Recurrence plots and unstable periodic orbits

Elizabeth Bradley and Ricardo Mantilla
Department of Computer Science, University of Colorado, Boulder, Colorado 80309-0430

A recurrence plot is a two-dimensional visualization technique for sequential data. These plots are useful in that they bring out correlations at all scales in a manner that is obvious to the human eye, but their rich geometric structure can make them hard to interpret. In this paper, we suggest that the unstable periodic orbits embedded in a chaotic attractor are a useful basis set for the geometry of a recurrence plot of those data. This provides not only a simple way to locate unstable periodic orbits in chaotic time-series data, but also a potentially effective way to use a recurrence plot to identify a dynamical system. © *2002 American Institute of Physics.* [DOI: 10.1063/1.1488255]

Chaotic attractors contain an infinity of unstable periodic orbits. These orbits are interesting not only because they represent an element of order within chaos, but also because they make up the skeleton of the attractor, and in an easily formalized way. An orbit on a chaotic attractor, in particular, is the closure of the set of unstable periodic orbits, and that set is a dynamical invariant. Their instability, however, makes them hard to find, and algorithms that do so are computationally complex. This paper proposes that the recurrence plot, a two-dimensional visualization technique for sequential data, is a useful and relatively inexpensive way to get around this problem. Moreover, thinking about unstable periodic orbits is a good way to understand the rich geometric structure that appears on recurrence plots of chaotic systems. These observations suggest not only a simple way to locate unstable periodic orbits in chaotic time-series data, but also a potentially effective way to use a recurrence plot to identify a dynamical system.

I. INTRODUCTION

A recurrence plot (RP) of a N-point sequence $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_N$ is a visualization of the recurrence matrix of that sequence: the pixels located at (i,j) and (j,i) on the recurrence plot are black if the distance between the ith and jth points in the time series falls within some *threshold corridor*:

$$\delta_l < ||\vec{x}_i - \vec{x}_j|| < \delta_h$$

for some appropriate choice of norm and white otherwise. See Fig. 1 for an example. This technique, first introduced by Eckmann, Kamphorst, and Ruelle[12] and developed further by Zbilut, Webber, and others,[15,22,29] can be applied to time-series data in order to bring out temporal correlations in a manner that is instantly apparent to the eye. Unlike an FFT, for instance, a recurrence plot lets the analyst see not only what frequencies are present, but exactly where the corresponding signals occur. This technique has another important practical advantage in that it can be used to visualize nonstationary data, making it a useful analysis tool for physiologi-

cal data[6,8,19,21,28,29] and driven systems,[9] among other things, and it is quite robust in the face of noise.[30] Better yet, an RP of time-series data from a dynamical system (i.e., a map or a flow that has been temporally discretized by sampling or numerical integration) preserves the invariants of the dynamics,[24] and its structure is at least to some extent independent of embedding.[20,30] Note that the ordering of the data points need not be temporal; RPs are also an effective means for analyzing other sequences, such as the amino acids in a protein.[16] Last, free software for computing and analyzing RPs is available on the World Wide Web.[1]

