Chapter 10: Vectors and the Geometry of Space

Course Introduction

Calculus III is takes concepts and techniques that we learned in Calculus I-II for functions of 1 variable and generalizes them to 2 and 3 variables.

The world is decidedly **NOT** one-dimensional. Sometimes we get away with $2D$, but usually the really good science and engineering is in $3D$.

$f(x)$

Derivatives: Slopes, linearizations of curves, extrema of single variable functions.

Integrals: Area under a curve, average value of functions, integration by parts.

$f(x, y, z)$

Derivatives: Grad/Div/Curl, linearization of surfaces, extrema of surfaces.

Integrals: Areas of complex 2D regions, volumes of 3D regions, integrals over surfaces, integrals over curves, Stokes/Green's Theorems.

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APPM 2350
Chapter 10
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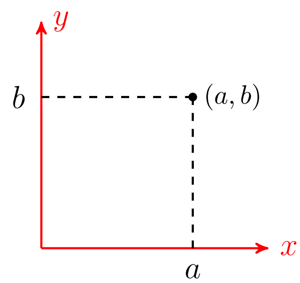
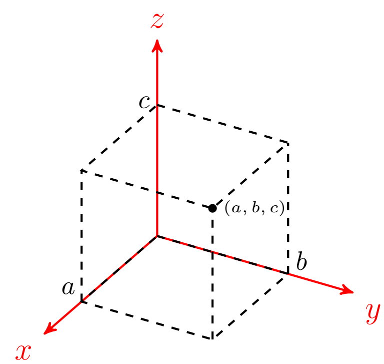
10.1 The Three Dimensional Coordinate System

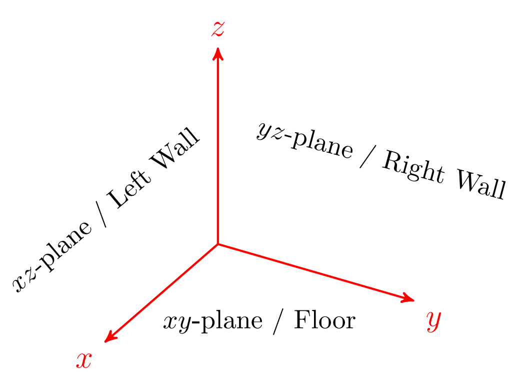
In the standard 2D Cartesian coordinate system we can represent the point $(a, b) \in \mathbb{R}^2$ as



Now, if we want to plot a point in 3D we need to add some height! A point in 3D looks like $(a, b, c) \in \mathbb{R}^3$. The first two coordinate are the same as 2D, indicating displacement in the x and y-directions, respectively. The third coordinate indicates how far off the *ground* the point is.



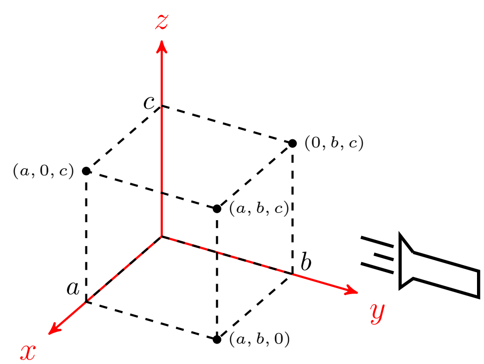
The region in the 3D coordinate system with x, y, $z > 0$ is called the *first octant*. It helps if we picture the first octant as a room:



We can characterize the coordinate planes mathematically in the following way

xy-plane: $\{(x, y, z) \mid z = 0\}$
yz-plane: $\{(x, y, z) \mid x = 0\}$
xz-plane: $\{(x, y, z) \mid y = 0\}$

We will often talk about the *projection* of points onto a coordinate plane. Think about this as shining a flashlight at a point in a direction perpendicular to the plain you're projecting onto. The coordinates of the projection are then the coordinates of the shadow of the point.



Notice that this is the same as setting the variable along which access you're projecting to 0.

OK, so now we know how to plot points in 3D. Let's think about how we could plot equations.

Example 1

In 2D, what curve does the equation $y = 3$ represent?

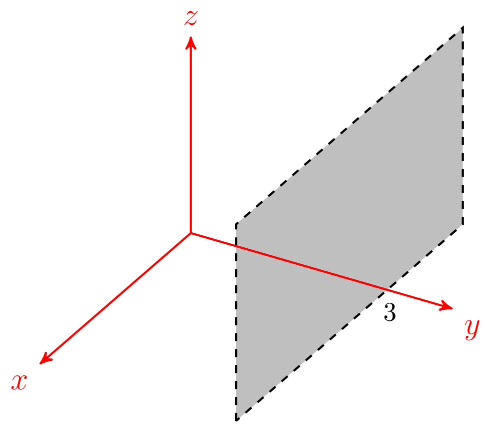
The equation tells us that all points on the curve must have y coordinate value of 3, but it makes no restrictions on the x value, so x can be anything we like. This means that the equation $y = 3$ represents all points (x, y) such that $y = 3$. Or, more mathematically, $\{(x, y) \,|\, y = 3\}$. This is a horizontal line in the 2D coordinate system.

So what happens in 3D?

Example 2

In 3D, what does the equation $y = 3$ represent?

In 3D points are described by triplets of the form (x, y, z). This equation tells us what the y-value is, but puts no restriction on the x or z-values. This means that these can be anything. The set of all points $(x, 3, z)$ form a plane in three dimensional space.



In general, any equation of the form $ax + by + cz = d$ describes a plane in three-space.

Example 3

Draw the plane $x + 2y + 3z = 6$ in the first octant.

We want to plot all points (x, y, z) that satisfy the equation $x + 2y + 3z = 6$. Since we know that the surface is a plane, and intersections of planes are described by straight lines, let's find the points where the plane intersects the three coordinate axes and then connect the dots.

The plane will intersect the z-axis precisely when $x = 0$ and $y = 0$. In order to satisfy the equation of the plane, we must then have

$$0 + 2\,(0) + 3z = 6 \quad \Rightarrow \quad 3z = 6 \quad \Rightarrow \quad z = 2 \quad \Rightarrow \quad (0, 0, 2)$$

Similarly, for the intersection with the x-axis we have $y = 0$ and $z = 0$ which gives

$$x + 2\,(0) + 3\,(0) = 6 \quad \Rightarrow \quad x = 6 \quad \Rightarrow \quad (6, 0, 0)$$

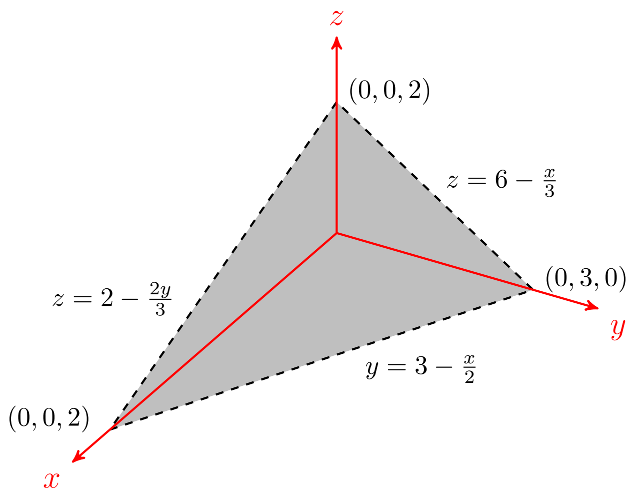
And finally, for the y-axis we have $x = 0$ and $z = 0$ which gives

$$0 + 2y + 3\,(0) = 6 \quad \Rightarrow \quad 2y = 6 \quad \Rightarrow \quad y = 3 \quad \Rightarrow \quad (0, 3, 0)$$

Plotting these three points and connecting the dots gives

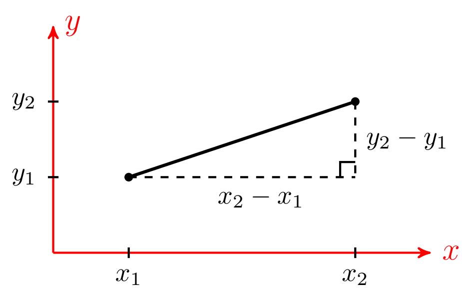
Note that we could also solve the problem by *projecting* the plane onto each of the coordinate planes to obtain lines.

