

University of Colorado  
Department of Computer Science  
Chaotic Dynamics – CSCI 4446/5446  
Spring 2022

Problem Set 6

Issued: 15 February 2022  
Due: 22 February 2022

**Reading:** *Kantz & Schreiber, section 3.5; Strogatz, section 8.7 (undergrads: skip the examples in 8.7); Parker&Chua, chapter 2.*

**Online assignment:** Tuesday: unit 4.4 and 5.5 videos. Thursday: unit 10.4 video. Friday: quiz 4.4.

**Bibliography:**

- E. Bradley and D. E. Straub, “Using Chaos to Improve the Capture Range of a Phase-Locked Loop: Experimental Verification,” *IEEE Trans. on Circuits and Systems*, **43**:914-922 (1996). On my webpage under publications.
- E. Bradley and J. Stuart, “Using Chaos to Generate Variations on Movement Sequences,” *Chaos* **8**:800-807 (1998). On my webpage under publications.
- D. S. Dabby, “Musical Variations from a Chaotic Mapping,” *Chaos* **6**:95-107 (1996).
- M. Hénon, “On the numerical computation of Poincaré maps,” *Physica D* **5**:412-414 (1982).

**Problems:**

1. Write a simplistic *temporal* Poincaré section program that takes a fixed trajectory (a list of state-space points, each annotated with the corresponding time), does a simple check on each point to see if the trajectory has pierced the hyperplane  $\Sigma : t = nT$  with  $n = 1, 2, \dots$ , and plots that point if such a crossing has occurred.

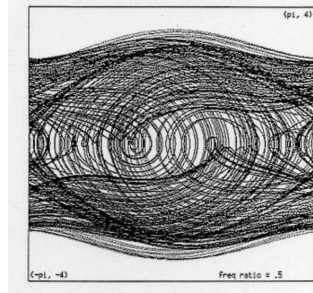
(a) Test this program on the trajectory from problem 2(b) on PS4. Use the same timestep as in PS4 and the surface of section  $\Sigma : t = nT_0$ , where  $T_0$  is the natural period of the linearized version of the system and  $n = 1, 2, \dots$ . Hint: if you have trouble figuring out  $T_0$ , generate a trajectory starting from the initial condition  $[\theta, \omega] = [0.01, 0]$  and see how long it takes to return.

Turn in a plot of the sectioned trajectory on the axes  $\omega$  versus  $\theta$  (i.e., not  $\theta$  modulo  $2\pi$ ). Is this what you expected to see? Why or why not? Hint: remember that the natural frequency of the linearized version of the pendulum is not *exactly* the frequency of the real pendulum. It's only exactly correct for  $\theta \rightarrow 0$ , and close iff  $\theta$  is small.

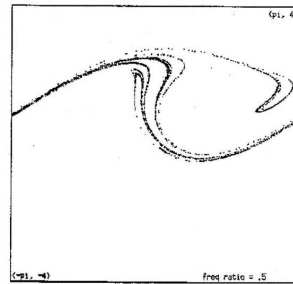
(b) Repeat (a), but with some different  $T$  that is not rationally related to  $T_0$ . Explain the differences between the plots in parts (a) and (b).

(c) Repeat (a) for one of the chaotic pendulum trajectories that you generated in problem 6 on PS4. Use  $\Sigma : t = nT_{drive}$ , where  $T_{drive}$  is the drive period, plot the results using  $\theta$  modulo  $2\pi$  on the horizontal axis. Make sure you use a long enough trajectory to get *at least* 1000 points on the plane of section, and that you have discarded the transient. This should look somewhat like the slide that I showed in class:

### Temporal section of a chaotic trajectory



Trajectory (driven damped pendulum)



sampled at the drive frequency

(d) Repeat (c), raising the step size that you use to generate the test trajectory—keeping the timespan covered by the trajectory constant—and describe the effects on the geometry of the Poincaré section. Why does this happen? (No need to turn in plots on this question.) Note: don't vary the stepsize so far as to induce numerical dynamics or you'll be seeing different effects. That's not what this question is about.

2. Now make your Poincaré section algorithm more intelligent about how it finds the point of intersection with the hyperplane, using a linear interpolation between the two trajectory points that fall on either side of  $\Sigma$  to approximate the actual point of piercing. Note: do *not* use this interpolated point as an initial condition for the next round of integrations! The interpolation error will severely affect the dynamics.

(a) Repeat parts (c) and (d) of problem 1 using the new algorithm. Describe and explain any differences between the two sets of plots.

3. (Required for people enrolled in CSCI 5446; optional for others) Modify your section algorithm from problem 2 to construct a *spatial* section. Use this code to construct sections of the  $r = 50$  Lorenz attractor at (a)  $y = 20$  and (b)  $y = 2x$ . These may look rather odd, particularly the latter.

**Final Note:** if a Poincaré section algorithm can control the numerical integrator (i.e., the trajectory isn't a fixed input), you can do much better, adjusting the stepsize so that the point falls *exactly* on the plane of section, as discussed in Henon's paper and in class.