RPs of chaotic time-series data, as is visible from Fig. 1(b), have a rich and highly characteristic structure. The first effort in the literature to define a quantitative metric for this structure was the *recurrence quantification analysis* (RQA) introduced in Ref. 32, which defines several statistical measures on the black points in a recurrence plot. These quantities measure many of the same dynamical properties as do the more-traditional measures like the Lyapunov exponent and the correlation dimension, and they provide a useful way to quantify RP structure, but their lumped statistical nature means that RQA cannot elucidate the spatiotemporal details of the dynamics. RQA results on structurally dissimilar RPs, for instance, can be virtually identical; see Fig. 4 of Ref. 20 for an illustration. Techniques for analyzing RP structure in more detail would be extremely useful: not only to understand the relationship between the intricacies of RP patterns and the dynamics of a system, but also to make it possible to turn that information to advantage (e.g., to compare two dynamical systems in a simple and yet meaningful way, using only their recurrence plots). The authors of the original RQA papers are taking a linear algebra approach to structural RP analysis, combining principal component analysis and RQA to pick out "important directions" and statistics.[31] The research described in this paper focuses on chaotic time-series data and takes a more geometric approach to RP structure classification. In particular, we claim that the set of unstable n-periodic orbits that lie within a chaotic attractor are a useful geometric basis for RPs of any orbit on that attractor.
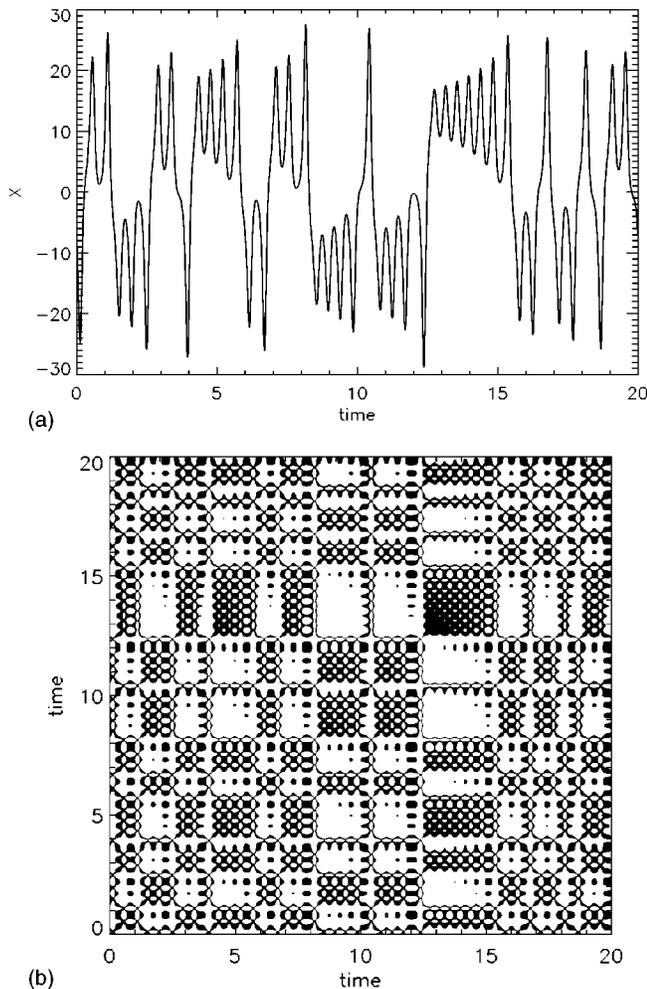
FIG. 1. A signal and its recurrence plot: (a) a time-series plot of the x variable from an 2000-point fourth-order Runge–Kutta integration of the Lorenz equations with $(a,r,b)=(16,45,4)$ and timestep $\Delta t = 0.01$ (b) a recurrence plot of that series with a threshold corridor of $[0,2]$, using the Euclidean 2-norm. Some RP formulations use a color-coding scheme to represent a range of distances according to hue.

II. UNSTABLE PERIODIC ORBITS

Unstable periodic orbits or UPOs make up the skeleton of a chaotic attractor. More formally, there are an infinite number of UPOs, of all periods, densely embedded in the attractor; indeed, an orbit on that attractor is the closure of the set of UPOs and the trajectory can be thought of as "hopping" from one UPO to the next.[3–5,11,13] Figure 2 shows several examples. The set of UPOs in an attractor is a dynamical invariant; their number, distribution, and properties "unfold" the structure of chaotic orbits, and they can be used to calculate other invariants, such as fractal dimension and topological entropy.[5] Algorithms for finding UPOs in time-series data are straightforward: one watches for close returns on a plane of section, then bins and averages several occurrences in order to reduce noise.[18] This procedure is quite time consuming, however, for two reasons: it not only involves an ensemble of nearest-neighbor searches, but also relies on the ergodicity of the orbit in order to visit each

UPO. One can accelerate matters somewhat by using estimates of the local dynamics,[26,27] but the computational complexity is largely inescapable.