Project onto xy-plane by setting $z = 0$: $x + 2y = 6 \Rightarrow y = 3 - x/2$ (line in the xy-plane)
Project onto yz-plane by setting $x = 0$: $2y + 3z = 6 \Rightarrow z = 2 - 2y/3$ (line in the yz-plane)
Project onto xz-plane by setting $y = 0$: $x + 3z = 6 \Rightarrow z = 6 - x/3$ (line in the xz-plane)



Distance Between Points

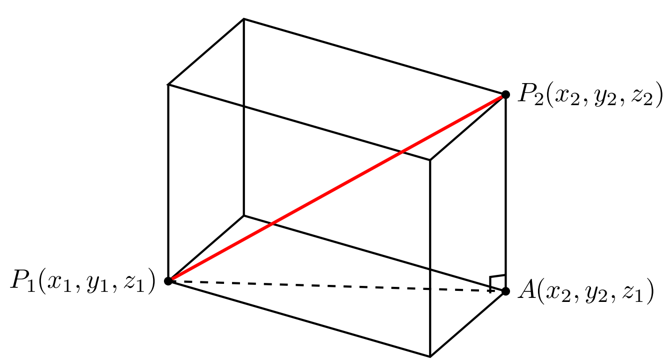
In 2D, the distance between points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, denoted $|P_1P_2|$ is



From the Pythagorean Theorem we have $|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

In 3D we want to find $|P_1P_2|$ for $P_1\,(x_1, y_1, z_1)$ and $P_2\,(x_2, y_2, z_2)$.

Instead of drawing a triangle we can draw a box



We'll use that auxiliary right-triangle to compute $|P_1P_2|$ using Pythagoras. We have

$$|P_1A| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Then, we have

$$|P_1P_2| = \sqrt{|P_1A|^2 + |AP_2|^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} = D$$

Spheres

Example 4

Find the equation for a sphere centered at the point $C(2, 4, 1)$ with radius 3.

The definition of a sphere is a surface made up of all points (x, y, z) that are distance 3 from the center point $C(2, 4, 1)$. We write down a general formula describing all points that are 3 units away from C.

$$D = \sqrt{(x - 2)^2 + (y - 4)^2 + (z - 1)^2} = 3$$

This is kind of a messy equation. Note that it's still true even if we square both sides, so a better expression is

$$(x - 2)^2 + (y - 4)^2 + (z - 1)^2 = 9$$

Definition: A sphere with center $C(h, k, \ell)$ and radius r is described by

$$(x - h)^2 + (y - k)^2 + (z - \ell)^2 = r^2$$

Example 5

Write down the expression for a sphere of radius 4 and center $(2, -1, 5)$

$$(x - 1)^2 + (y - (-1))^2 + (z - 5)^2 = 16 \quad \Rightarrow \quad (x - 1)^2 + (y + 1)^2 + (z - 5)^2 = 16$$

Example 6

Describe the surface $x^2 + y^2 + z^2 + 4x - 2z = 0$

Rule of Thumb: If the equation has x^2, y^2, and z^2 as it's highest powers and they all have the same sign, then the surface is a sphere.

To put the equation in the general form of a sphere we have to complete the square

$$\begin{aligned}
\left(x^2 + 4x \qquad\right)^2 + y^2 + \left(z^2 - 2z \qquad\right)^2 &= 0 \\
\left(x^2 + 4x + 4\right)^2 + y^2 + \left(z^2 - 2z + 1\right)^2 &= 4 + 1 \\
(x+2)^2 + y^2 + \left(z^2 - 1\right)^2 &= 5
\end{aligned}$$

So the surface is a sphere centered at $(-2, 0, 1)$ with radius $\sqrt{5}$.

Example 7

What is described by the expression $x^2 + y^2 + z^2 \leq 3$?

All points such that the distance from the origin is less than or equal to 3. So, everything inside a sphere of radius 3 centered at the origin.

Example 8

Find the equation of a sphere with center $(2, -6, 4)$ and radius 5. Find the distance from the sphere center to the origin. Describe the sphere's intersection with each of the coordinate planes.

$$(x-2)^2 + (y+6)^2 + (z-4)^2 = 25$$

$$D = \sqrt{2^2 + 6^2 + 4^2} = \sqrt{4 + 36 + 16} = \sqrt{56}$$

xy-plane $\Rightarrow$ $z = 0$ $\Rightarrow$ $(x-2)^2 + (y+6)^2 = 9$

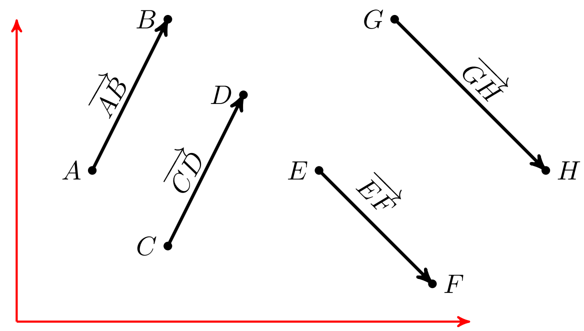
xz-plane $\Rightarrow$ $y = 0$ $\Rightarrow$ $(x-2)^2 + (z-4)^2 = -11$ (Not possible, sphere doesn't intersect)

yz-plane $\Rightarrow$ $x = 0$ $\Rightarrow$ $(y+6)^2 + (z-4)^2 = 21$

10.2 Vectors

Lots of things get measured in amounts, e.g. mass, time, temperature

More things have a magnitude **and** and direction, e.g. displacement, velocity, force. Quantities with both a magnitude and a direction can be represented by vectors. It's helpful to think of vectors by drawing them as arrows on paper.



$\overrightarrow{AB}$ and $\overrightarrow{CD}$ have the same direction and magnitude.

$\overrightarrow{AB}$ and $\overrightarrow{EF}$ have the same magnitude but different directions.

$\overrightarrow{EF}$ and $\overrightarrow{GH}$ have the same direction but different magnitudes.

We say that two vectors are **equal** if they have the same magnitude and direction (e.g. $\overrightarrow{AB}$ and $\overrightarrow{CD}$ above).

We say that two vectors are **parallel** if they have the same direction

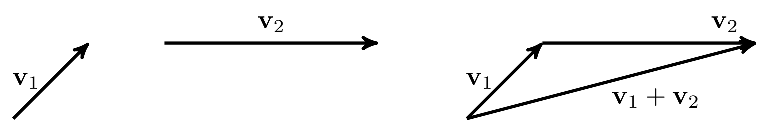
The **zero vector** (denoted by $\mathbf{0}$ or $\overrightarrow{0}$) has magnitude 0 be **NO** direction.

There are two main arithmetic operations for combining and modifying vectors. These are **vector addition** and **scalar multiplication**.

Vector Addition

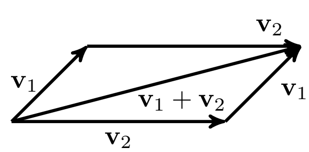
Example 9

Displacement Say someone walks NE from their home for 50 meters. From there they walk East 100 meters. Represent their displacement by a vector.



So, graphically, we represent the addition of two vectors by setting them tip-to-tail, and then drawing the new vector from the starting point of the first, to the end point of the second.

What happens if the person goes East first for 100 meters and then goes 50 meters NE?



Note that the result is the same! $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_2 + \mathbf{v}_1$ This result is called **The Parallelogram Law**.

Scalar Multiplication

Again it's helpful to think about displacement:

Example 10

$\mathbf{v} \quad \Rightarrow \quad$ Man walks 100 meters NE

$2\mathbf{v} \quad \Rightarrow \quad$ Man walks 200 meters NE

Scalar multiplication changes the **magnitude** of a vector **NOT** the direction (kinda).

Example 11

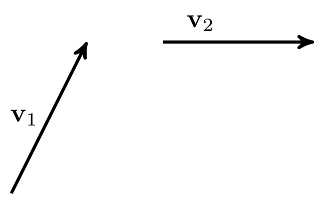
$-\mathbf{v} \quad \Rightarrow \quad$ Man walks -100 meters NE (LOL WUT?)

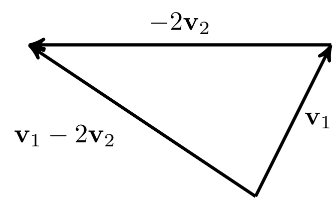
$\quad \Rightarrow \quad$ Man walks 100 meters SW

Note: Two vectors are parallel if one can be written as a scalar multiple of the other.

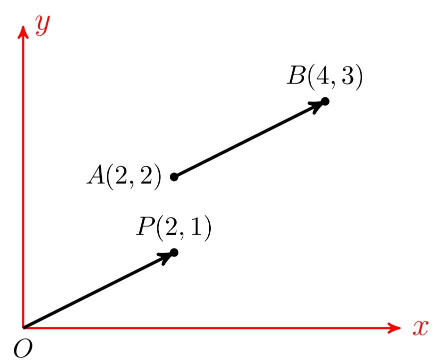
Example 12

Given the following vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, draw $\mathbf{v}_1 - 2\mathbf{v}_2$





OK, so manipulating arrows on paper is pretty cool, but we need a way to talk about vectors without drawing. Consider the following vectors in 2D



The vectors $\overrightarrow{AB}$ and $\overrightarrow{OP}$ are the same (i.e. same dir and mag) and we can represent them both as $\langle 2, 1\rangle$ (Note the difference between $(2, 1)$, which is a point).

For point $P(2, 1)$ the vector $\langle 2, 1\rangle = \overrightarrow{OP}$ has a particular name: The **position vector** of point $P(2, 1)$.

