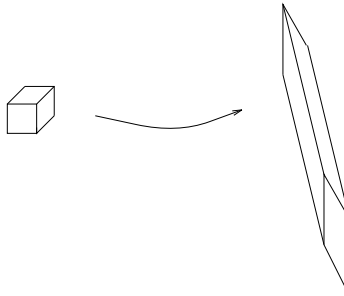# The Variational Equation
Notes for CSCI4446/5446

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The variational equation is the mathematics that goes with pictures like this:

This equation describes how pertubations (or state-space volumes) evolve along a trajectory. This kind of information is essential to any kind of in-depth understanding of the landscapes of dynamical systems: stability, Liouville's theorem and dissipation, un/stable manifolds, λs, etc.

Given some state-space point $\vec{x}_0$ and some dynamical system $\vec{x}' = \vec{F}(\vec{x})$, it's easy to generate the state-space trajectory $\phi(\vec{x}_0)$ that emanates from that point; just use any old ODE solver like RK4. What we'd like to know, now, is how that would *change* if someone wiggled $\vec{x}$. Consider the Rössler system

$$\vec{F}(\vec{x}) = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} f_x(x,y,z) \\ f_y(x,y,z) \\ f_z(x,y,z) \end{bmatrix} = \begin{bmatrix} -(y+z) \\ x + 0.398y \\ 2 + z(x-4) \end{bmatrix} \tag{1}$$

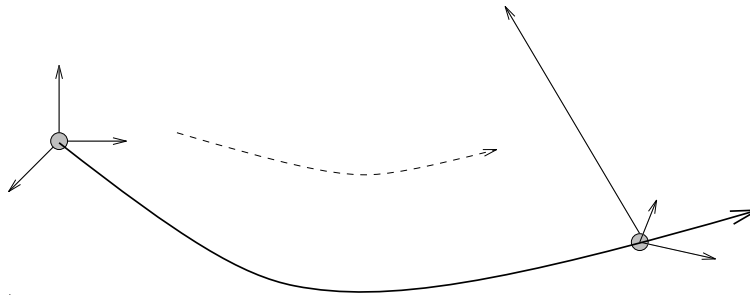with $\vec{x}_0 = [0, 1, 2]^T$. The state-space trajectory $\phi(\vec{x}_0)$ from that point is shown below:

$\vec{x}_0$

$\phi(\vec{x}_0)$

Imagine, now, that we add a small perturbation to $\vec{x}_0$, in the x-direction. That perturbation will evolve as the system moves along the trajectory:
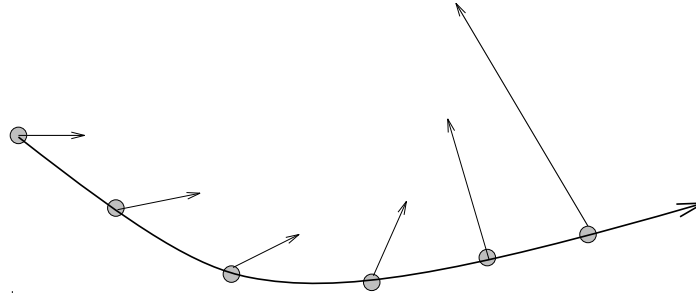
The same is also true of perturbations in the other two directions:

The change in length and orientation of each arrow, as well as the resultant to which all three add up in the end (which is equal to the volume into which the original box has grown), depends on the *differential slope* of the ODE "landscape" around the trajectory:

- If a particular perturbation points out over a sharp dropoff, that arrow will lengthen

- If it points over a gentle *rise*, that arrow will shrink

- If it points out diagonally from a saddle point (not in the direction of either the stable or the unstable direction), it will be pulled sideways: it will grow along the unstable direction and shrink along the stable one

- If it points out over a region of constant slope, it will not change length.

The formal way to describe these differential slopes to which the variations react is with partial derivatives:

$$\frac{\partial f_x}{\partial y}$$

for instance, is how much the x-slope (f_x) changes if you wiggle y. In n dimensions, there are n variables and n slopes, so you need n^2 of these different partials to describe everything. One way to write these down is in an $n \times n$ matrix called the Jacobian:

$$D_x \vec{F} = \begin{bmatrix} \frac{\partial f_x}{\partial x} & \frac{\partial f_x}{\partial y} & \frac{\partial f_x}{\partial z} \\ \frac{\partial f_y}{\partial x} & \frac{\partial f_y}{\partial y} & \frac{\partial f_y}{\partial z} \\ \frac{\partial f_z}{\partial x} & \frac{\partial f_z}{\partial y} & \frac{\partial f_z}{\partial z} \end{bmatrix}$$

For the Rössler system, the Jacobian is

$$\begin{bmatrix} 0 & -1 & -1 \\ 1 & 0.398 & 0 \\ z & 0 & x-4 \end{bmatrix}$$

The last thing we need to set up the variational equation is a way to describe the variations. The notation is a little complicated; δ_{xy} is the component of the x variation that came from the previous y variation, and so on. Again, there are nine of these:

$$[\delta] = \begin{bmatrix} \delta_{xx} & \delta_{yx} & \delta_{zx} \\ \delta_{xy} & \delta_{yy} & \delta_{zy} \\ \delta_{xz} & \delta_{yz} & \delta_{zz} \end{bmatrix}$$

The column sums of this matrix are the lengths of the x, y, and z coordinates (respectively) of the evolved variation. The *rows* are the coordinates of the vectors into which the original x, y, and z components of the variation have evolved.

