Linear Algebra Representation for "Quantum Telepathy Saves the World"

by Michael Main

This supplement to "Quantum Telepathy Saves the World"¹ recasts the representation of quantum states and programs in terms of linear algebra. It is intended for students who have studied linear algebra and tensor products.

In the quantum telepathy paper, Alice, Bob and Charlie create a quantum computer with three qubits, each of which can have the value of κ or γ . The start state of the machine is:

 $\frac{1}{\sqrt{2}}|\kappa\kappa\kappa\rangle + \frac{1}{\sqrt{2}}|\gamma\gamma\gamma\rangle$

In this way of writing the states, each term has an amplitude (the number $\frac{1}{\sqrt{2}}$ in this example) and a triplet of qubit values $|\kappa\kappa\kappa\rangle$). The amplitudes can be any complex number, and there are eight possible base states (2³ since each of the three components of the base state has two possible values, κ or γ).

Using the start state written above, the probabilities work out to:

50% chance of $|\kappa\kappa\kappa\rangle$ 0% chance of $|\kappa\kappa\gamma\rangle$ 0% chance of $|\kappa\gamma\kappa\rangle$ 0% chance of $|\kappa\gamma\gamma\rangle$ 0% chance of $|\gamma\kappa\kappa\rangle$ 0% chance of $|\gamma\kappa\gamma\rangle$ 0% chance of $|\gamma\gamma\kappa\rangle$ 50% chance of $|\gamma\gamma\gamma\rangle$

These are the probabilities of finding particular states when the qubits are examined. Prior to the examination, the qubits are not in any particular state. In quantum mechanics, it is the act of examining the qubits that causes them to enter a particular state.

In the linear algebra formation, a quantum state of a three-qubit quantum computer is represented as a column vector of eight complex coefficients. Each coefficient is the amplitude of one of the eight triplets, as illustrated in this example:

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[1/2]	amplitude for $ \kappa\kappa\kappa\rangle$
0	amplitude for $ \kappa\kappa\gamma\rangle$
0	amplitude for $ \kappa\gamma\kappa\rangle$
1/2	amplitude for $ \kappa\gamma\gamma\rangle$
1/2	amplitude for $ \gamma \kappa \kappa \rangle$
0	amplitude for $ \gamma\kappa\gamma\rangle$
0	amplitude for $ \gamma\gamma\kappa\rangle$
-1/2	amplitude for $ \gamma\gamma\gamma\rangle$

The advantage of this representation is that each quantum program can be represented as an 8×8 linear transformation.² For example, Alice's smooth stone program is the matrix:

$$Alice_{SMOOTH} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

If Alice happens to get a smooth stone, then she would apply this matrix to the current state via a matrix multiplication. For example, consider the start state:

² In order to be physically realizable, quantum physicists require the matrix M to be a unitary transformation, which means that the matrix product of M with its own adjoint is the identity matrix.



If the quantum computer is in this start state, and Alice runs her smooth program, the resulting state after the program runs is the column vector on the far right in this matrix equation:

 $Alice_{BLUE}(S) =$

$$\begin{bmatrix} y_{\sqrt{2}} & 0 & 0 & 0 & y_{\sqrt{2}} & 0 & 0 & 0 \\ 0 & y_{\sqrt{2}} & 0 & 0 & 0 & y_{\sqrt{2}} & 0 & 0 \\ 0 & 0 & y_{\sqrt{2}} & 0 & 0 & 0 & y_{\sqrt{2}} & 0 \\ 0 & 0 & 0 & y_{\sqrt{2}} & 0 & 0 & 0 & y_{\sqrt{2}} \\ y_{\sqrt{2}} & 0 & 0 & 0 & -y_{\sqrt{2}} & 0 & 0 \\ 0 & y_{\sqrt{2}} & 0 & 0 & 0 & -y_{\sqrt{2}} & 0 & 0 \\ 0 & 0 & y_{\sqrt{2}} & 0 & 0 & 0 & -y_{\sqrt{2}} & 0 & 0 \\ 0 & 0 & y_{\sqrt{2}} & 0 & 0 & 0 & -y_{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & y_{\sqrt{2}} & 0 & 0 & 0 & -y_{\sqrt{2}} & 0 \\ 0 & 0 & 0 & y_{\sqrt{2}} & 0 & 0 & 0 & -y_{\sqrt{2}} \end{bmatrix} \begin{bmatrix} y_{\sqrt{2}} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ y_{\sqrt{2}} \end{bmatrix}$$