In 3D everything is the same. The position vector of $Q(3, 2, -1)$ is written as $\langle 3, 2, -1\rangle$.

We can define a vector by the directed line segment from point $A(x_1, y_1, z_1)$ to $B(x_2, y_2, z_2)$.

$$\mathbf{v} = \overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1\rangle$$

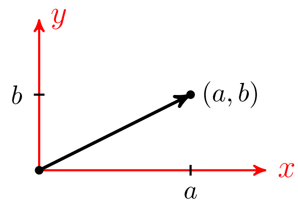
Example 13

Write down the vector represented by the directed line segment from initial point $A(1, -2, 4)$ to terminal point $B(-2, 4, -1)$.

$$\mathbf{v} = \overrightarrow{AB} = \langle -2 - 1, 4 + 2, -1 - 4\rangle = \langle -3, 6, -5\rangle$$

Magnitude of a Vector

Let $\mathbf{v} = \langle a, b \rangle$. The magnitude is written as $|\mathbf{v}|$ or sometimes $\|\mathbf{v}\|$. To find the magnitude we need to think in terms of the position vector corresponding to $\mathbf{v}$.



$$|\mathbf{v}| = \sqrt{a^2 + b^2}$$

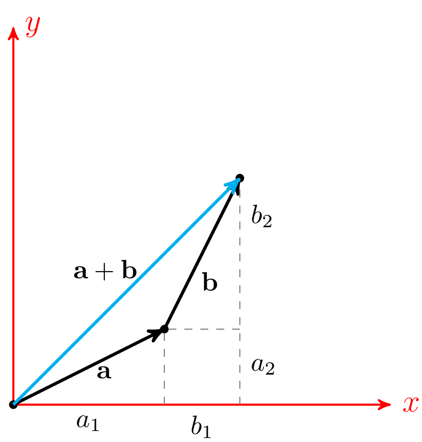
In 3D for $\mathbf{v} = \langle a, b, c \rangle$ we have $|\mathbf{v}| = \sqrt{a^2 + b^2 + c^2}$

Example 14

Let $\mathbf{v} = \langle 3, 4 \rangle$, then $|\mathbf{v}| = \sqrt{3^2 + 4^2} = 5$.

Adding Vectors Algebraically

Let $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$

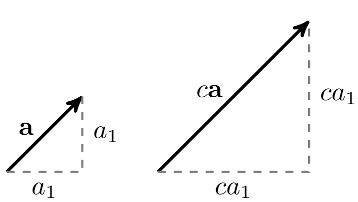


$$\mathbf{a}+\mathbf{b} = \langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle = \langle a_1 + b_1, a_2 + b_2 \rangle$$

Addition is componentwise!

Scalar Multiplication

Let $\mathbf{a} = \langle a_1, a_2 \rangle$. We want to find $c\mathbf{a}$ where c is a scalar.

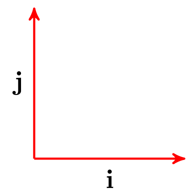


$$c\mathbf{a} = c \langle a_1, a_2 \rangle = \langle ca_1, ca_2 \rangle$$

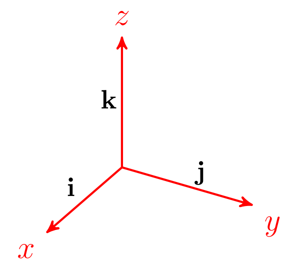
Scalar multiplication is also componentwise!

Yet Another Notation (From Physics)

Consider vectors of length 1 pointing along the coordinate axes.

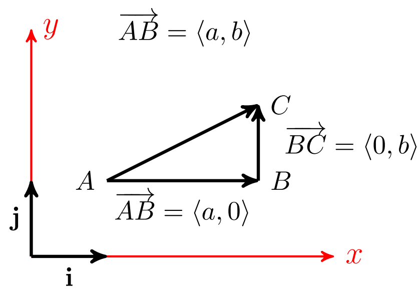


$\mathbf{i} = \langle 1, 0 \rangle$, $\mathbf{j} = \langle 0, 1 \rangle$



$\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, $\mathbf{k} = \langle 0, 0, 1 \rangle$

We can now decompose any vector into a sum of vectors that are aligned with the coordinate axes.



$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$

$$\begin{aligned} \langle a, b \rangle &= \langle a, 0 \rangle + \langle 0, b \rangle \\ &= a \langle 1, 0 \rangle + b \langle 0, 1 \rangle \\ &= a\mathbf{i} + b\mathbf{j} \end{aligned}$$

So in 2D we can write $\langle a, b \rangle = a\mathbf{i} + b\mathbf{j}$ and in 3D $\langle a, b, c \rangle = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

Example 15

If $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = -2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ then

$$\mathbf{a} + \mathbf{b} = (3 - 2)\,\mathbf{i} + (-2 + 3)\,\mathbf{j} + (1 + 2)\,\mathbf{k} = \mathbf{i} + \mathbf{j} + 3\mathbf{k}$$

A **unit vector** is a vector of magnitude 1.

$\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are all unit vectors.

Suppose $\mathbf{v}$ is some vector. How can we find a unit vector in the same direction as $\mathbf{v}$?

$\mathbf{v}$ has magnitude $|\mathbf{v}|$, so $\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$ is a unit vector.

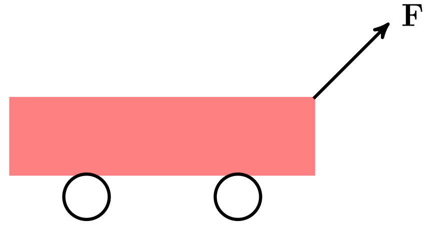
Example 16

Find a unit vector in the direction of $\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$.

$$|\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}| = \sqrt{1^2 + (-2)^2 + 2^2} = 3 \quad \Rightarrow \quad \mathbf{u} = \frac{\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}}{3} = \frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$$

Example 17

Suppose we pull a wagon by applying some force vector $\mathbf{F}$ on the handle at a 45° angle with the horizon. Find $\mathbf{F}$ if we know that $|\mathbf{F}| = 10$.



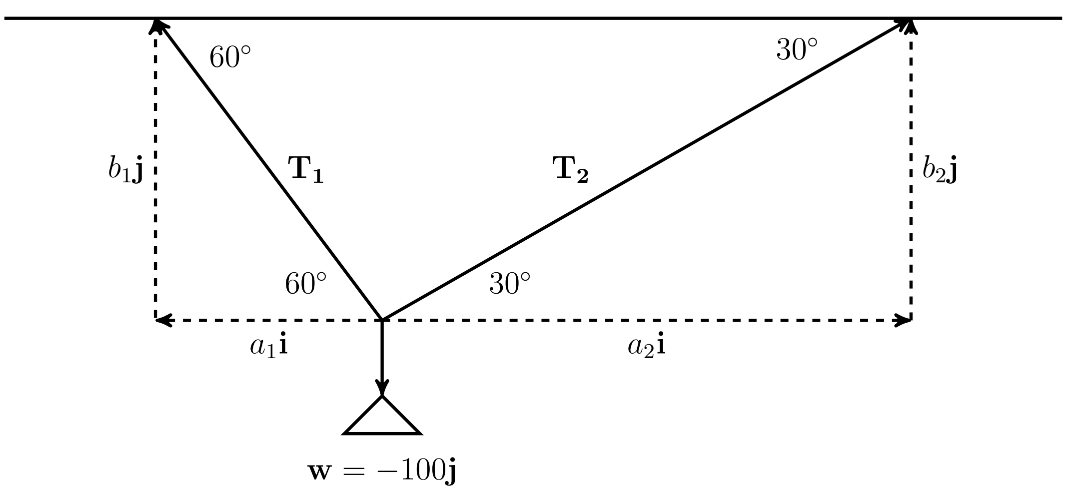
Assume that $\mathbf{F}$ has the form $\mathbf{F} = \mathbf{ai} + \mathbf{bj}$. Since the angle of the handle is 45 ° we know that $a = b \quad \Rightarrow \quad \mathbf{F} = \mathbf{ai} + \mathbf{aj}$. Then, since $|\mathbf{F}| = 10$ we have

$$|\mathbf{F}| = \sqrt{a^2 + a^2} = \sqrt{2}a = 10 \quad \Rightarrow \quad a = \frac{10}{\sqrt{2}} = \sqrt{50} \quad \Rightarrow \quad \mathbf{F} = \sqrt{50}\mathbf{i} + \sqrt{50}\mathbf{j}$$

Note that only the portion of $\mathbf{F}$ in the $\mathbf{i}$-direction actually acts to pull the wagon. The component of the force in the $\mathbf{j}$-direction attempts to pull the wagon off the ground!

Example 18

Supppose a 100 KG weight is suspended by a sequence of cables as depicted in the figure below. Find the tension in the cables represented by the vectors $\mathbf{T_1}$ and $\mathbf{T_2}$.



