Inference in Bayesian networks

Outline

- ♢ Exact inference by enumeration
- ♢ Exact inference by variable elimination
- ♢ Approximate inference by stochastic simulation
- ♢ Approximate inference by Markov chain Monte Carlo

Inference tasks

Simple queries: compute posterior marginal $\mathbf{P}(X_i|\mathbf{E}=\mathbf{e})$
e.g., $P(NoGas|Gauge=empty, Lights=on, Starts=false)$

Conjunctive queries: $\mathbf{P}(X_i, X_j|\mathbf{E}=\mathbf{e}) = \mathbf{P}(X_i|\mathbf{E}=\mathbf{e})\mathbf{P}(X_j|X_i, \mathbf{E}=\mathbf{e})$

Optimal decisions: decision networks include utility information;
probabilistic inference required for $P(outcome|action, evidence)$

Value of information: which evidence to seek next?

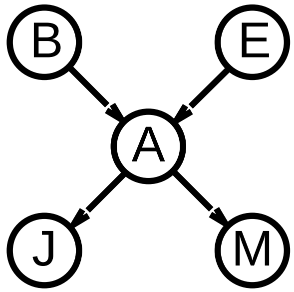
Sensitivity analysis: which probability values are most critical?

Explanation: why do I need a new starter motor?

Inference by enumeration

Slightly intelligent way to sum out variables from the joint without actually constructing its explicit representation

Simple query on the burglary network:
$\mathbf{P}(B|j,m)$
$= \mathbf{P}(B,j,m)/P(j,m)$
$= \alpha\mathbf{P}(B,j,m)$
$= \alpha\Sigma_e\Sigma_a\mathbf{P}(B,e,a,j,m)$



Rewrite full joint entries using product of CPT entries:
$\mathbf{P}(B|j,m)$
$= \alpha\Sigma_e\Sigma_a\mathbf{P}(B)P(e)\mathbf{P}(a|B,e)P(j|a)P(m|a)$
$= \alpha\mathbf{P}(B)\Sigma_e P(e)\Sigma_a\mathbf{P}(a|B,e)P(j|a)P(m|a)$

Recursive depth-first enumeration: $O(n)$ space, $O(d^n)$ time

Enumeration algorithm

function ENUMERATION-ASK(X, $\mathbf{e}$, bn) **returns** a distribution over X
  inputs: X, the query variable
    $\mathbf{e}$, observed values for variables $\mathbf{E}$
    bn, a Bayesian network with variables $\{X\} \cup \mathbf{E} \cup \mathbf{Y}$

  $\mathbf{Q}(X) \leftarrow$ a distribution over X, initially empty
  for each value x_i of X **do**
    extend $\mathbf{e}$ with value x_i for X
    $\mathbf{Q}(x_i) \leftarrow$ ENUMERATE-ALL(ORDERED-VARS[bn], $\mathbf{e}$)
  return NORMALIZE($\mathbf{Q}(X)$)