III. RECURRENCE PLOTS OF UPOS

The notion of a chaotic attractor as the closure of the set of the UPOs embedded within it—a decomposition that has been used to understand systems ranging from semiconductors[10] to neurons[25]—is patently obvious if one uses recurrence plots to examine a trajectory on that attractor. Figure 3, for example, shows RPs of the x component of the trajectories in Fig. 2. Note how the repeated patterns in parts (b), (c), and (d) of Fig. 3—that is, the RPs of the UPOs—are building blocks in the RP of the overall attractor that is shown in Fig. 3(a). A two-by-two copy of the crosshatch pattern in part (b) appears about three-fifths of the way up the diagonal of part (a), for instance, and the two-cycle pattern in part (c)—which resembles a slice of bread with a cross superimposed upon it—appears at least four times along the diagonal of part (a), as well as in several other places. These blocks simply reflect time intervals when the trajectory is travelling on or near the corresponding UPO. Note that the time scales of the RPs have to be identical for this kind of overlay analysis to make sense, so the trajectories used to construct parts (b), (c), and (d) of Fig. 3 involve repeated transits around the corresponding UPOs. [The trajectory in part (a) is 1000 integrator steps long, but the period of the two-cycle in part (c) is roughly 189 timesteps. In order to generate RPs with identical axes, we followed that UPO for 5.29 orbits to obtain a 1000-point time series.] Comparison of parts (c) and (d) shows yet another layer of compositionality: the five-cycle appears to contain an instance of the two-cycle. We are currently investigating the mathematics behind this effect.

These conclusions are largely independent of the parameters used to construct the recurrence plots. The choice of the x component, for instance, is made without loss of generality; all of the discussion in the previous paragraph holds if one uses the y or z components instead, or if one uses full state-space (xyz) trajectories. See our website for associated images.[2] The choice of norm and threshold corridor, while unimportant from a theoretical standpoint, do matter for practical purposes. Different threshold corridors make the features on an RP thicker or thinner; this does not destroy one's ability to compare these features, but comparing two finely filigreed structures is easier than matching the edges of two black blobs. Norms can have similar effects; using the maximum norm on an attractor that has a high aspect ratio, for instance, will effectively obscure dynamics along the attractor's thin direction. Again, see our webpage for graphical images that demonstrate these effects.