Let $\mathbf{T_1} = -a_1\mathbf{i} + b_1\mathbf{j}$ and $\mathbf{T_2} = a_2\mathbf{i} + b_2\mathbf{j}$ where

$$\begin{aligned} a_1 &= \cos 60° \, |\mathbf{T_1}| \\ b_1 &= \sin 60° \, |\mathbf{T_1}| \\ a_2 &= \cos 30° \, |\mathbf{T_2}| \\ b_2 &= \sin 30° \, |\mathbf{T_2}| \end{aligned}$$

This gives us

$$\begin{aligned}\mathbf{T}_1 &= -\cos 60^\circ\,|\mathbf{T}_1|\,\mathbf{i} + \sin 60^\circ\,|\mathbf{T}_1|\,\mathbf{j} \\ \mathbf{T}_2 &= \cos 30^\circ\,|\mathbf{T}_2|\,\mathbf{i} + \sin 30^\circ\,|\mathbf{T}_2|\,\mathbf{j}\end{aligned}$$

From the expressions above it's clear that to determine $\mathbf{T_1}$ and $\mathbf{T_2}$ we just need to find their magnitudes. Since the entire system is at rest, we can find the magnitudes by assuming that the sum of all of the force vectors in the system is equal to the zero vector.

$$\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{w} = \mathbf{0}$$

which gives

$$(-\cos 60^\circ\,|\mathbf{T}_1|\,\mathbf{i} + \sin 60^\circ\,|\mathbf{T}_1|\,\mathbf{j}) + (\cos 30^\circ\,|\mathbf{T}_2|\,\mathbf{i} + \sin 30^\circ\,|\mathbf{T}_2|\,\mathbf{j}) - 100\mathbf{j} = \mathbf{0}$$

Now, if the expression on the left-hand side of the equation is really the zero vector, it must be the case that both of the terms in front of the $\mathbf{i}$ and $\mathbf{j}$ are zero. From this we can separate the vector equation into two scalar equations in the unknowns $|\mathbf{T_1}|$ and $|\mathbf{T_2}|$.

$$\begin{aligned}-\cos 60^\circ\,|\mathbf{T}_1| + \cos 30^\circ\,|\mathbf{T}_2| &= 0 \qquad (1)\\ \sin 60^\circ\,|\mathbf{T}_1| + \sin 30^\circ\,|\mathbf{T}_2| - 100 &= 0 \qquad (2)\end{aligned}$$

Solving (1) for $|\mathbf{T}_1|$ gives

$$|\mathbf{T}_1| = \frac{\cos 30^\circ}{\cos 60^\circ}\,|\mathbf{T}_2| \qquad (3)$$

Substituting (3) into (2) gives

$$\frac{\sin 60^\circ \cos 30^\circ}{\cos 60^\circ}\,|\mathbf{T}_2| + \sin 30^\circ\,|\mathbf{T}_2| = 100 \qquad (4)$$

Solving (4) for $|\mathbf{T}_2|$ yields $|\mathbf{T}_2| = 50$. Then from (3) we have $|\mathbf{T}_1| = 50\sqrt{3}$.

Then plugging these into the definition of $\mathbf{T}_1$ and $\mathbf{T}_2$ we have

$$\begin{aligned}\mathbf{T}_1 &= -43.3\mathbf{i} + 75\mathbf{j} \\ \mathbf{T}_2 &= 43.3\mathbf{i} + 25\mathbf{j}\end{aligned}$$

10.3 Dot Products

So far we've added vectors together and multiplied a vector by a scalar. But what about multiplying vectors together? It turns out that there are several vector-vector operations that are similar to multiplication. The first such operation is the **dot product**.

Definition: If $\mathbf{A} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{B} = \langle b_1, b_2, b_3 \rangle$, then the **dot product** of $\mathbf{A}$ and $\mathbf{B}$ is the number $\mathbf{A} \cdot \mathbf{B}$ given by

$$\mathbf{A} \cdot \mathbf{B} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Note that the result of the dot product is a scalar, not a vector.

Example 19

$$\begin{aligned}
\langle 1, -2 \rangle \cdot \langle 3, 4 \rangle &= 1\,(3) + (-2)\,(4) = -3 \\
\langle 2, -1, 3 \rangle \cdot \langle 0, -1, 2 \rangle &= 2\,(0) + (-1)\,(-1) + 3\,(2) = 7 \\
(2\mathbf{i} - \mathbf{j} + \mathbf{k}) \cdot (\mathbf{i} + 2\mathbf{j} - \mathbf{k}) &= 2\,(1) + (-1)\,(2) + 1\,(-1) = -1
\end{aligned}$$

Properties of the Dot Product

1. $\mathbf{0} \cdot \mathbf{A} = 0$
2. $\mathbf{A} \cdot \mathbf{A} = |\mathbf{A}|^2$
3. $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$
4. $c\,(\mathbf{A} \cdot \mathbf{B}) = (c\mathbf{A}) \cdot \mathbf{A} = \mathbf{A} \cdot (c\mathbf{B})$
5. $\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = (\mathbf{A} \cdot \mathbf{B}) + (\mathbf{A} \cdot \mathbf{C}) \quad \text{or} \quad (\mathbf{A} + \mathbf{B}) \cdot \mathbf{C} = (\mathbf{A} \cdot \mathbf{C}) + (\mathbf{B} \cdot \mathbf{C})$

These properties are easy to prove using the definition of the dot product

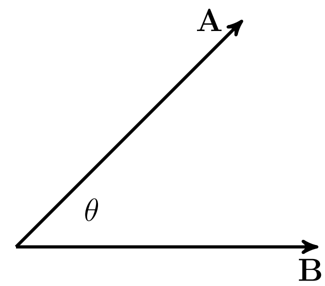
Proof of 2: $\mathbf{A} \cdot \mathbf{A} = a_1^2 + a_2^2 + a_3^2 = |\mathbf{A}|^2$

Proof of 3: $\mathbf{A} \cdot \mathbf{B} = a_1 b_1 + a_2 b_2 + a_3 b_3 = b_1 a_1 + b_2 a_2 + b_3 a_3 = \mathbf{B} \cdot \mathbf{A}$

Proof of 4: $c\,(\mathbf{A} \cdot \mathbf{B}) = c\,(a_1 b_1 + a_2 b_2 + a_3 b_3) = (ca_1)\,b_1 + (ca_2)\,b_2 + (ca_3)\,b_3 = (c\mathbf{A}) \cdot \mathbf{B}$

Geometric intuition

Take two vectors $\mathbf{A}$ and $\mathbf{B}$, and place them on the same initial point

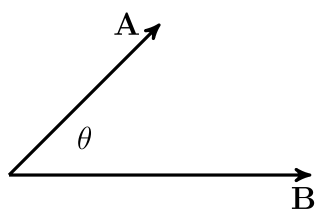


Theorem: $\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}|\,|\mathbf{B}| \cos\theta$

The dot product gives us an idea of how much two vectors line up with one another.

Example 20

Let $\mathbf{A} = \mathbf{i} + \mathbf{j}$ and $\mathbf{B} = 2\mathbf{i}$.



$$|\mathbf{A}| = \sqrt{2} \qquad |\mathbf{B}| = 2$$

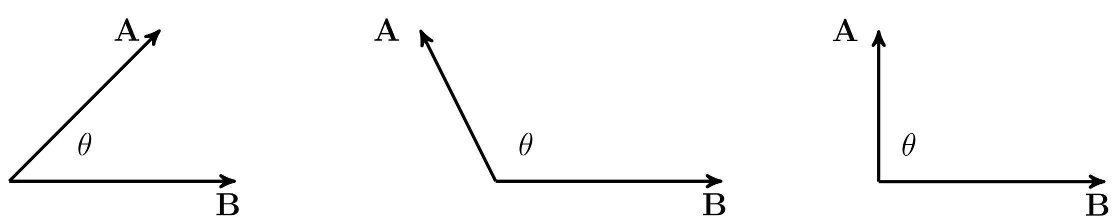
$$\mathbf{A} \cdot \mathbf{B} = \left(2\sqrt{2}\right) \cos\theta$$

But $\theta = 45^\circ = \frac{\pi}{4} \quad \Rightarrow \quad \cos\theta = \frac{\sqrt{2}}{2}$

Then $\mathbf{A} \cdot \mathbf{B} = \left(2\sqrt{2}\right) \frac{\sqrt{2}}{2} = 2$

Likewise, if we use the definition of a dot product we obtain $\mathbf{A} \cdot \mathbf{B} = 1\,(2) + 1\,(0) = 2$.

The sign on $\mathbf{A} \cdot \mathbf{B}$ can tell us whether the vectors are pointing in the same or opposite directions.



θ acute $\Rightarrow \mathbf{A} \cdot \mathbf{B} > 0$ $\qquad$ θ obtuse $\Rightarrow \mathbf{A} \cdot \mathbf{B} < 0$ $\qquad$ $\theta = 90^\circ \Rightarrow \mathbf{A} \cdot \mathbf{B} = 0$

When vectors $\mathbf{A}$ and $\mathbf{B}$ are such that $\mathbf{A} \cdot \mathbf{B} = 0$ we say that $\mathbf{A}$ and $\mathbf{B}$ are **orthogonal**.