Tensor products of matrixes provide us with a final piece of convenience. We start by defining just two 2×2 matrixes:

$$SMOOTH = \begin{bmatrix} y_{\sqrt{2}} & y_{\sqrt{2}} \\ y_{\sqrt{2}} & -y_{\sqrt{2}} \end{bmatrix} \qquad \qquad JAGGED = \begin{bmatrix} y_{\sqrt{2}} & y_{\sqrt{-2}} \\ y_{\sqrt{2}} & -y_{\sqrt{-2}} \end{bmatrix}$$

Now, suppose that Alice and Bob get red stones, while Charlie's is blue. Using two tensor products, we can combine two copies of the JAGGED matrix (for Alice and Bob) with one copy of the SMOOTH matrix (for Charlie):

$\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{-2}}$	$\frac{1}{2\sqrt{-2}}$	$\frac{1}{2\sqrt{-2}}$	$\frac{1}{2\sqrt{-2}}$	$-\frac{1}{2\sqrt{2}}$	$-\frac{1}{2\sqrt{2}}$
$\frac{1}{2\sqrt{2}}$	$-\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{-2}}$	$-\frac{1}{2\sqrt{-2}}$	$\frac{1}{2\sqrt{-2}}$	$-\frac{1}{2\sqrt{-2}}$	$-\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$
$\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$	$-\frac{1}{2\sqrt{-2}}$	$-\frac{1}{2\sqrt{-2}}$	$\frac{1}{2\sqrt{-2}}$	$\frac{1}{2\sqrt{-2}}$	$\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$
$\frac{1}{2\sqrt{2}}$	$-\frac{1}{2\sqrt{2}}$	$-\frac{1}{2\sqrt{-2}}$	$\frac{1}{2\sqrt{-2}}$	$\frac{1}{2\sqrt{-2}}$	$-\frac{1}{2\sqrt{-2}}$	$\frac{1}{2\sqrt{2}}$	$-\frac{1}{2\sqrt{2}}$
$\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{-2}}$	$\frac{1}{2\sqrt{-2}}$	$-\frac{1}{2\sqrt{-2}}$	$-\frac{1}{2\sqrt{-2}}$	$\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$
$\frac{1}{2\sqrt{2}}$	$-\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{-2}}$	$-\frac{1}{2\sqrt{-2}}$	$-\frac{1}{2\sqrt{-2}}$	$\frac{1}{2\sqrt{-2}}$	$\frac{1}{2\sqrt{2}}$	$-\frac{1}{2\sqrt{2}}$
$\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$	$-\frac{1}{2\sqrt{-2}}$	$-\frac{1}{2\sqrt{-2}}$	$-\frac{1}{2\sqrt{-2}}$	$-\frac{1}{2\sqrt{-2}}$	$-\frac{1}{2\sqrt{2}}$	$-\frac{1}{2\sqrt{2}}$
$\frac{1}{2\sqrt{2}}$	$-\frac{1}{2\sqrt{2}}$	$-\frac{1}{2\sqrt{-2}}$	$\frac{1}{2\sqrt{-2}}$	$-\frac{1}{2\sqrt{-2}}$	$\frac{1}{2\sqrt{-2}}$	$-\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$

 $JAGGED \otimes JAGGED \otimes SMOOTH =$

If we want to figure out the possible outcomes with this combination of stones, we multiple the JAGGED \otimes JAGGED \otimes SMOOTH matrix times the start state and the result is:

With this final state, there is a 25% chance of finding each of four triplets:

0% chance of $|\kappa\kappa\kappa\rangle$ 25% chance of $|\kappa\kappa\gamma\rangle$ 25% chance of $|\kappa\gamma\kappa\rangle$ 0% chance of $|\kappa\gamma\gamma\rangle$ 25% chance of $|\gamma\kappa\kappa\rangle$ 0% chance of $|\gamma\kappa\kappa\rangle$ 0% chance of $|\gamma\gamma\kappa\rangle$ 25% chance of $|\gamma\gamma\gamma\rangle$

Each of the failing triplets (those with an odd number of kappas) has zero probability because of destructive interference, so Alice, Bob and Charlie always pass the test in this particular 1-smooth case. You can work out the matrixes for yourself to see that each of the other possible cases also guarantees a win.