This obvious geometric decomposition of recurrence plot block structure suggests several interesting and potentially useful applications. Not only do RPs of UPOs play the role of geometric basis elements for the structure of an RP of a chaotic attractor, but they are also a useful way to *identify* those UPOs—a task that is algorithmically complex and computationally expensive. To find a UPO, one would sim-
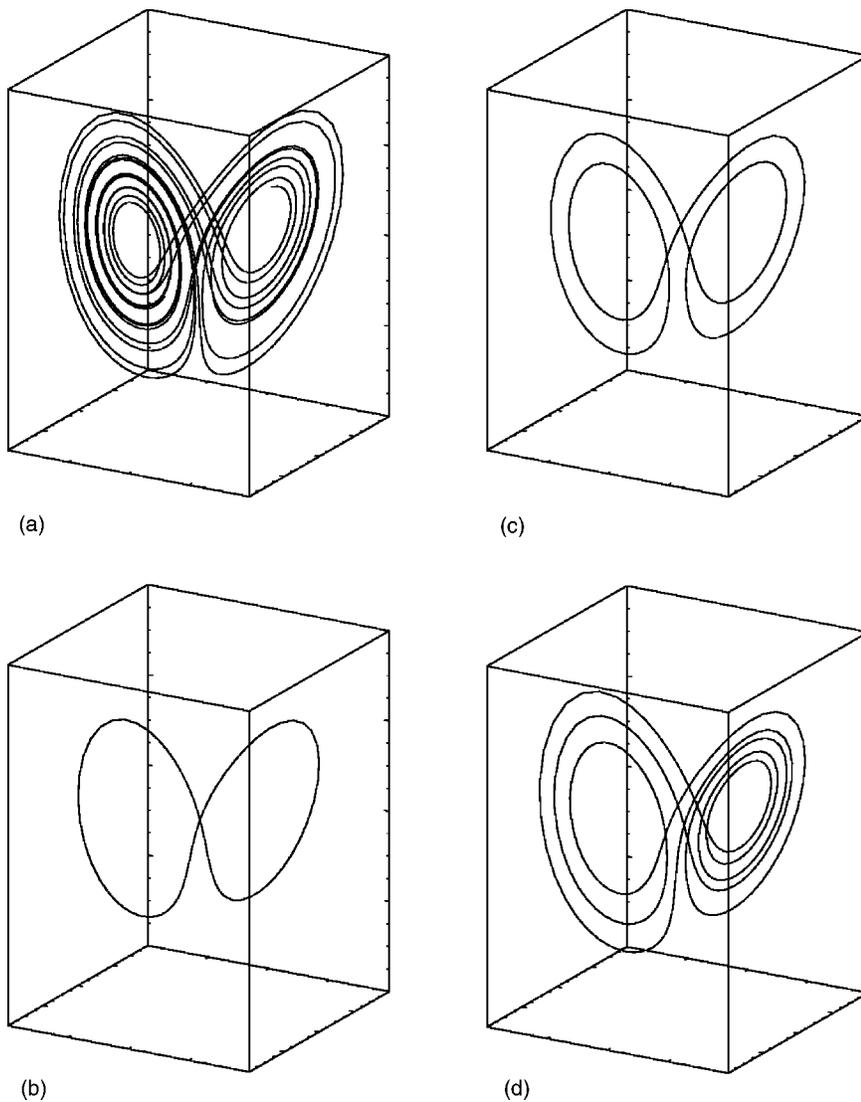
FIG. 2. Unstable periodic orbits embedded within the Lorenz attractor: (a) the full attractor, (b) an unstable one-cycle starting from $(x,y,z) = (10, -2.341\,99, 53.8658)$, (c) an unstable two-cycle starting from $(x,y,z) = (10, -4.080\,38, 55.2704)$, (d) an unstable five-cycle starting from $(x,y,z) = (10, -4.323\,95, 55.4605)$. Same system and integration parameters as Fig. 1. In all images, axis ranges are $x=[-40, 40]$, $y=[-40, 40]$, $z=[0.80]$, and the plane of section that defines the orbit period is $x=10$.

ply construct the RP of a trajectory on a chaotic attractor, look for repeated structures, and use that information to index into the trajectory and find the associated state variable values. This does not sidestep all of the complexity, of course; the time required to construct an RP grows as the square of the number N of points if it is coded naively—or $O(N \log N)$ if coded intelligently, using $k-d$ trees or some other appropriate data structure.[14] The basic algorithms[18,26,27] for finding UPOs in experimental data, however, are at least $O(N^3)$, so using RPs is much faster. The geometric decomposition suggested by Fig. 3 also provides a useful way to do a quick, qualitative comparison of two chaotic systems. The set of UPOs in an attractor uniquely identifies it, and in a well-defined way. Thus, if RPs of two trajectories have different building blocks in their structure, the trajectories are probably not from the same system; conversely, identical block RP structure suggests identical dynamics.