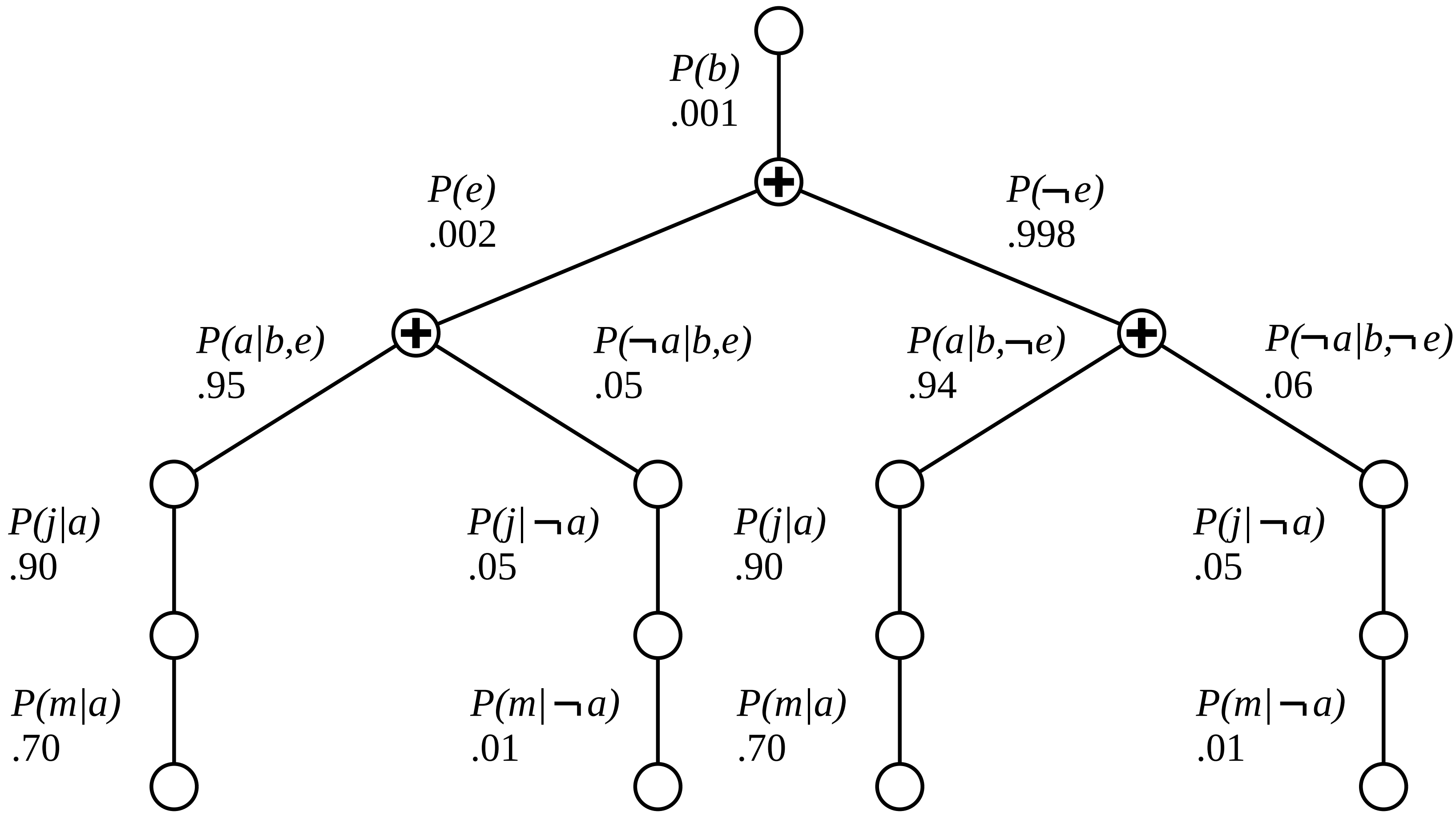
---

function ENUMERATE-ALL($vars$, $\mathbf{e}$) **returns** a real number
  if EMPTY?($vars$) **then return** 1.0
  $Y \leftarrow$ FIRST($vars$)
  if Y has value y in $\mathbf{e}$
    then return $P(y \mid Pa(Y)) \times$ ENUMERATE-ALL(REST($vars$), $\mathbf{e}$)
    else return $\Sigma_y\ P(y \mid Pa(Y)) \times$ ENUMERATE-ALL(REST($vars$), $\mathbf{e}_y$)
      where $\mathbf{e}_y$ is $\mathbf{e}$ extended with $Y = y$

Evaluation tree

Enumeration is inefficient: repeated computation

e.g., computes $P(j|a)P(m|a)$ for each value of e



Inference by variable elimination

Store intermediate results, a.k.a. factors, to avoid recomputation.

Variable elimination: carry out computations right-to-left

$$\mathbf{P}(B|j,m) = \alpha \underbrace{\mathbf{P}(B)}_{B} \Sigma_e \underbrace{P(e)}_{E} \Sigma_a \underbrace{\mathbf{P}(a|B,e)}_{A} \underbrace{P(j|a)}_{J} \underbrace{P(m|a)}_{M}$$

$$\begin{aligned}
&= \alpha \mathbf{P}(B) \Sigma_e P(e) \Sigma_a \mathbf{P}(a|B,e) P(j|a) f_M(a) \\
&= \alpha \mathbf{P}(B) \Sigma_e P(e) \Sigma_a \mathbf{P}(a|B,e) f_J(a) f_M(a) \\
&= \alpha \mathbf{P}(B) \Sigma_e P(e) \Sigma_a f_A(a,b,e) f_J(a) f_M(a) \\
&= \alpha \mathbf{P}(B) \Sigma_e P(e) f_{\bar{A}JM}(b,e) \text{ (sum out } A) \\
&= \alpha \mathbf{P}(B) f_{\bar{E}\bar{A}JM}(b) \text{ (sum out } E) \\
&= \alpha f_B(b) \times f_{\bar{E}\bar{A}JM}(b)
\end{aligned}$$

$$\mathbf{f}_M(A) = \begin{pmatrix} P(m|a) \\ P(m|\neg a) \end{pmatrix} \mathbf{f}_J(A) = \begin{pmatrix} P(j|a) \\ P(j|\neg a) \end{pmatrix} \mathbf{f}_{JM}(A) = \begin{pmatrix} P(j|a)P(m|a) \\ P(j|\neg a)P(m|\neg a) \end{pmatrix}$$

$$\mathbf{f}_A(A,B,E) = \begin{pmatrix} P(\neg a|\neg b,\neg e) & P(\neg a|\neg b,e) & / & P(a|\neg b,\neg e) & P(a|\neg b,e) \\ P(\neg a|b,\neg e) & P(\neg a|b,e) & / & P(a|b,\neg e) & P(a|b,e) \end{pmatrix}$$

Variable elimination: Basic operations

Summing out a variable from a product of factors:

move any constant factors outside the summation

add up submatrices in pointwise product of remaining factors

$$\Sigma_x f_1 \times \cdots \times f_k = f_1 \times \cdots \times f_i \, \Sigma_x \, f_{i+1} \times \cdots \times f_k = f_1 \times \cdots \times f_i \times f_{\bar{X}}$$

assuming $f_1, \ldots, f_i$ do not depend on X

Pointwise product of factors f_1 and f_2:

$$f_1(x_1, \ldots, x_j, y_1, \ldots, y_k) \times f_2(y_1, \ldots, y_k, z_1, \ldots, z_l)$$
$$= f(x_1, \ldots, x_j, y_1, \ldots, y_k, z_1, \ldots, z_l)$$

E.g., $f_1(a, b) \times f_2(b, c) = f(a, b, c)$

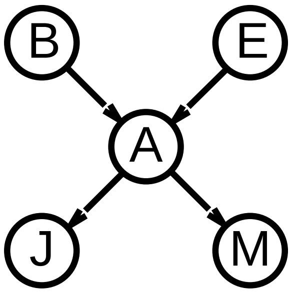
Variable elimination algorithm

```
function ELIMINATION-ASK(X, e, bn) returns a distribution over X
   inputs: X, the query variable
           e, evidence specified as an event
           bn, a belief network specifying joint distribution P(X_1, ..., X_n)

   factors ← [ ]; vars ← REVERSE(VARS[bn])
   for each var in vars do
      factors ← [MAKE-FACTOR(var, e)|factors]
      if var is a hidden variable then factors ← SUM-OUT(var, factors)
   return NORMALIZE(POINTWISE-PRODUCT(factors))
```

Irrelevant variables

Consider the query $P(JohnCalls|Burglary = true)$



$$P(J|b) = \alpha P(b) \sum_e P(e) \sum_a P(a|b,e) P(J|a) \sum_m P(m|a)$$

Sum over $m = 1$.

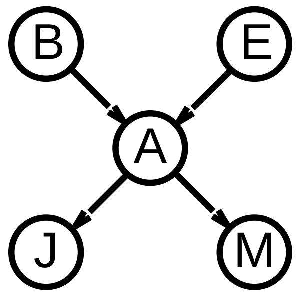
Therefore, $MaryCalls$ is irrelevant to the query.