Example 21

Let $\mathbf{A} = 2\mathbf{i} - \mathbf{j} - 3\mathbf{k}$ and $\mathbf{B} = 2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$.

$$\mathbf{A} \cdot \mathbf{B} = (2)\,(2) + (-1)\,(-2) + (-3)\,(2) = 4 + 2 - 6 = 0$$

So **A** and **B** are orthogonal vectors.

Example 22

The coordinate vectors **i**, **j** and **k** are all mutually orthogonal. That is

$\mathbf{i}\cdot\mathbf{i}=1$	$\mathbf{i}\cdot\mathbf{j}=0$	$\mathbf{i}\cdot\mathbf{k}=0$
$\mathbf{j}\cdot\mathbf{i}=0$	$\mathbf{j}\cdot\mathbf{j}=1$	$\mathbf{j}\cdot\mathbf{k}=0$
$\mathbf{k}\cdot\mathbf{i}=0$	$\mathbf{k}\cdot\mathbf{j}=0$	$\mathbf{k}\cdot\mathbf{k}=1$

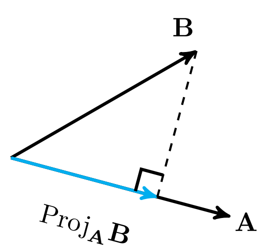
We can even find the angle between two vectors **A** and **B** using the dot product.

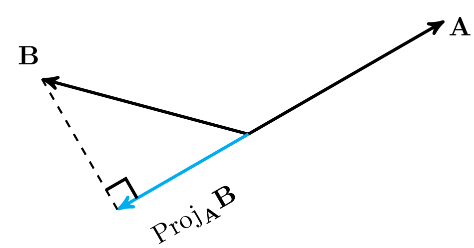
$$\cos\theta = \frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}|\,|\mathbf{B}|} \qquad \Rightarrow \qquad \theta = \cos^{-1}\left(\frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}|\,|\mathbf{B}|}\right)$$

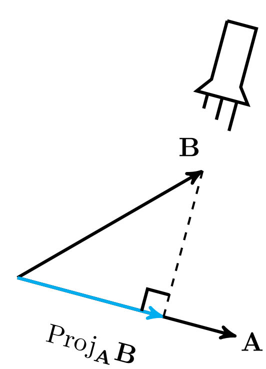
Big Picture: The size of $\mathbf{A}\cdot\mathbf{B}$ tells us how much how much of the vector **A** is aligned with vector **B**. Conversely, it also tells us how much of **B** is aligned with **A**. The sign of $\mathbf{A}\cdot\mathbf{B}$ tells us if the vectors point in the same general direction or not.

Projections

What is a projection?



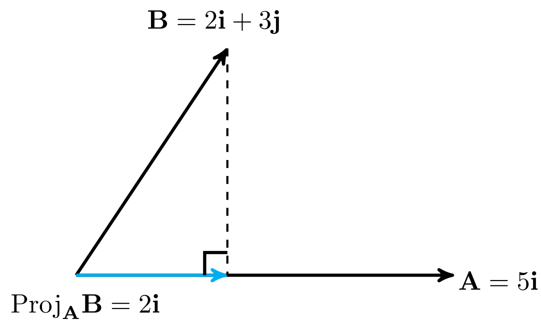




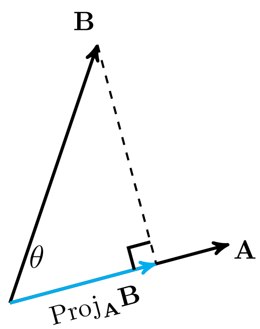
Again, you can think of $\mathrm{Proj}_{\mathbf{B}}\mathbf{A}$ as the vector that **B**'s shadow would make if you shined a flashlight above **B** in the direction orthongal to **A**.

Example 23

Let $\mathbf{B} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{A} = 5\mathbf{i}$. Compute $\text{Proj}_{\mathbf{A}}\mathbf{B}$.



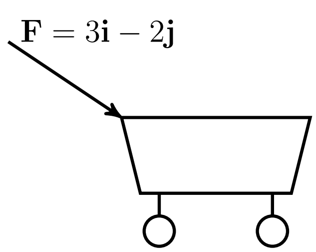
Now, how can we compute a projection for general vectors $\mathbf{A}$ and $\mathbf{B}$?



$$\begin{aligned}
\text{proj}_{\mathbf{A}}\mathbf{B} &= \left(|\mathbf{B}|\cos\theta\right)\frac{\mathbf{A}}{|\mathbf{A}|} \\
\text{proj}_{\mathbf{A}}\mathbf{B} &= \left(|\mathbf{A}|\,|\mathbf{B}|\cos\theta\right)\frac{\mathbf{A}}{|\mathbf{A}|^2} \\
\text{proj}_{\mathbf{A}}\mathbf{B} &= \left(\frac{\mathbf{A}\cdot\mathbf{B}}{\mathbf{A}\cdot\mathbf{A}}\right)\mathbf{A}
\end{aligned}$$

Example 24

Consider applying a force $\mathbf{F} = 3\mathbf{i} - 2\mathbf{j}$ to push a cart. What is the horizontal component of $\mathbf{F}$?

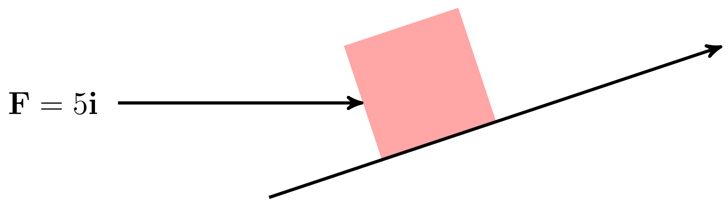


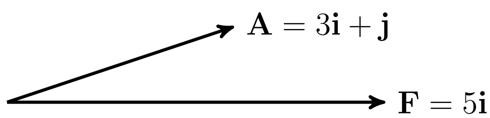
Duh, it's $3\mathbf{i}$. But it's also $\text{proj}_{+\mathbf{i}}\mathbf{F}$

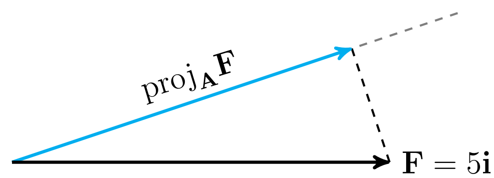
$$\text{proj}_{+\mathbf{i}}\mathbf{F} = \left[\frac{\mathbf{i}\cdot\mathbf{F}}{\mathbf{i}\cdot\mathbf{i}}\right]\mathbf{i} = \left[\frac{\mathbf{i}\cdot(3\mathbf{i}-2\mathbf{j})}{\mathbf{i}\cdot\mathbf{i}}\right]\mathbf{i} = 3\mathbf{i}$$

Example 25

Suppose you push a block up a ramp with a slope of $1/3$ using a horizontal force of $\mathbf{F} = 5\mathbf{i}$. Find the component of $\mathbf{F}$ that acts to move the block up the ramp.



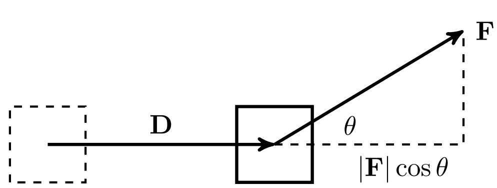




$$
\begin{aligned}
\operatorname{proj}_{\mathbf{A}}\mathbf{F} &= \left[\frac{\mathbf{A}\cdot\mathbf{F}}{\mathbf{A}\cdot\mathbf{A}}\right]\mathbf{A} \\
&= \left[\frac{(3\mathbf{i}+\mathbf{j})\cdot(5\mathbf{i})}{(3\mathbf{i}+\mathbf{j})\cdot(3\mathbf{i}+\mathbf{j})}\right](3\mathbf{i}+\mathbf{j}) \\
&= \left[\frac{15}{9+1}\right](3\mathbf{i}+\mathbf{j}) \\
&= \left[\frac{3}{2}\right](3\mathbf{i}+\mathbf{j}) = \frac{9}{2}\mathbf{i} + \frac{3}{2}\mathbf{j} \quad \text{or} \quad 4.5\mathbf{i} + 1.5\mathbf{j}
\end{aligned}
$$

Work

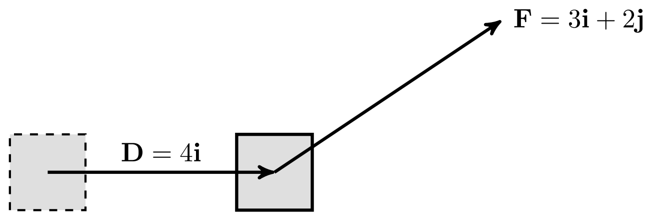
Suppose we apply a force $\mathbf{F}$ to move a block along some displacement vector $\mathbf{D}$.