In practice, of course, there are several caveats. No two trajectories on a chaotic attractor will visit exactly the same UPOs, and no finite-length trajectory will visit *all* UPOs. A UPO's stability properties dictate how trajectories travel upon and around it; consider a ball bearing rolling around the apex of a bagel versus one rolling around the rim of a fine porcelain teacup. The one-cycle in Figs. 2(b) and 3(b), for instance, appears very frequently in any trajectory on this attractor, while the five-cycle in Figs. 2(d) and 3(d) is comparatively rare. One can quantify these effects by integrating the variational equation along the orbit and observing the behavior of the transverse component, as is very nicely depicted in the work of Helwig Löffelman and other members of the Institut für Computergraphik in Vienna.[23] If the Lyapunov exponents and stable/unstable manifold geometry of the dynamical system are such that a small perturbation off the UPO grows very quickly—the teacup situation—then trajectories are not only more likely to diverge from that orbit but also less likely to visit it in the first place, and so that UPO will leave its signature neither in the trajectory nor on the RP. High-period UPOs pose particular challenges in this regard, as their length provides more opportunity for the unavoidable (i.e., floating-point arithmetic on a computer or noise in an experiment) transverse perturbations to grow. This means that the claims made in this paper, while true in general, cannot be used in practice to find *every* UPO. Using a longer trajectory improves matters, of course, and there is evidence that a system's short, low-period UPOs provide "good" descriptions of its dynamics,[5] so this limitation is by
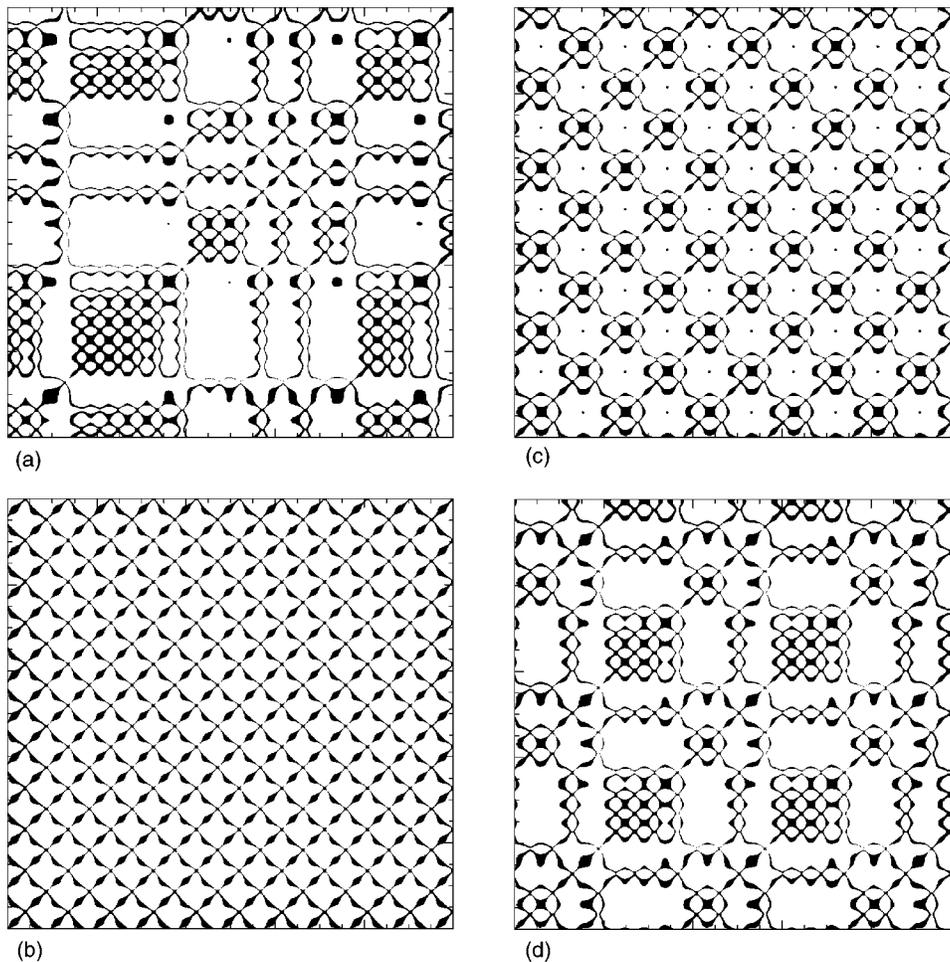
(a)