Theorem: Y is irrelevant unless $Y \in Ancestors(\{X\} \cup \mathbf{E})$

Here, $X = JohnCalls$, $\mathbf{E} = \{Burglary\}$, and
$Ancestors(\{X\} \cup \mathbf{E}) = \{Alarm, Earthquake\}$
so $MaryCalls$ is irrelevant

Another example

Consider the query $P(A|m, \neg e)$

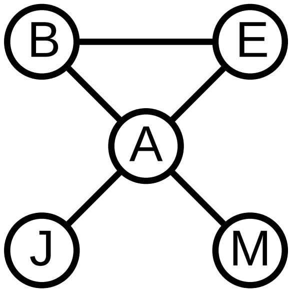


Irrelevant variables contd.

Defn: moral graph of Bayes net: marry all parents and drop arrows

Defn: $\mathbf{Y}$ is m-separated from $\mathbf{X}$ by $\mathbf{E}$ iff separated by $\mathbf{E}$ in the moral graph

Theorem: Y is irrelevant if m-separated from X by $\mathbf{E}$



For $P(JohnCalls|Alarm = true)$, both $Burglary$ and $Earthquake$ are irrelevant

Complexity of exact inference

Singly connected networks (or polytrees)

– any two nodes are connected by at most one (undirected) path

– time and space cost of variable elimination are $O(d^k n)$

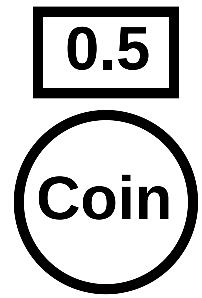
Multiply connected networks

– NP-hard (actual #P-complete)

Inference by stochastic simulation

Basic idea:

1) Draw N samples from a sampling distribution S
2) Compute an approximate posterior probability $\hat{P}$
3) Show this converges to the true probability P



Outline:

– Sampling from an empty network
– Rejection sampling: reject samples disagreeing with evidence
– Likelihood weighting: use evidence to weight samples
– Markov chain Monte Carlo (MCMC): sample from a stochastic process whose stationary distribution is the true posterior

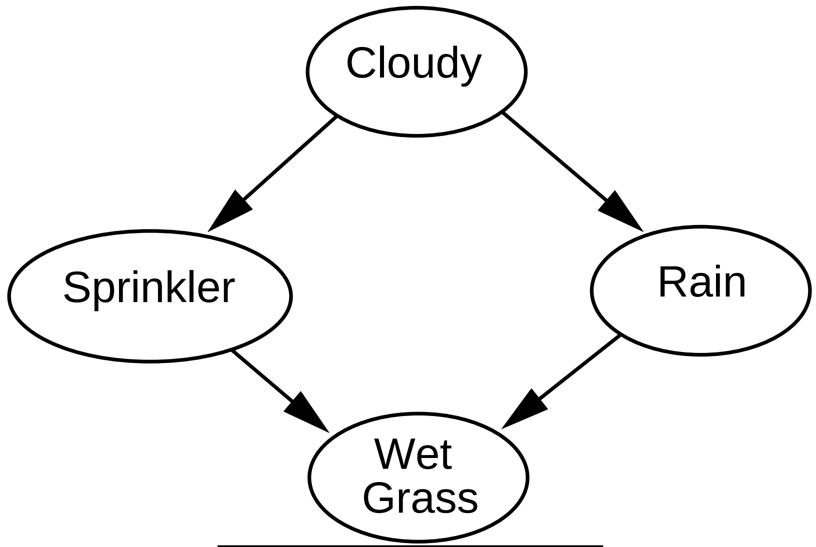
Sampling from an empty network

function PRIOR-SAMPLE(bn) **returns** an event sampled from bn
  inputs: bn, a belief network specifying joint distribution $\mathbf{P}(X_1, \ldots, X_n)$

  $\mathbf{x} \leftarrow$ an event with n elements
  for $i = 1$ **to** n **do**
    $x_i \leftarrow$ a random sample from $\mathbf{P}(X_i \mid Parents(X_i))$
  return $\mathbf{x}$

Example

P(C)
.50



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T	.10
F	.50

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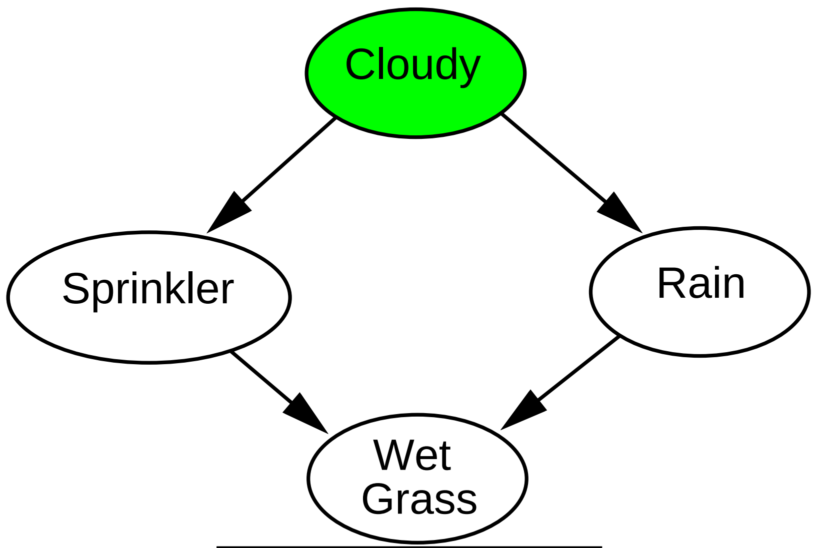

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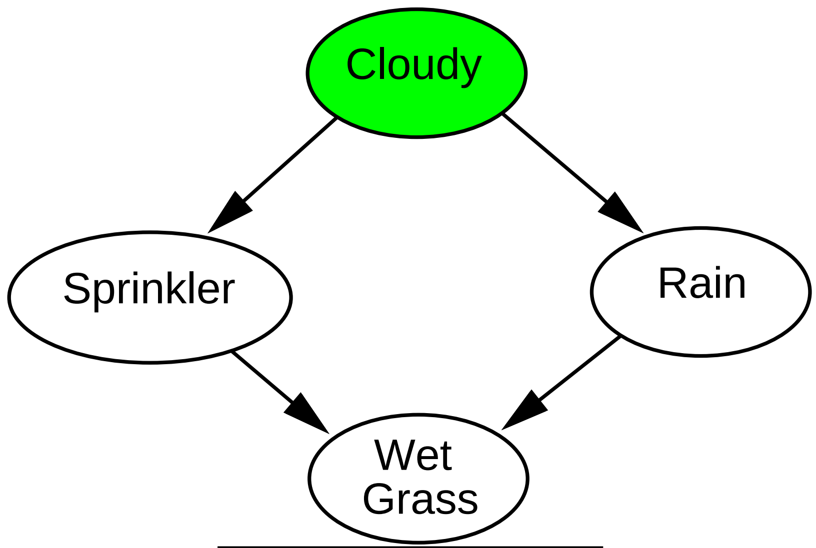

