ) with...

```
   Function WordProb($\mathbf{u}$,$w$):
*Returns the probability that the next word after context $\mathbf{u}$ will be $w$.*
If $\mathbf{u} = 0$, return $G_0(w)$. Else return
$\frac{c_{\mathbf{u}w\cdot} - d_{|\mathbf{u}|}t_{\mathbf{u}w\cdot}}{\alpha_{|\mathbf{u}|} + c_{\mathbf{u}\cdot\cdot}} + \frac{\alpha_{|\mathbf{u}|} + d_{|\mathbf{u}|}t_{\mathbf{u}\cdot\cdot}}{\alpha_{|\mathbf{u}|} + c_{\mathbf{u}\cdot\cdot}}$ WordProb($\pi(\mathbf{u})$,$w$).
```

Estimate P($\mathcal{S}$,Θ|$\mathcal{D}$) with MCMC

Gibbs Sampling of Seat Assignments

For a given individual (l) in a given restaurant (u), there are only a few possible choices for the seat assignment (k_{ul})

- If the individual’s meal is already assigned to a table, they can sit at one of those tables.
- A new table can be created that serves the individual’s meal.

$$p(k_{\mathbf{u}l} = k \mid \mathcal{S}^{-\mathbf{u}l}, \Theta) \propto \frac{\max(0, c^{-\mathbf{u}l}_{\mathbf{u}x_{\mathbf{u}l}k} - d)}{\alpha + c^{-\mathbf{u}l}_{\mathbf{u}\cdot\cdot}}$$

$$p(k_{\mathbf{u}l} = k^{\text{new}} \text{ with } y_{\mathbf{u}k^{\text{new}}} = x_{\mathbf{u}l} \mid \mathcal{S}^{-\mathbf{u}l}, \Theta) \propto \frac{\alpha + d t^{-\mathbf{u}l}_{\mathbf{u}\cdot\cdot}}{\alpha + c^{-\mathbf{u}l}_{\mathbf{u}\cdot\cdot}} p(x_{\mathbf{u}l} \mid \pi(\mathbf{u}), \mathcal{S}^{-\mathbf{u}l}, \Theta)$$

Sampling of Parameters Θ

- “Parameters are sampled using an auxiliary variable sampler as detailed in Teh (2006)”

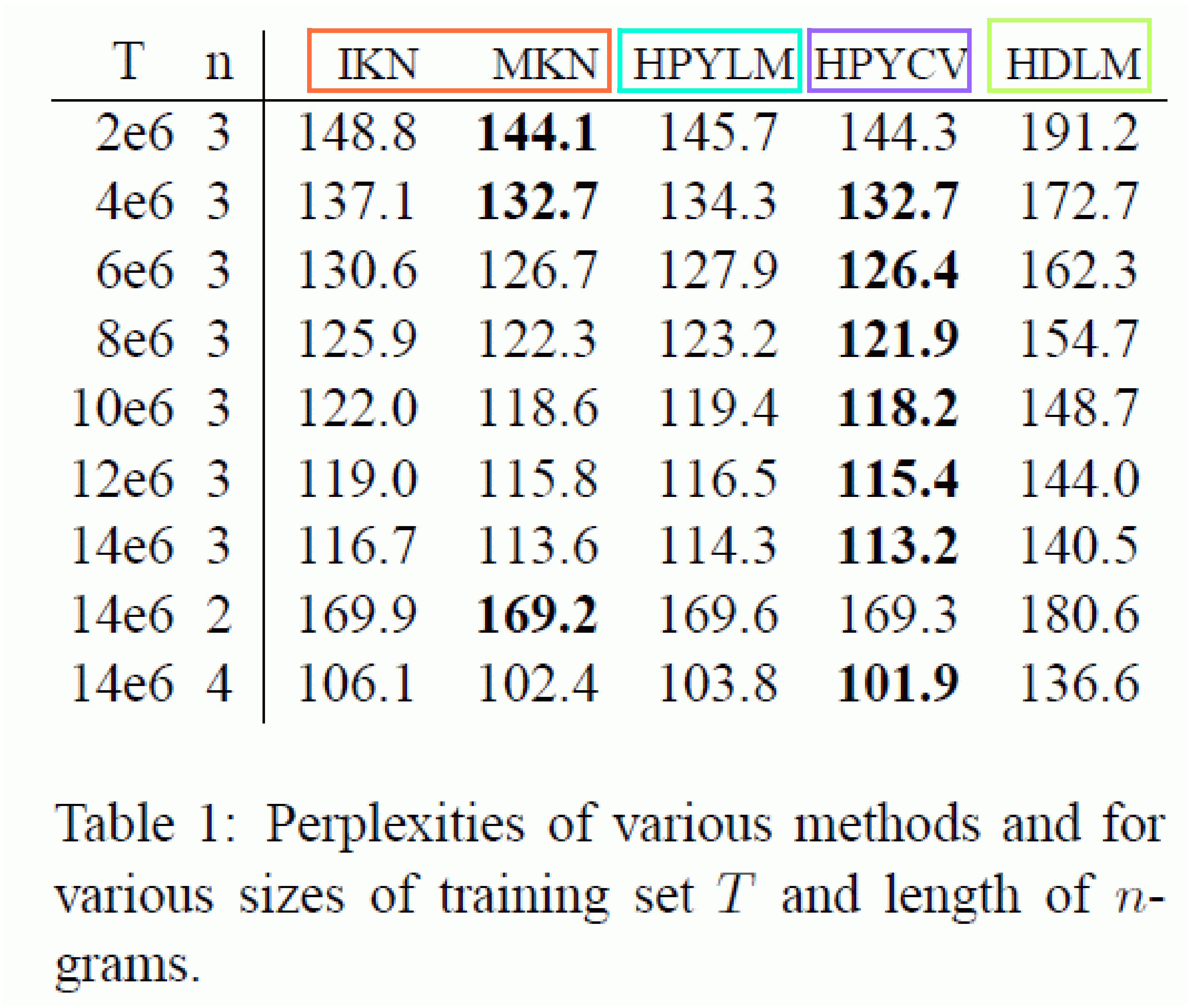
Implementation

$G_0(w) = 1/V$ for all w

$d_i \sim \text{Uniform}(0,1)$

$\alpha_i \sim \text{Gamma}(1,1)$

Results



T	n	IKN	MKN	HPYLM	HPYCV	HDLM
2e6	3	148.8	**144.1**	145.7	144.3	191.2
4e6	3	137.1	**132.7**	134.3	**132.7**	172.7
6e6	3	130.6	126.7	127.9	**126.4**	162.3
8e6	3	125.9	122.3	123.2	**121.9**	154.7
10e6	3	122.0	118.6	119.4	**118.2**	148.7
12e6	3	119.0	115.8	116.5	**115.4**	144.0
14e6	3	116.7	113.6	114.3	**113.2**	140.5
14e6	2	169.9	**169.2**	169.6	169.3	180.6
14e6	4	106.1	102.4	103.8	**101.9**	136.6

Table 1: Perplexities of various methods and for various sizes of training set T and length of n-grams.

IKN, MKN = traditional methods in language; HPYLM = hierarchical Pitman-Yor with Gibbs sampling; HPYCV= hierarchical Pitman-Yor with some cross validation hack to determine parameters; HDLM = hierarchical Dirichlet language model