$$
\begin{aligned}
\text{Work} &= \begin{pmatrix}\text{Magnitude of Force}\\ \text{in Direction of}\\ \text{Displacement}\end{pmatrix}\begin{pmatrix}\text{Length}\\ \text{of}\\ \text{Displacement}\end{pmatrix} \\
&= (|\mathbf{F}|\cos\theta)\,|\mathbf{D}| \\
&= |\mathbf{F}|\,|\mathbf{D}|\cos\theta \\
&= \mathbf{F}\cdot\mathbf{D}
\end{aligned}
$$

Example 26

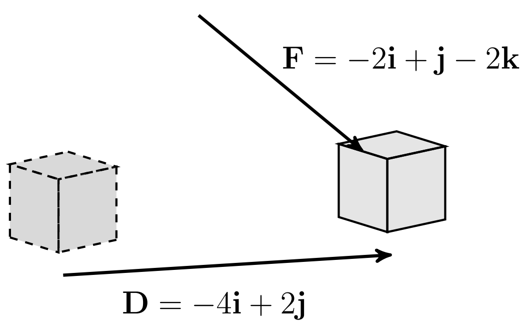
How much work is done if we apply a force of $\mathbf{F} = 3\mathbf{i} + 2\mathbf{j}$ Newtons to drag a block along a displacment vector $\mathbf{D} = 4\mathbf{i}$ meters?



$$W = \mathbf{F} \cdot \mathbf{D} = (3\mathbf{i} + 2\mathbf{j}) \cdot (4\mathbf{i}) = 12 \text{ N m}$$

Example 27

Suppose a force of $\mathbf{F} = -2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ is applied to push a large stone block along the floor with a displacement vector of $\mathbf{D} = -4\mathbf{i} + 2\mathbf{j}$. How much work is done?

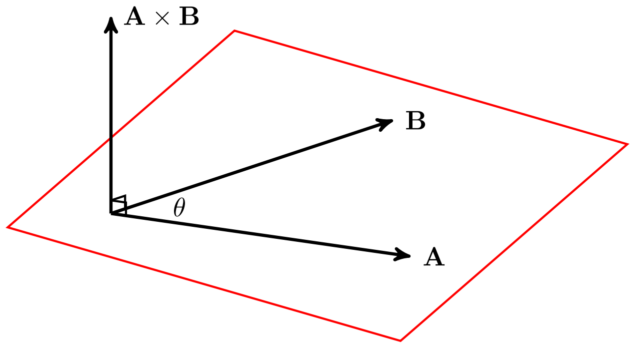


$$W = \mathbf{F} \cdot \mathbf{D} = (-2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) \cdot (-4\mathbf{i} + 2\mathbf{j}) = 10\text{N m}$$

10.4 The Cross Product

We've already seen one form of vector multiplication: the dot product. The dot product of two vectors is a scalar and is bigger the more aligned two vectors are.

Today, we talk about another form of vector multiplication: the **cross product**. Consider the plane formed by two vectors $\mathbf{A}$ and $\mathbf{B}$.



The cross product of $\mathbf{A}$ and $\mathbf{B}$, written $\mathbf{A} \times \mathbf{B}$ is a **vector** that is orthogonal to BOTH $\mathbf{A}$ and $\mathbf{B}$. In particular, it is orthogonal to **any** vector that lies in the same plane as $\mathbf{A}$ and $\mathbf{B}$. We say that the cross product is **normal** to the plane defined by $\mathbf{A}$ and $\mathbf{B}$.

The direction that the cross product points away from the plane is determined by the **Right Hand Rule**.

Definition: Let $\mathbf{A} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{B} = \langle b_1, b_2, b_3 \rangle$

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle \\ &= (a_2 b_3 - a_3 b_2)\,\mathbf{i} + (a_3 b_1 - a_1 b_3)\,\mathbf{j} + (a_1 b_2 - a_2 b_1)\,\mathbf{k} \end{aligned}$$

There is an easy way to remember the formula for the cross product, but it requires some new machinery from linear algebra

Definition: The **determinant** of a 2×2 matrix is given by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} 2 & -1 \\ 4 & 3 \end{vmatrix} = (2)\,(3) - (-1)\,(4) = 6 + 4 = 10$$

Definition: The **determinant** of a 3×3 matrix is given by

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Then, another way to compute $\mathbf{A} \times \mathbf{B}$ is by computing the following 3×3 determinant:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example 28

Compute $\mathbf{A} \times \mathbf{B}$ where $\mathbf{A} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and $\mathbf{B} = -\mathbf{i} + 5\mathbf{k}$. We have

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -2 \\ -1 & 0 & 5 \end{vmatrix} = \begin{vmatrix} 3 & -2 \\ 0 & 5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -2 \\ -1 & 5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix} \mathbf{k} = 15\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$$

Let's check to see if it's really true that $\mathbf{A} \times \mathbf{B}$ is orthogonal to both $\mathbf{A}$ and $\mathbf{B}$.

$$\begin{aligned} \mathbf{A} \cdot (\mathbf{A} \times \mathbf{B}) &= (\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) \cdot (15\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}) = 15 - 9 - 6 = 0 \quad \checkmark \\ \mathbf{B} \cdot (\mathbf{A} \times \mathbf{B}) &= (-\mathbf{i} + 5\mathbf{k}) \cdot (15\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}) = -15 + 15 = 0 \quad \checkmark \end{aligned}$$

Alternate Method for Computing $\mathbf{A} \times \mathbf{B}$

$$\left| \begin{array}{ccc|cc} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{i} & \mathbf{j} \\ 1 & 3 & -2 & 1 & 3 \\ -1 & 0 & 5 & -1 & 0 \end{array} \right|$$

Multiply along arrows. Down arrows get a $(+)$ and up arrows get a $(-)$.

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= 15\mathbf{i} + 2\mathbf{j} + 0\mathbf{k} - (-3)\,\mathbf{k} - 0\mathbf{i} - 5\mathbf{j} \\ &= 15\mathbf{i} - 3\mathbf{j} + 3\mathbf{k} \end{aligned}$$

Similar to the dot product, the magnitude of $\mathbf{A} \times \mathbf{B}$ is determined by the size of the angle between $\mathbf{A}$ and $\mathbf{B}$, but the relationship is flipped.

Recall that the dot product of two vectors was bigger if the angle between the vectors was smaller. The magnitude of the cross product is **smaller** if the vectors are more aligned with each other.

Theorem: $|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}|\,|\mathbf{B}| \sin\theta$

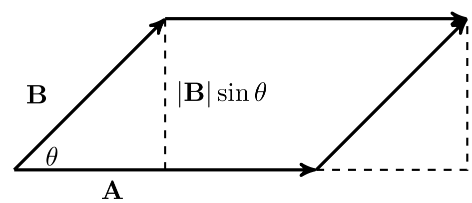
Dot Product | **Cross Product**

$$\frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}|\,|\mathbf{B}|} = \cos\theta \qquad\qquad \frac{|\mathbf{A}\times\mathbf{B}|}{|\mathbf{A}|\,|\mathbf{B}|} = \sin\theta$$

$\mathbf{A}\cdot\mathbf{B} = 0$ if $\mathbf{A}\perp\mathbf{B}$
$\mathbf{A}\cdot\mathbf{B}$ maximized if $\mathbf{A}\parallel\mathbf{B}$

$|\mathbf{A}\times\mathbf{B}|$ maximized if $\mathbf{A}\perp\mathbf{B}$
$|\mathbf{A}\times|\,\mathbf{B} = 0$ if $\mathbf{A}\parallel\mathbf{B}$

Fact: $|\mathbf{A}\times\mathbf{B}| = |\mathbf{A}|\,|\mathbf{B}|\sin\theta =$ Area of Parallelogram formed by $\mathbf{A}$ and $\mathbf{B}$.



Example 29

Compute the area of the parallelogram formed by $\mathbf{A}$ and $\mathbf{B}$ and verify that $\mathbf{A}\perp(\mathbf{A}\times\mathbf{B})$ and $\mathbf{B}\perp(\mathbf{A}\times\mathbf{B})$.