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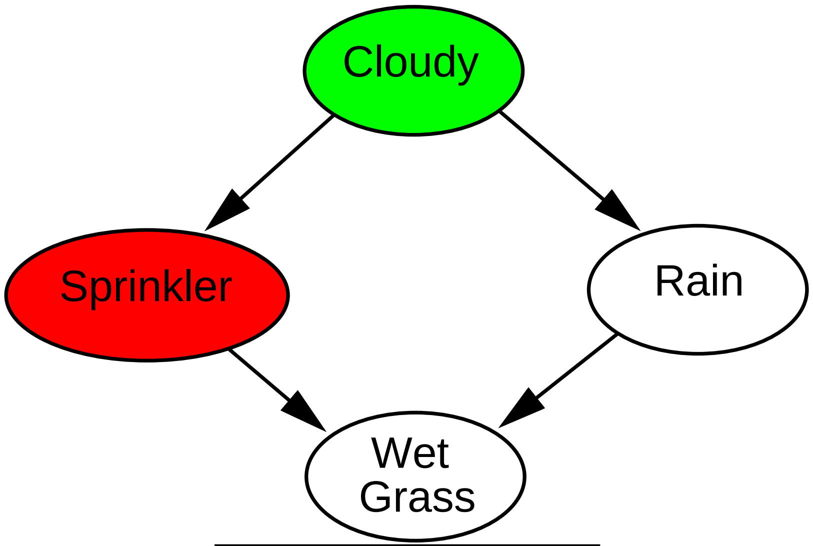

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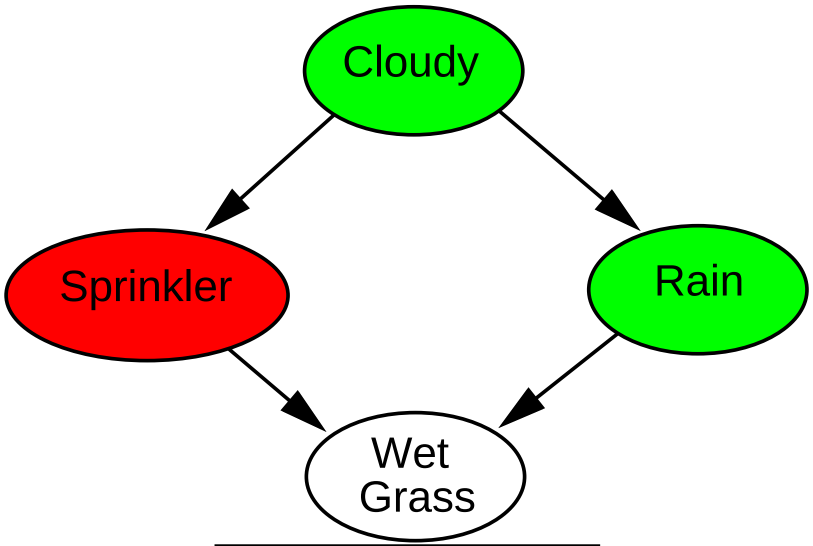

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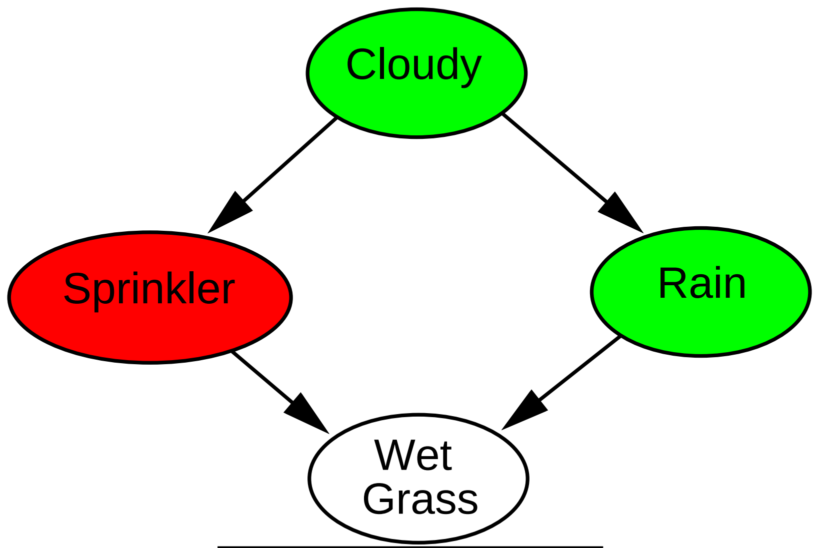

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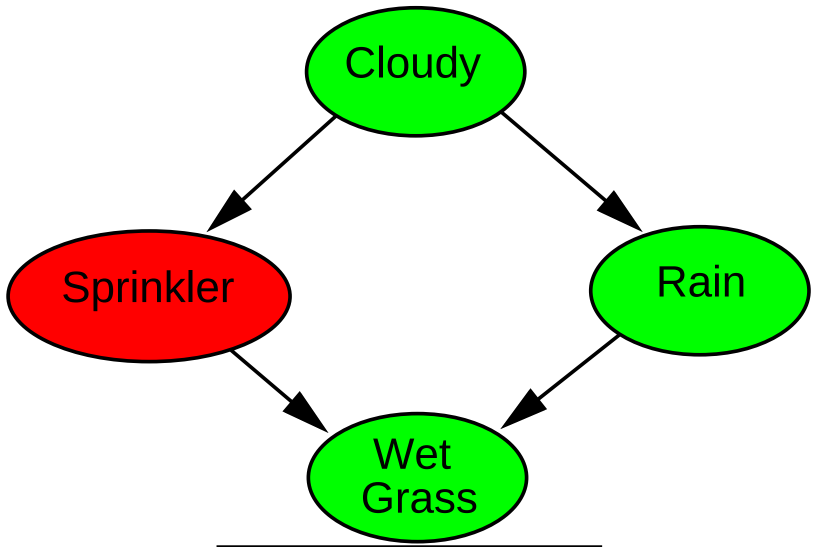

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Sampling from an empty network contd.