(b)

(c)

(d)

FIG. 3. Recurrence plots of the *x* component of the trajectories in Fig. 2 constructed with a threshold corridor of [0,2] and the Euclidean 2-norm. All time series are 1000 points long, and hence all RPs are 1000 points square.

no means fatal. Incidentally, the algorithms in Refs. 18, 26, and 27 suffer from the same problems, since they too rely on the system dynamics to cause the trajectory to visit each orbit. In order to find high-period and/or highly unstable UPOs, one must fall back on analytical methods (e.g., Ref. 17 for flows or Ref. 7 for maps), but such methods cannot be used unless one has the system equations and thus are all but useless in experimental situations.

IV. SUMMARY

Unstable periodic orbits are a useful geometric basis for the complex structure of a recurrence plot of a trajectory on a chaotic attractor. Their locations in state space and in time, as well as the compositional nature with which they make up the structure of the attractor, are immediately apparent to the eye, and in a manner that aids one's understanding of a chaotic attractor as the closure of the UPOs that are embedded within it. These ideas suggest several practical applications. The RP representation is useful not only for identifying and locating UPOs—a task that is computationally demanding—but also for comparing one dynamical system to another: the practical task termed "modeling" by engineers, "scientific discovery" by artificial intelligence practitioners, and "system identification" by control theorists. Because RPs preserve a system's invariant dynamical structure, and because the UPOs that appear so prominently on a RP are dynamical invariants, one can draw sensible qualitative conclusions about the similarity or difference between two systems from comparisons of the obvious block structures in RPs of their trajectories.

[1] http://home.netcom.com/~eugenek/download.html
[2] http://www.cs.colorado.edu/~lizb/rps.html
[3] R. Artuso, E. Aurell, and P. Cvitanovic, "Recycling of strange sets: I. Cycle expansions," Nonlinearity **3**, 325–359 (1990).
[4] R. Artuso, E. Aurell, and P. Cvitanovic, "Recycling of strange sets: II. Applications," Nonlinearity **3**, 361–386 (1990).
[5] D. Auerbach, P. Cvitanovic, J.-P. Eckmann, G. Gunaratne, and I. Procaccia, "Exploring chaotic motion through periodic orbits," Phys. Rev. Lett. **58**, 2387–2389 (1987).
[6] R. Balocchi, A. DiGarbo, C. Michelassi, S. Chillemi, M. Varanini, M. Barbi, J. Legramente, G. Raimondi, and J. Zbilut, "Heart rate and blood pressure response to short-term head-down bed rest: A nonlinear approach," Methods Inf. Med. **39**, 157–159 (2000).
[7] O. Biham and W. Wenzel, "Characterization of unstable periodic orbits in chaotic attractors and repellers," Phys. Rev. Lett. **63**, 819–822 (1989).
[8] J. C. Webber, "Rhythmogenesis of deterministic breathing patterns," in *Rhythms in Physiological Systems*, edited by H. Haken and H. Koepchen (Springer-Verlag, Berlin, 1991).

[9] M. Casdagli, "Recurrence plots revisited," Physica D **108**, 12–44 (1997).

[10] F. Christiansen and P. Cvitanovic, "Periodic orbit quantization of the anisotropic Kepler problem," Chaos **2**, 61–69 (1992).

[11] P. Cvitanovic, "Invariant measurement of strange sets in terms of circles," Phys. Rev. Lett. **61**, 2729–2732 (1988).

[12] J.-P. Eckmann, S. Kamphorst, and D. Ruelle, "Recurrence plots of dynamical systems," Europhys. Lett. **4**, 973–977 (1987).