Area of Parallelogram $= |\mathbf{A}\times\mathbf{B}| = |15\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}| = \sqrt{225+9+8} = \sqrt{243}$

$$\begin{aligned}
\mathbf{A}\cdot(\mathbf{A}\times\mathbf{B}) &= (\mathbf{i} + 3\mathbf{j} - 2\mathbf{k})\cdot(15\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}) = 15 - 9 - 6 = 0 \quad \checkmark \\
\mathbf{A}\cdot(\mathbf{A}\times\mathbf{B}) &= (-\mathbf{i} + 5\mathbf{k})\cdot(15\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}) = -15 + 15 = 0 \quad \checkmark
\end{aligned}$$

Properties of the Cross Product

Anti-Commutativity: Is $\mathbf{A}\times\mathbf{B} = \mathbf{B}\times\mathbf{A}$? Think about RHR! $\quad \mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A}$

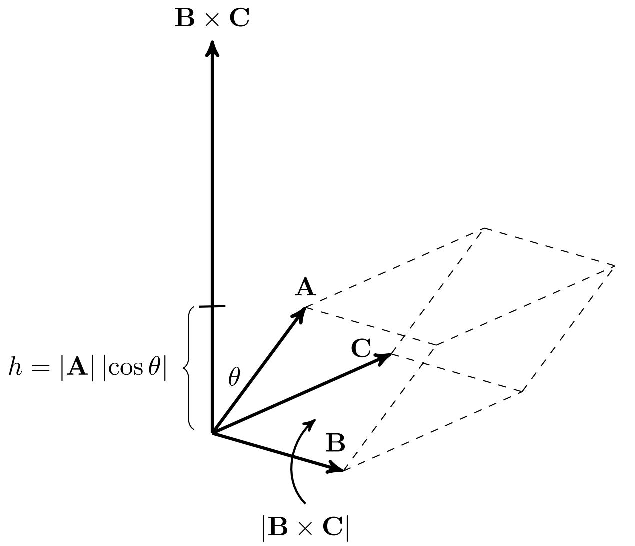
Scalar Multiplication: $(c\mathbf{A})\times\mathbf{B} = c\,(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\times(c\mathbf{B})$

Distribution: $\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = (\mathbf{A}\times\mathbf{B}) + (\mathbf{A}\times\mathbf{C})$

Triple Products: $\begin{cases} \text{Scalar} & \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} \\ \text{Vector} & \mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\,\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\,\mathbf{C} \end{cases}$

Example 30

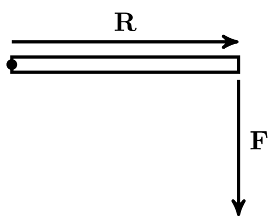
The volume of a parallel piped formed by the vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ is given by the scalar triple product $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$.



$$V = |\mathbf{B} \times \mathbf{C}|\,|\mathbf{A}|\,|\cos\theta| = |\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})|$$

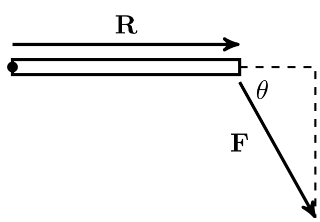
Example 31

Torque measures the tendency of an object to rotate around a pivot (twisting power).



$$|\tau| = \left(\begin{array}{c}\text{Magnitude of Force}\\ \perp \text{ to lever arm}\end{array}\right)\left(\begin{array}{c}\text{Length of}\\ \text{lever arm}\end{array}\right)$$

But what if the force applied is not orthogonal to $\mathbf{R}$?

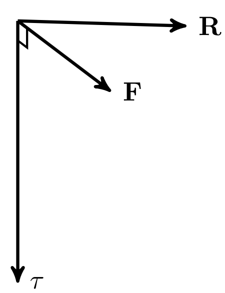


Magnitude of Force $\perp$ to $\mathbf{R}$ is $|\mathbf{F}|\sin\theta$

Length of $\mathbf{R}$ is $|\mathbf{R}|$

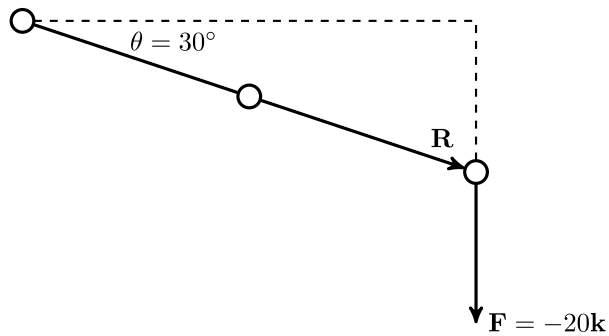
So $|\tau| = |\mathbf{R}|\,|\mathbf{F}|\sin\theta = |\mathbf{R} \times \mathbf{F}|$

We can think of torque as a vector τ where $\tau = \mathbf{R} \times \mathbf{F}$. The torque vector has a direction orthogonal to the plane of rotation and sign given by the RHR.



Example 32

Consider an arm with 1ft between the shoulder and the elbow and 1ft between the elbow and the hand extended in the $\mathbf{j}$ direction and holding a 20lb dumbbell at $30°$ below the horizon. Calculate the magnitude of the torque on both the elbow and the shoulder.



Shoulder: We have $\mathbf{R}_s = 2\cos 30°\mathbf{j} - 2\sin 30°\mathbf{k} = \sqrt{3}\mathbf{j} - \mathbf{k}$

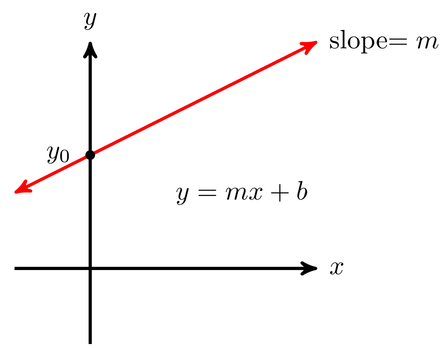
$$\tau_s = \mathbf{R}_s \times \mathbf{F} = \left| \begin{array}{ccc|cc} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{i} & \mathbf{j} \\ 0 & \sqrt{3} & -1 & 0 & \sqrt{3} \\ 0 & 0 & -20 & 0 & 0 \end{array} \right| = -20\sqrt{3}\mathbf{i}$$

Elbow: We have $\mathbf{R}_e = \frac{1}{2}\mathbf{R}_s \quad \Rightarrow \quad \tau_e = \mathbf{R}_e \times \mathbf{F} = \frac{1}{2}\mathbf{R}_s \times \mathbf{F} = \frac{1}{2}\tau_s = -10\sqrt{3}\mathbf{i}$ so...

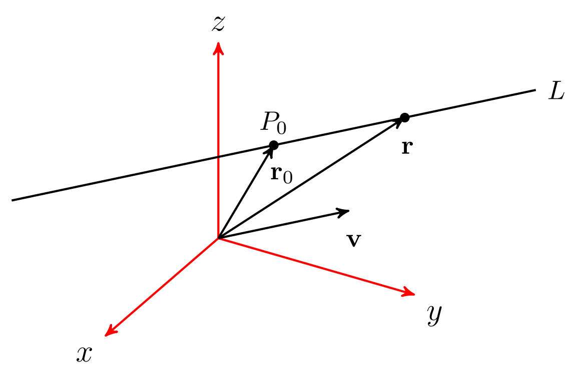
$$|\tau_s| = 20\sqrt{3} \text{ ft-lbs} \qquad \text{and} \qquad |\tau_e| = 10\sqrt{3} \text{ ft-lbs}$$

10.5 Equations of Lines and Planes

In 2D a line is defined by
a point and a slope



In 3D a line is defined by
a point and a direction vector



$\mathbf{r}_0$ is the position vector of $P_0(x_0, y_0, z_0)$, i.e. $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$.

To get to a new point on line L we add a scalar multiple of $\mathbf{v}$. The position vector $\mathbf{r}$ traces out the line.

$$L: \quad \mathbf{r} = \mathbf{r}_0 + t\mathbf{v} \qquad -\infty \leq t \leq \infty$$

Suppose the direction vector $\mathbf{v}$ is given by $\mathbf{v} = \langle a, b, c \rangle$. ($a$, b, c are called the *directional numbers of* L). Then

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v} \quad \Leftrightarrow \quad \langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

$$\text{Vector Equation of L} : \langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

Example 33

Find the vector equation of the line that pass through the point $(1, 3, 2)$ and is $\|$ to vector $\mathbf{v} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

$$\langle x, y, z \rangle = \langle 1 + 2t, 3 - 2t, 2 + t \rangle$$

We can also write the equation of the line in terms of a parameterization of each coordinate

$$\text{Parametric Equation:} \quad x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct$$

Example 34

We can also write L as $x = 1 + 2t$, $y = 3 - 2t$, $z = 2 + t$.

Each value of the parameter t gives a new point on line L

Example 35

Find 2 other points on the line L

$$\begin{array}{rcl} t = \;\;\,1 & \Rightarrow & x = \;\;\,3,\; y = 1,\; z = 4 \quad (3,1,4) \\ t = -1 & \Rightarrow & x = -1,\; y = 5,\; z = 1 \quad (-1,5,1) \end{array}$$

Vector and parametric equations of lines are not unique. We could choose another point on the line and another vector that points in the same direction as $\mathbf{v}$ and obtain different equations describing the same line.