Probability that PriorSample generates a particular event

$S_{PS}(x_1 \ldots x_n) = \prod_{i=1}^{n} P(x_i | Parents(X_i)) = P(x_1 \ldots x_n)$

i.e., the true prior probability

E.g., $S_{PS}(t, f, t, t) = 0.5 \times 0.9 \times 0.8 \times 0.9 = 0.324 = P(t, f, t, t)$

Let $N_{PS}(x_1 \ldots x_n)$ be the number of samples generated for event $x_1, \ldots, x_n$

Then we have

$$\begin{aligned} \lim_{N \to \infty} \hat{P}(x_1, \ldots, x_n) &= \lim_{N \to \infty} N_{PS}(x_1, \ldots, x_n)/N \\ &= S_{PS}(x_1, \ldots, x_n) \\ &= P(x_1 \ldots x_n) \end{aligned}$$

That is, estimates derived from PriorSample are consistent

Shorthand: $\hat{P}(x_1, \ldots, x_n) \approx P(x_1 \ldots x_n)$

Rejection sampling

$\hat{\mathbf{P}}(X|\mathbf{e})$ estimated from samples agreeing with $\mathbf{e}$

```
function Rejection-Sampling(X, e, bn, N) returns an estimate of P(X|e)
   local variables: N, a vector of counts over X, initially zero

   for j = 1 to N do
       x ← Prior-Sample(bn)
       if x is consistent with e then
           N[x] ← N[x]+1 where x is the value of X in x
   return Normalize(N[X])
```

E.g., estimate $\mathbf{P}(Rain|Sprinkler = true)$ using 100 samples
27 samples have $Sprinkler = true$
Of these, 8 have $Rain = true$ and 19 have $Rain = false$.

$\hat{\mathbf{P}}(Rain|Sprinkler = true) = \textsc{Normalize}(\langle 8, 19\rangle) = \langle 0.296, 0.704\rangle$

Similar to a basic real-world empirical estimation procedure

Analysis of rejection sampling

$\hat{\mathbf{P}}(X|\mathbf{e}) = \alpha \mathbf{N}_{PS}(X, \mathbf{e})$ (algorithm defn.)
$= \mathbf{N}_{PS}(X, \mathbf{e})/N_{PS}(\mathbf{e})$ (normalized by $N_{PS}(\mathbf{e})$)
$\approx \mathbf{P}(X, \mathbf{e})/P(\mathbf{e})$ (property of PriorSample)
$= \mathbf{P}(X|\mathbf{e})$ (defn. of conditional probability)

Hence rejection sampling returns consistent posterior estimates

Problem: hopelessly expensive if $P(\mathbf{e})$ is small

$P(\mathbf{e})$ drops off exponentially with number of evidence variables!

Likelihood weighting

Idea: fix evidence variables, sample only nonevidence variables,
and weight each sample by the likelihood it accords the evidence

function LIKELIHOOD-WEIGHTING(X, $\mathbf{e}$, bn, N) **returns** an estimate of $P(X|\mathbf{e})$
  local variables: $\mathbf{W}$, a vector of weighted counts over X, initially zero

  for $j = 1$ to N **do**
    $\mathbf{x}$, $w \leftarrow$ WEIGHTED-SAMPLE(bn)
    $\mathbf{W}[x] \leftarrow \mathbf{W}[x] + w$ where x is the value of X in $\mathbf{x}$
  return NORMALIZE($\mathbf{W}[X]$)

---

function WEIGHTED-SAMPLE(bn, $\mathbf{e}$) **returns** an event and a weight

  $\mathbf{x} \leftarrow$ an event with n elements; $w \leftarrow 1$
  for $i = 1$ **to** n **do**
    if X_i has a value x_i in $\mathbf{e}$
      then $w \leftarrow w \times P(X_i = x_i \mid Parents(X_i))$
      else $x_i \leftarrow$ a random sample from $\mathbf{P}(X_i \mid Parents(X_i))$
  return $\mathbf{x}$, w

Likelihood weighting example

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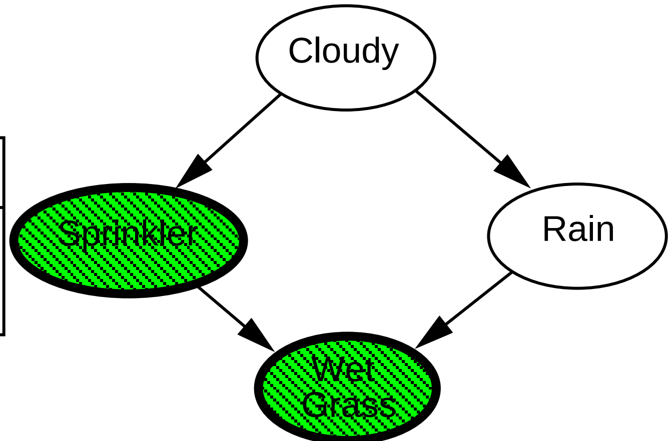
.50



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$w = 1.0$

Likelihood weighting example

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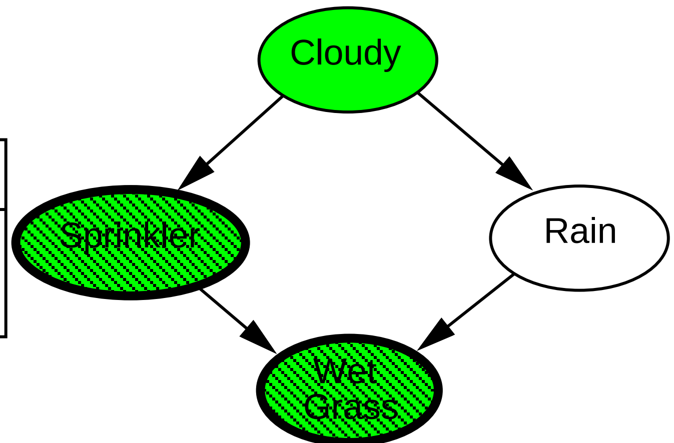