[13] J.-P. Eckmann and D. Ruelle, "Ergodic theory of chaos and strange attractors," Rev. Mod. Phys. **57**, 617–656 (1985).

[14] J. H. Friedman, J. L. Bentley, and R. A. Finkel, "An algorithm for finding best matches in logarithmic expected time," ACM Trans. Math. Softw. **3**, 209–226 (1977).

[15] J. Gao and H. Cai, "On the structures and quantification of recurrence plots," Phys. Lett. A **270**, 75–87 (2000).

[16] A. Giuliani, R. Benigni, P. Sirabella, J. Zbilut, and A. Colosimo, "Exploiting the information content of protein sequences using time-series methods: A case study in rubredoxins," Biophys. J. **78**, 136–149 (1998).

[17] J. Guckenheimer and B. Meloon, "Computing periodic orbits and their bifurcations with automatic differentiation," SIAM J. Sci. Comput. (USA) **22**, 951–985 (2000).

[18] G. H. Gunaratne, P. S. Linsay, and M. J. Vinson, "Chaos beyond onset: A comparison of theory and experiment," Phys. Rev. Lett. **63**, 1 (1989).

[19] S. Ikagawa, M. Shinohara, T. Fukunaga, J. Zbilut, and C. Webber, "Nonlinear time-course of lumbar muscle fatigue using recurrence quantifications," Biol. Cybern. **82**, 373–382 (2000).

[20] J. Iwanski and E. Bradley, "Recurrence plots of experimental data: To embed or not to embed?" Chaos **8**, 861–871 (1998).

[21] P. Kaluzny and R. Tarnecki, "Recurrence plots of neuronal spike trains," Biol. Cybern. **68**, 527–534 (1993).

[22] M. Koebbe and G. Mayer-Kress, "Use of recurrence plots in the analysis of time-series data," in *Nonlinear Modeling and Forecasting*, edited by M. Casdagli and S. Eubank (Addison Wesley, New York, 1992).

[23] H. Löffelman, Ph.D. thesis, Vienna Technical University, November 1998. See also http://www.cg.tuwien.ac.at/~helwig/diss/node42.htm

[24] G. McGuire, N. Azar, and M. Shelhammer, "Recurrence matrices and the preservation of dynamical properties," Phys. Lett. A **237**, 43–47 (1997).

[25] P. So, J. Francis, T. Netoff, B. Gluckman, and S. Schiff, "Periodic orbits: A new language for neuronal dynamics," Biophys. J. **74**, 2776–2785 (1998).

[26] P. So, E. Ott, T. Sauer, B. Gluckman, C. Grebogi, and S. Schiff, "Extracting unstable periodic orbits from chaotic time-series data," Phys. Rev. E **55**, 5398–5417 (1997).

[27] P. So, E. Ott, S. Schiff, D. Kaplan, T. Sauer, and C. Grebogi, "Detecting unstable periodic orbits in chaotic experimental data," Phys. Rev. Lett. **76**, 4705–4708 (1996).

[28] C. Webber, M. Schmidt, and J. Walsh, "Influence of isometric loading on biceps EMG dynamics as assessed by linear and nonlinear tools," J. Appl. Physiol. **78**, 814–822 (1995).

[29] C. Webber and J. Zbilut, "Dynamical assessment of physiological systems and states using recurrence plot strategies," J. Appl. Physiol. **76**, 965–973 (1994).

[30] J. Zbilut, A. Giuliani, and C. Webber, "Detecting deterministic signals in exceptionally noisy environments using cross-recurrence quantification," Phys. Lett. A **246**, 122–128 (1998).

[31] J. Zbilut, A. Giuliani, and C. Webber, "Recurrence quantification analysis and principal components in the detection of short complex signals," Phys. Lett. A **237**, 131–135 (1998).

[32] J. Zbilut and C. Webber, "Embeddings and delays as derived from recurrence quantification analysis," Phys. Lett. A **171**, 199–203 (1992).