The variational equation is $[\dot{\delta}] = D_x \vec{F} [\delta]$, or

$$\begin{bmatrix} \dot{\delta}_{xx} & \dot{\delta}_{yx} & \dot{\delta}_{zx} \\ \dot{\delta}_{xy} & \dot{\delta}_{yy} & \dot{\delta}_{zy} \\ \dot{\delta}_{xz} & \dot{\delta}_{yz} & \dot{\delta}_{zz} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_x}{\partial x} & \frac{\partial f_x}{\partial y} & \frac{\partial f_x}{\partial z} \\ \frac{\partial f_y}{\partial x} & \frac{\partial f_y}{\partial y} & \frac{\partial f_y}{\partial z} \\ \frac{\partial f_z}{\partial x} & \frac{\partial f_z}{\partial y} & \frac{\partial f_z}{\partial z} \end{bmatrix} \begin{bmatrix} \delta_{xx} & \delta_{yx} & \delta_{zx} \\ \delta_{xy} & \delta_{yy} & \delta_{zy} \\ \delta_{xz} & \delta_{yz} & \delta_{zz} \end{bmatrix}$$

For the Rössler system, this looks like:

$$\begin{bmatrix} \dot{\delta}_{xx} & \dot{\delta}_{yx} & \dot{\delta}_{zx} \\ \dot{\delta}_{xy} & \dot{\delta}_{yy} & \dot{\delta}_{zy} \\ \dot{\delta}_{xz} & \dot{\delta}_{yz} & \dot{\delta}_{zz} \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0.398 & 0 \\ z & 0 & x-4 \end{bmatrix} \begin{bmatrix} \delta_{xx} & \delta_{yx} & \delta_{zx} \\ \delta_{xy} & \delta_{yy} & \delta_{zy} \\ \delta_{xz} & \delta_{yz} & \delta_{zz} \end{bmatrix}$$

To figure out what happens to the variations, you have to solve the variational equation and the system equation *simultaneously* (because the variations react to the differential slopes at each point along the trajectory). To do this, you work with a new $n + n^2$ length *augmented* state vector:

$$(x \; y \; z \; \delta_{xx} \; \delta_{xy} \; \delta_{xz} \; \delta_{yx} \; \delta_{yy} \; \delta_{yz} \; \delta_{zx} \; \delta_{zy} \; \delta_{zz})$$

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and its derivative:

$$(\dot{x}\ \dot{y}\ \dot{z}\ \dot{\delta}_{xx}\ \dot{\delta}_{xy}\ \dot{\delta}_{xz}\ \dot{\delta}_{yx}\ \dot{\delta}_{yy}\ \dot{\delta}_{yz}\ \dot{\delta}_{zx}\ \dot{\delta}_{zy}\ \dot{\delta}_{zz})$$

...and you stack the system equation $\vec{x}' = \vec{F}\vec{x}$ on top of the variational equation $[\dot{\delta}] = D_x\vec{F}[\delta]$. The equations are:

$$
\begin{aligned}
\dot{x} &= -(y+z) \\
\dot{y} &= x + 0.398y \\
\dot{z} &= 2 + z(x-4) \\
\dot{\delta}_{xx} &= -\delta_{xy} - \delta_{xz} \\
\dot{\delta}_{xy} &= \delta_{xx} + 0.398\delta_{xy} \\
\dot{\delta}_{xz} &= z\delta_{xx} + (x-4)\delta_{xz} \\
\dot{\delta}_{yx} &= -\delta_{yy} - \delta_{yz} \\
\dot{\delta}_{yy} &= \delta_{yx} + 0.398\delta_{yy} \\
\dot{\delta}_{yz} &= z\delta_{yx} + (x-4)\delta_{yz} \\
\dot{\delta}_{zx} &= -\delta_{zy} - \delta_{zz} \\
\dot{\delta}_{zy} &= \delta_{zx} + 0.398\delta_{zy} \\
\dot{\delta}_{zz} &= z\delta_{zx} + (x-4)\delta_{zz}
\end{aligned}
$$

If you wrote your RK4 for arbitrary length vectors and derivatives, you should have no problem here; it's just a 12D system instead of Rössler's three.

For example, if you want to see how unit variations grow near the point $[0, 1, 2]^T$, you'd fire up your ODE solver starting at

$$[0\ 1\ 2\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1]$$

and let it go. Say, for the purposes of this example, that after 100 steps at 0.05 seconds, the new state of this augmented system were:

$$[13.087\ 23.455\ 24.204\ 0.79\ 0.95\ 0.89\ 0.61\ 1.23\ 0.79\ -0.14\ -0.48\ 0.66$$

To figure out the actual variations from this, you need to look at the appropriate rows or columns. The total x variation at $t = 5sec$ is $\delta_{xx} + \delta_{xy} + \delta_{xz} = 0.79 + 0.95 + 0.89$; the original x perturbation (at $t = 0$) has grown into the vector $\delta_{xx}\hat{x} + \delta_{yx}\hat{y} + \delta_{zx}\hat{z} = 0.79\hat{x} + 0.61\hat{y} - 0.14\hat{z}$: