Introduction to the Renormalization Group

Gregory Petropoulos

University of Colorado Boulder

March 4, 2015

Summary

- Flavor of Statistical Physics
- Universality / Critical Exponents
- Ising Model
- Renormalization Group
- Connection to Neural Networks

Punchline

- The Renormalization Group builds up relevant long distance physics by course graining short distance fluctuations.
- Deep Neural Networks seem to do the same thing for tasks like image recognition.

Section 1

Crash Course Statistical Mechanics

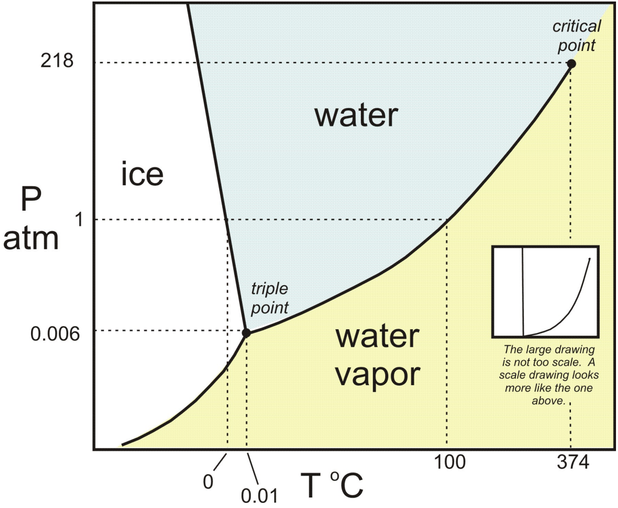
What is Statistical Mechanics?

- Study of systems with many degrees of freedom
- No longer concerned with deterministic behavior of the system at a microscopic level
- Interested in probabilistic properties of ensemble constituents
- System best described by macroscopic properties

Some Important Ideas

- *Microstate*: a state of the system where all the parameters of the constituents are specified (position, momentum)
- *Macrostate*: a state of the system where the distribution of particles over the energy levels is specified (pressure, volume, temperature)
- The *equilibrium* macrostate contains the overwhelming majority of microstates available to the system (nature maximises entropy)
- Probability distribution of a state with energy, E, at temperature, T, follows the *Boltzmann Distribution* $P(E) \propto \exp(-E/k_b T)$

Phase Transitions



[0]http://d32ogoqmya1dw8.cloudfront.net/images/research$_e$*ducation/equilibria/h2o*$_p$

Critical Points

At critical points:

- the correlation length of the system diverges
- system becomes scale invariant
- the properties of the system are characterized by critical exponents

Many disparate physical systems have the same critical exponents, this is known as universality.

Definitions

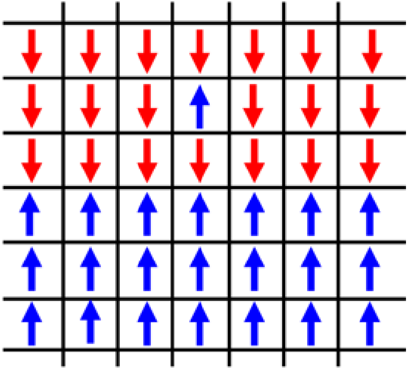
- Boltzmann Constant: $k_b = 1.3806488 \times 10^{-23} \frac{J}{K}$
- Boltzmann Factor: $\exp(\frac{-E}{k_b T})$
- Partition Function: $Z(T) = \sum_s \exp(\frac{-E_s}{k_b T})$
- Probability of state with energy E_s: $P(E_s) = \frac{\exp(\frac{-E_s}{k_b T})}{Z}$
- Average Energy: $\langle E \rangle = \frac{\sum_s E_s \exp(E_s / k_b T)}{Z} = k_b T^2 \frac{\partial \log Z}{\partial T}$

Section 2

Interlude: The Ising Model

The Ising Model

- Simple model of a ferromagnet
- Spins on a lattice (either up or down)
- Nearest neighbor interactions only
- $H = -\frac{1}{2} J \sum_{\langle ij \rangle} s_i s_j - B \sum_i s_i$



[0] http://www.thebrokendesk.com/post/monte-carlo-simulation-of-the-ising-model-using-python/

Phase Diagram

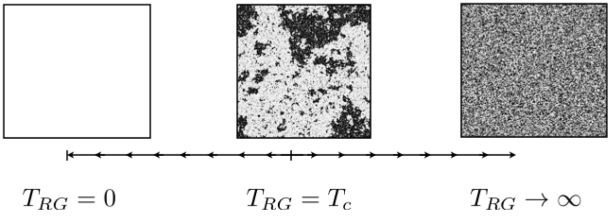
Consider $B = 0$

$T = 0$

- spins are completely aligned
- 2 degenerate ground states
- average magnetization is unitary

$T = \infty$

- spins are random
- a large number of equilibrium states
- average magnetization is zero



[0] http://www.kineticallyconstrained.com/2012_4_1_archive.html

Critical Temperature

- At the critical temperature $T = T_c$ the correlation length diverges
- Scale invariance (video)
- Correlation is measured by the two point correlation function $G_c(i,j)$
- $G_c(i,j) = \langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle$
- This is a measure of long range interactions that were not part of the local Hamiltonian!

Section 3

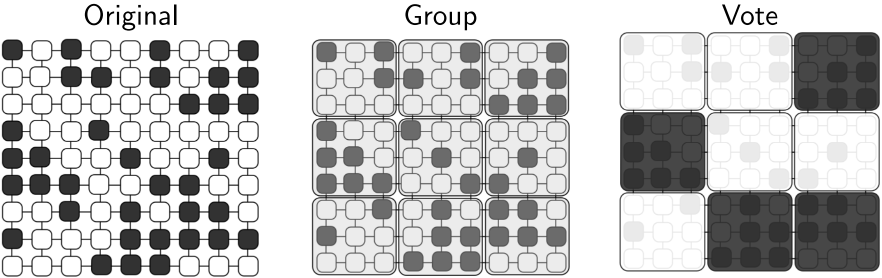
Renormalization Group

Real Space Renormalization Group

- We want to integrate out the short distance fluctuations to expose long distance properties.
- Accomplish this by iteratively blocking by a scale factor s
- lattice spacing: $a \to a' \equiv sa$
- degrees of freedom: $D \to D' \equiv \frac{D}{s^d}$
- correlation length: $\xi \to \xi' \equiv \frac{\xi}{s}$

Blocking

2 dimensional lattice with a scale factor of $s = 3$



[0] http://www.kineticallyconstrained.com/2012$_0$4$_0$1$_a$*rchive.html*

Renormalization Group Flow

- Blocked lattice is Ising model with effective Hamiltonian H' at temperature T'
- For RG transformation R: $H^{(n)} = R(H^{(n-1)})$
- At fixed point $H = R(H)$
- Flow in phase space

Calculating The Critical Temperature for the Ising Model

- At critical point correlation length $\xi = \infty$ on both lattices
- At critical point both effective Hamiltonians are identical
- At critical point all observables are identical

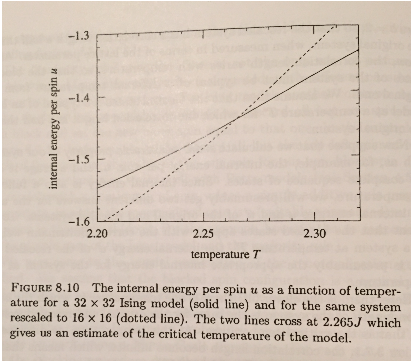


FIGURE 8.10 The internal energy per spin u as a function of temperature for a 32×32 Ising model (solid line) and for the same system rescaled to 16×16 (dotted line). The two lines cross at $2.265J$ which gives us an estimate of the critical temperature of the model.

[0] *Monte Carlo Methods in Statistical Physics*, Newman & Barkema

Section 4

Connection To Neural Networks

Vague Similarities

- Renormalization group transformation course grains short distance scales
- Neural network for image recognition built up features of an object

Paper

An exact mapping between the Variational Renormalization Group and Deep Learning: http://arxiv.org/pdf/1410.3831.pdf

- Derive a mapping from the Renormalization Group to Restricted Boltzmann Machines
- Give an example for the 1-D Ising Model
- Give an example for the 2-D Ising Model
- Authors' neural network appears to have learned how to perform block transformations on its own

Just Wondering Out loud

- Is there a deep connection here?
- Can this improve neural net design?
- Maybe images of cats are in a universality class?
- Can neural nets contribute to physics?