University of Colorado  
Department of Computer Science  
Chaotic Dynamics – CSCI 4446/5446  
Spring 2021  
Problem Set 5

Due: 2 March 2021

Reading: Kantz & Schreiber, section 3.5; Strogatz, chapter 9 and sections 4.4-4.6, 6.7, 8.7, 10.6, 12.3, and 12.4 (undergrads: skip the examples in 8.7). Parts of sections 4.5, 4.6, 9.1 and 9.2 in Strogatz are mathematically quite dense. Read for content here; look at the equations, but don’t worry about understanding the derivations or the deep mathematical implications. Section 3 of Liz’s TSA Notes may also be useful.

Bibliography:

Problems:

1. Write a simplistic temporal Poincaré section program that takes a fixed trajectory (a list of state-space points, each annotated with the corresponding time), does a simple check on each point to see if the trajectory has pierced the hyperplane \( \Sigma : t = nT \) with \( n = 1, 2, \ldots \), and plots that point if such a crossing has occurred.

(a) Test this program on the trajectory from problem 2(b) on PS4. Use the same timestep as in PS4 and the surface of section \( \Sigma : t = nT_0 \), where \( T_0 \) is the natural period of the linearized version of the system and \( n = 1, 2, \ldots \). (Hint: if you have trouble figuring out \( T_0 \), generate a trajectory starting from the initial condition \([\theta, \omega] = [0.01, 0]\) and see how long it takes to return.)

Turn in a plot of the sectioned trajectory on the axes \( \omega \) versus \( \theta \) (i.e., not \( \theta \) modulo \( 2\pi \)). Is this what you expected to see? Why or why not? (Hint: remember that the natural frequency of the linearized version of the pendulum is not exactly the frequency of the real pendulum. It’s only exactly correct for \( \theta \to 0 \), and close iff \( \theta \) is small.)

(b) Repeat (a), but with some different \( T \) that is not rationally related to \( T_0 \). Explain the differences between the plots in parts (a) and (b).

(c) Repeat (a) for one of the chaotic pendulum trajectories that you generated in problem 6 on PS4. Make sure you use a long enough trajectory to get at least 1000 points on the plane of section. It should look something like this:

Make sure that you have discarded the transient, too, or you’ll get some stray (non-attractor) points on the section.

For your section, use \( \Sigma : t = nT_{drive} \), where \( T_{drive} \) is the drive period, plot the results using \( \theta \) modulo \( 2\pi \) on the horizontal axis. Those results should look something like the section that I showed in class on the slide with the image above.

(d) Repeat (c), raising the step size that you use to generate the test trajectory—keeping the timespan covered by the trajectory constant—and describe the effects on the geometry of the Poincaré section. Why does this happen? (No need to turn in plots on this question.) Note: don’t vary the stepsize so far as to induce numerical dynamics or you’ll be seeing different effects. That’s not what this question is about.
2. Now make your Poincaré section algorithm more intelligent about how it finds the point of intersection with the hyperplane, using a linear interpolation between the two trajectory points that fall on either side of Σ to approximate the actual point of piercing. Note: do not use this interpolated point as an initial condition for the next round of integrations! The interpolation error will severely affect the dynamics.

(a) Repeat parts (c) and (d) of problem 1 using the new algorithm. Describe and explain any differences between the two sets of plots.

3. *(Required for people enrolled in CSCI 5446; optional for others)* Modify your section algorithm from problem 2 to construct a spatial section. Use this code to construct sections of the $r = 50$ Lorenz attractor at (a) $y = 20$ and (b) $y = 2x$. These may look rather odd, particularly the latter.

**Final Note:** if a Poincaré section algorithm can control the numerical integrator (i.e., the trajectory isn’t a fixed input), you can do much better, adjusting the stepsize so that the point falls *exactly* on the plane of section, as discussed in Henon’s paper and in class.