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Likelihood weighting example

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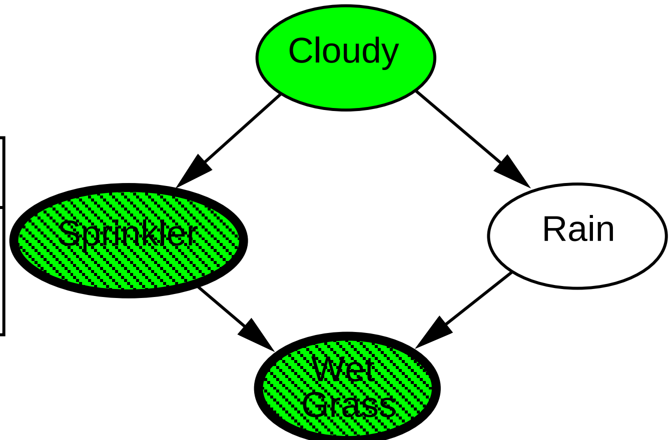

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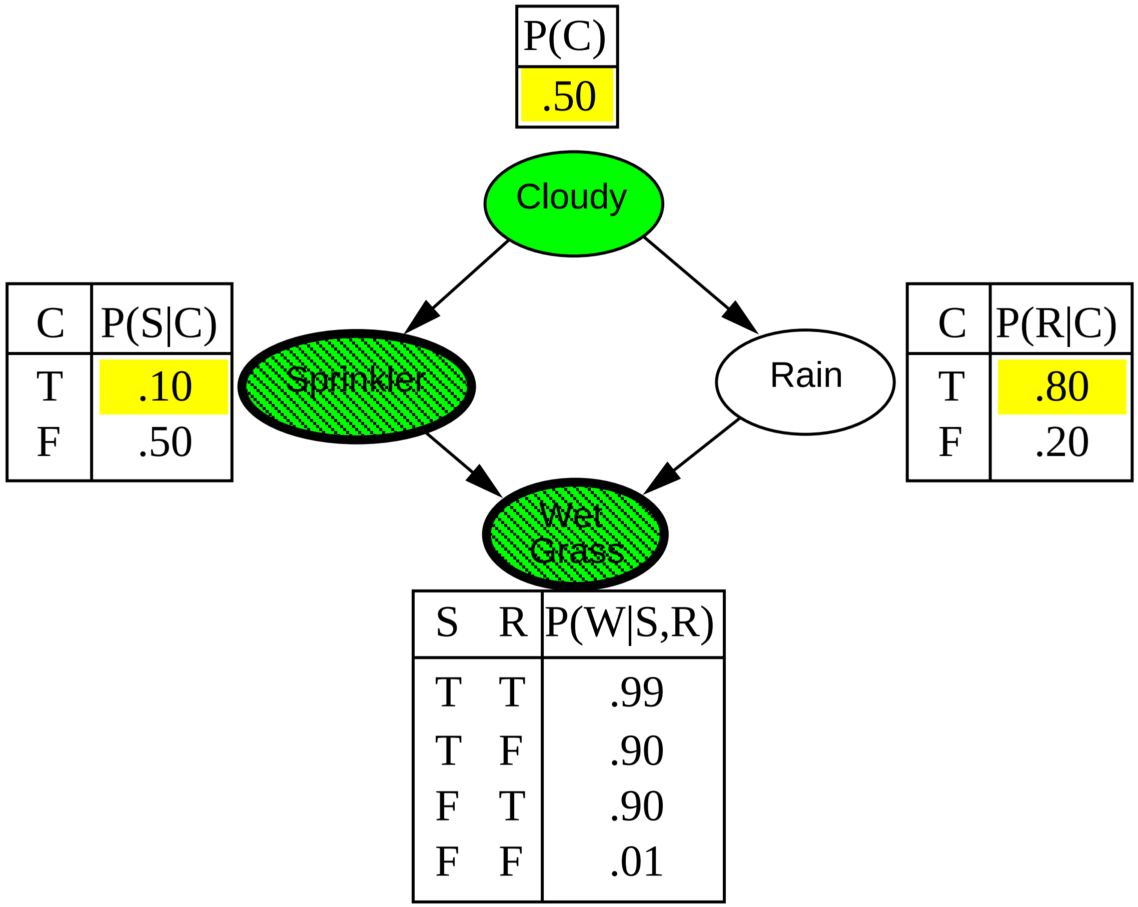

$w = 1.0 \times 0.1$

Likelihood weighting example

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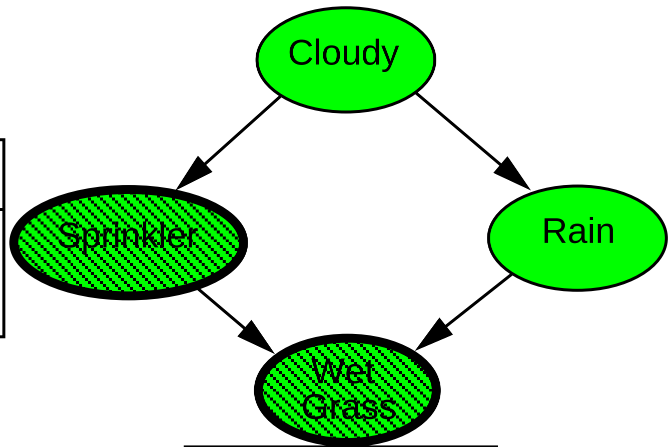

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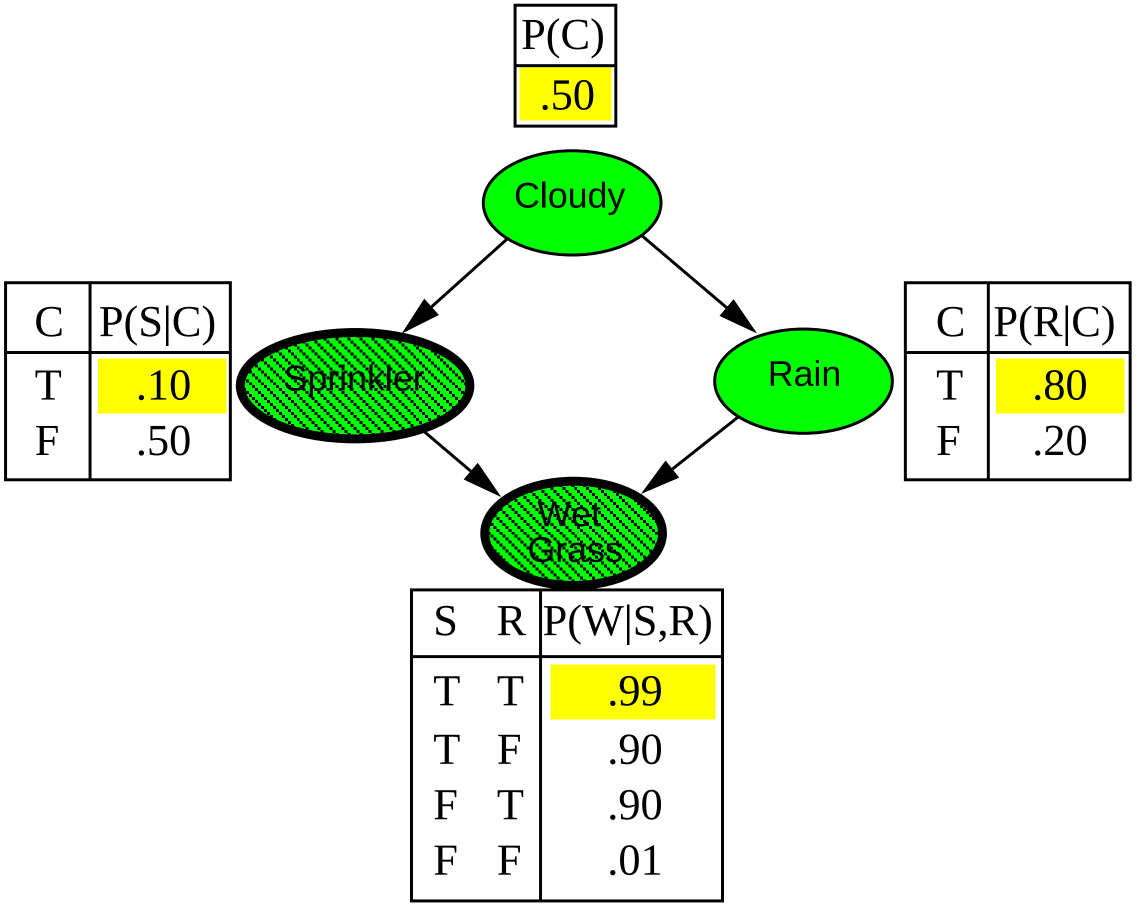

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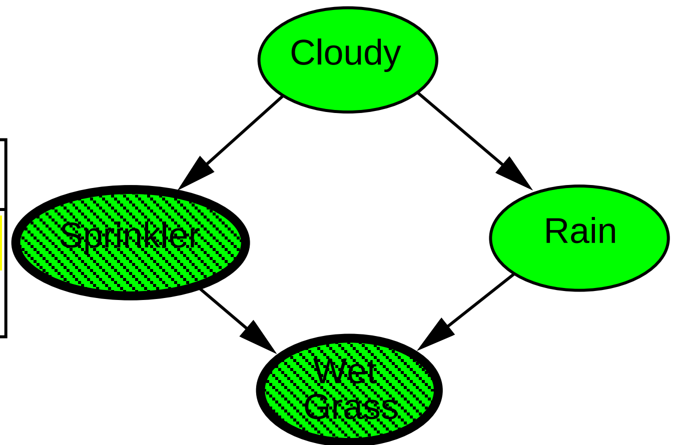
Likelihood weighting example



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$w = 1.0 \times 0.1 \times 0.99 = 0.099$

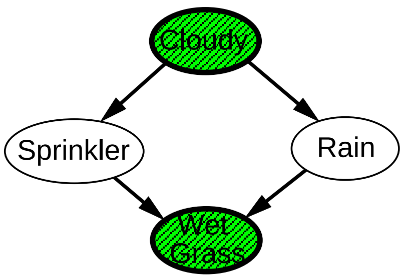
Likelihood weighting analysis

Sampling probability for WEIGHTEDSAMPLE is

$S_{WS}(\mathbf{z},\mathbf{e}) = \prod_{i=1}^{l} P(z_i|Parents(Z_i))$

Note: pays attention to evidence in **ancestors** only

$\Rightarrow$ somewhere "in between" prior and posterior distribution



Weight for a given sample $\mathbf{z},\mathbf{e}$ is

$w(\mathbf{z},\mathbf{e}) = \prod_{i=1}^{m} P(e_i|Parents(E_i))$

Weighted sampling probability is

$S_{WS}(\mathbf{z},\mathbf{e})w(\mathbf{z},\mathbf{e})$

$= \prod_{i=1}^{l} P(z_i|Parents(Z_i)) \;\; \prod_{i=1}^{m} P(e_i|Parents(E_i))$

$= P(\mathbf{z},\mathbf{e})$ (by standard global semantics of network)

Hence likelihood weighting returns consistent estimates
but performance still degrades with many evidence variables
because a few samples have nearly all the total weight

Approximate inference using MCMC

Start by assigning all variables in network random values.

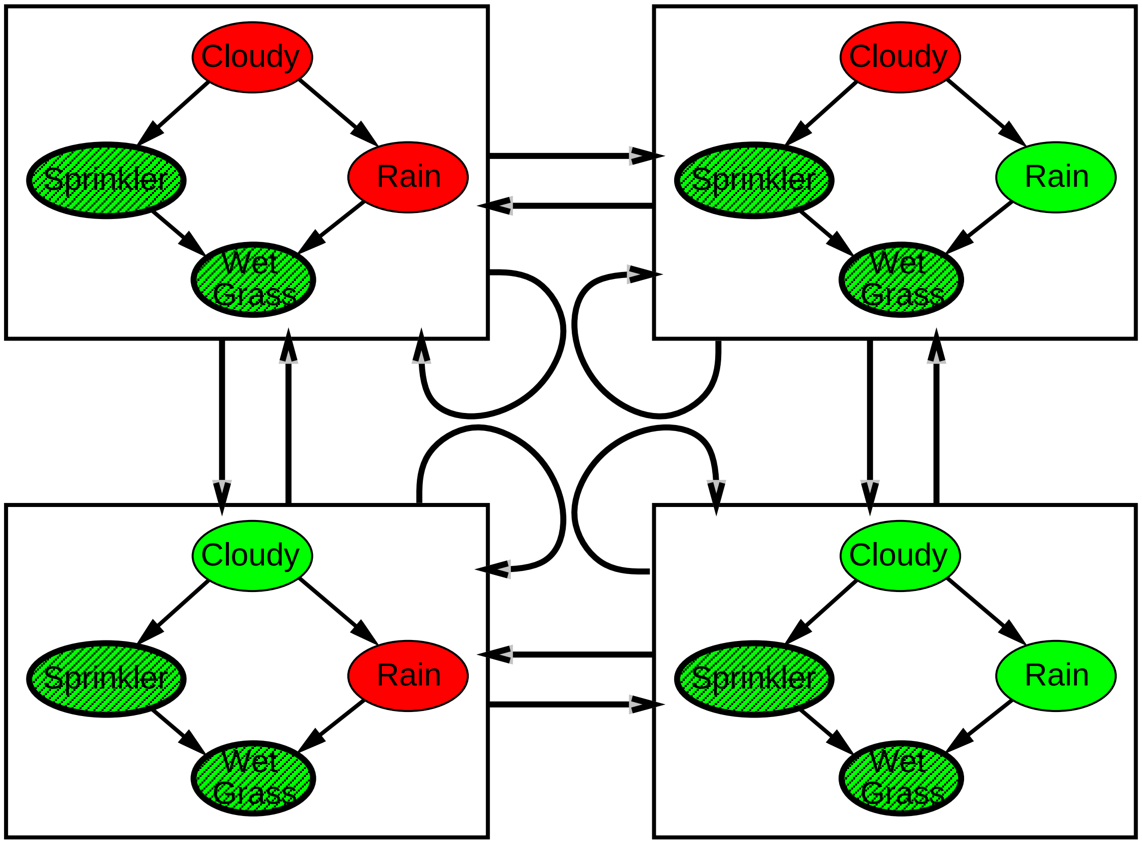
"State" of network = current assignment to all variables.

Generate sequence of states by sampling one variable at a time, and choosing new value for variable given its Markov blanket.

Variables can be sampled sequentially or at random, keeping evidence fixed.

The Markov chain

With $Sprinkler = true, WetGrass = true$, there are four states:



Wander about for a while, average what you see

MCMC example contd.

Estimate $\mathbf{P}(Rain|Sprinkler = true, WetGrass = true)$

Sample $Cloudy$ or $Rain$ given its Markov blanket, repeat.
Count number of times $Rain$ is true and false in the samples.

E.g., visit 100 states
  31 have $Rain = true$, 69 have $Rain = false$

$\hat{\mathbf{P}}(Rain|Sprinkler = true, WetGrass = true)$
  $= \text{Normalize}(\langle 31, 69 \rangle) = \langle 0.31, 0.69 \rangle$

Theorem: chain approaches stationary distribution: over a long run, fraction of time spent in each state is exactly posterior probability

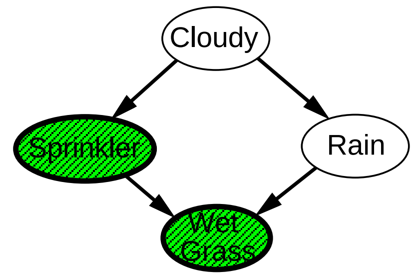
MCMC Algorithm

function MCMC-Ask(X, $\mathbf{e}$, bn, N) **returns** an estimate of $P(X|\mathbf{e})$
  local variables: $\mathbf{N}[X]$, a vector of counts over X, initially zero
    $\mathbf{Z}$, the nonevidence variables in bn
    $\mathbf{x}$, the current state of the network, initially copied from $\mathbf{e}$

  initialize $\mathbf{x}$ with random values for the variables in $\mathbf{Y}$
  for $j = 1$ to N **do**
    $\mathbf{N}[x] \leftarrow \mathbf{N}[x] + 1$ where x is the value of X in $\mathbf{x}$
    for each Z_i in $\mathbf{Z}$ **do**
      sample the value of Z_i in $\mathbf{x}$ from $\mathbf{P}(Z_i|MB(Z_i))$
        given the values of $MB(Z_i)$ in $\mathbf{x}$
  return Normalize($\mathbf{N}[X]$)

Markov blanket sampling

Markov blanket of $Cloudy$ is
$Sprinkler$ and $Rain$
Markov blanket of $Rain$ is
$Cloudy$, $Sprinkler$, and $WetGrass$



Probability given the Markov blanket is calculated as follows:

$$P(x_i'|MB(X_i)) = P(x_i'|Parents(X_i))\prod_{Z_j \in Children(X_i)} P(z_j|Parents(Z_j))$$

Main computational problems:
1) Difficult to tell if convergence has been achieved
2) Can be wasteful if Markov blanket is large:
$P(X_i|MB(X_i))$ won't change much

Sampling with Markov blanket called Gibbs sampler.

Other sampling schemes are used in AI (e.g., simulated annealing, Metropolis).

Summary

Exact inference by variable elimination:
- polytime on polytrees
- NP-hard on general graphs
- very sensitive to topology

Approximate inference by LW, MCMC:
- LW does poorly when there is lots of (downstream) evidence
- LW, MCMC generally insensitive to topology
- Convergence can be very slow with probabilities close to 1 or 0
- Can handle arbitrary combinations of discrete and continuous variables