Example 36

Letting $P_0 = (3,1,4)$ and $\mathbf{v} = 8\mathbf{i} - 8\mathbf{j} + 4\mathbf{k}$ gives $\langle x, y, z\rangle = \langle 3 + 8t, 1 - 8t, 4 + 4t\rangle$. Then

$$\left.\begin{array}{lcl} t = 0 & \Rightarrow & (3,1,4) \\ t = -1/2 & \Rightarrow & (-1,5,1) \end{array}\right\} \quad \text{SAME LINE!}$$

Example 37

Find the Line that passes through the point $P\,(-1,0,2)$ and $Q\,(2,1,4)$.

Need a point and a direction vector. $\mathbf{v} = \overrightarrow{PQ} = \langle 2 - (-1)\,, 1 - 0, 4 - 2\rangle = \langle 3,1,2\rangle$.

$$x = 2 + 3t,\; y = 1 + t,\; z = 4 + 2t \quad \text{or} \quad x = -1 + 3t,\; y = t,\; z = 2 + 2t$$

Example 38

Where does L intersect the xy-plane?

We're in the xy-plane if $z = 0 \quad \Rightarrow \quad 0 = 2 + 2t \quad \Rightarrow \quad t = -1$

$\Rightarrow \quad x = 2 + 3\,(-1) = -1$ and $y = 1 - 1 = 0 \quad \Rightarrow \quad (-1,0,0)$

Example 39

What if we only want to know the equation of the **finite line segment** between P and Q?

$$x = -1 + 3t,\ y = t,\ z = 2 + 2t, \quad 0 \leq t \leq 1$$

Example 40

Determine whether the following line segments are parrallel. Do they intersect?

$$\begin{aligned} L_1 &:= \langle x, y, z \rangle = \langle 1 - t, 2 + t, 2t - 1 \rangle \\ L_2 &:= \langle x, y, z \rangle = \langle 2 + 3s, 5 + 3s, s \rangle \end{aligned}$$

The direction vectors associated with each line are

$$\mathbf{v}_1 = \langle -1, 1, 2 \rangle \qquad \text{and} \qquad \mathbf{v}_2 = \langle 1, 3, 1 \rangle \quad \Rightarrow \quad \text{Not Parallel!}$$

To see if the two lines intersect we need to see if there are values of t and s that make the three coordinates of the points on the line equal.

$$\begin{aligned} 1 - t &= 2 + s \\ 2 + t &= 5 + 3s \\ 2t - 1 &= s \end{aligned}$$

We'll use two equations to solve for t and s, and then see if the computed values satisfy the third equation. Adding the first and second equations we have

$$3 = 7 + 4s \qquad \Rightarrow \qquad s = -1 \qquad \Rightarrow \qquad t = 0$$

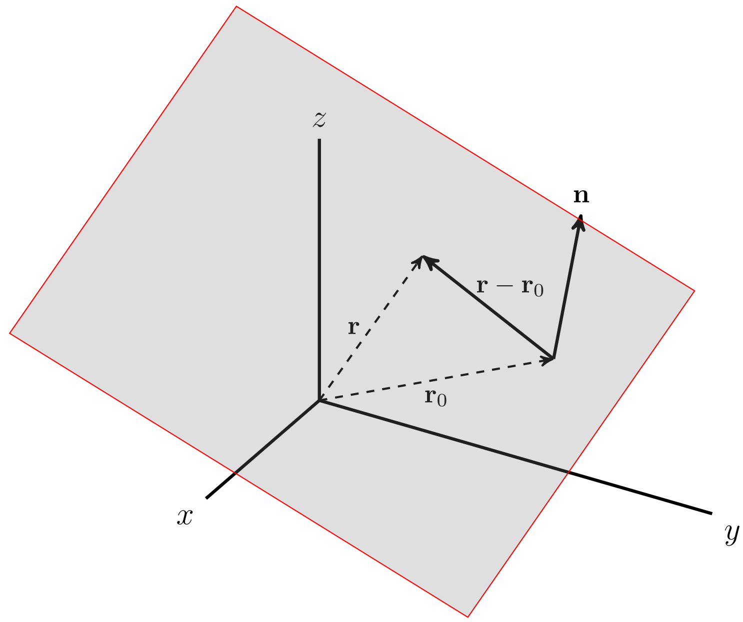
Plugging these into the third equation we have $-1 = -1 \;\checkmark$. Therefore the point of intersection of the two lines occurs when $t = 0$ in L_1 or $s = -1$ in L_2, which gives $(1, 2, -1)$.

Planes

Earlier we saw that the equation of a plane has the form $Ax + By + Cz = D$. We also noted that described a vector **normal** to a plane as a vector that is orthogonal to any vector lying in the plane. We'll use these facts to determine the equation of a specific plane.

In 2D we know that we can find the equation of a line if we know 2 points on the line, or if we know 1 points and the slope. Notice the slope of a line in 2D is sort of like a direction.

Analogously, we can find the equation of a plane if we know 3 points on the plane, or 1 point and a vector that is normal to the plane. Let $P_0(x_0, y_0, z_0)$ be a point on a plane with $\mathbf{n}$.



Then $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0 \quad \Leftrightarrow \quad \mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0$ which is the so-called vector equation of a plane.

If the normal vector is given by $\mathbf{n} = \langle a, b, c\rangle$ and the position vector $\mathbf{r} = \langle x, y, z\rangle$ then we have

$$\langle a, b, c\rangle \cdot \langle x, y, z\rangle = \langle a, b, c\rangle \cdot \langle x_0, y_0, z_0\rangle \quad \Rightarrow \quad ax + by + cz = ax_0 + by_0 + cz_0$$

which we can rearrange to get the more familiar equation for a plane

$$a\,(x - x_0) + b\,(y - y_0) + c\,(z - z_0) = 0$$

Example 41

Find the equation of a plane containing the point $(-1, 1/2, 3)$ with normal vector $\mathbf{n} = \mathbf{i}+4\mathbf{j}+\mathbf{k}$.

$$(x + 1) + 4\left(y - \frac{1}{2}\right) + (z - 3) = 0 \quad \Leftrightarrow \quad x + 4y + z = 5$$

Example 42

Find the equation of the plane containing the points $P_1\,(1, 2, 1)$, $P_2\,(0, -1, 0)$, and $P_3\,(2, -1, 3)$.

To find the equation of the plane we need at least one point and a normal vector. We already have a point. To find the normal vector we can use the three points to find two vectors in the plane and cross them.

$$v_1 = \overrightarrow{P_1P_2} = \langle -1,-3,-1\rangle \quad \text{and} \quad v_2 = \overrightarrow{P_2P_3} = \langle 2,0,3\rangle$$

Then

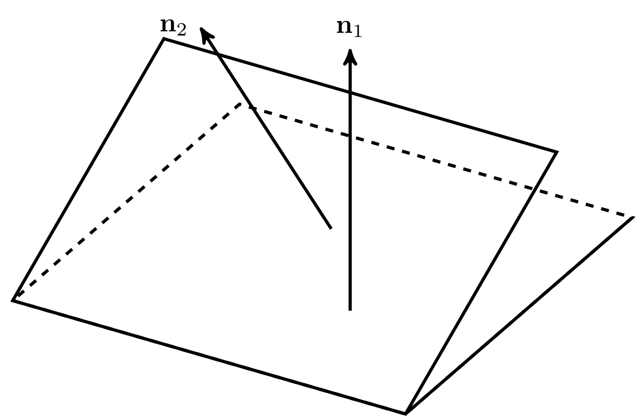
$$\mathbf{n} = v_1 \times v_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -3 & -1 \\ 2 & 0 & 3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} -3 & -1 \\ 0 & 3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} -1 & -1 \\ 2 & 3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -1 & -3 \\ 2 & 0 \end{vmatrix} = -9\mathbf{i} + \mathbf{j} + 6\mathbf{k}$$

Then, choosing (arbitrarily) P_1 as the point we get

$$-9\,(x-1) + (y-2) + 6\,(z-1) = 0$$

Note that if two planes have the same normal vector then we say the planes are parallel.

If two planes are not parallel then they must intersect somewhere and the intersection is a line.



Example 43

Find the parametric equation of the line of intersection of the planes $x + y + z = 1$ and $x + 2y + 2z = 1$.

To find the equation of a line we need a point and a vector parallel to the line. To find the point, let's look for the point where the line of intersection intersects the xy-plane. To do this we set $z = 0$ in both planes and then solve simultaneously for x and y.

$$\begin{array}{rcl} x + y &=& 1 \\ -(x + 2y &=& 1) \\ \hline -y &=& 0 \end{array}$$

Then $y = 0 \;\Rightarrow\; x = 1$ which gives us the point $(1, 0, 0)$.

The vector parallel to the line of intersection will be orthogonal to the normal vector for both planes. We can find this by taking their cross-product.

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = -\mathbf{j} + \mathbf{k}$$

Then together with the point we found we have

$$\langle x, y, z \rangle = \langle 1, -t, t \rangle \quad \text{or} \quad x = 1,\ y = -t,\ z = t$$

Sometimes it's also useful to know the angle between two planes. This is simply the angle between the planes' normal vectors, which we can find using the dot product

$$\theta = \cos^{-1}\left[\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right]$$

Example 44

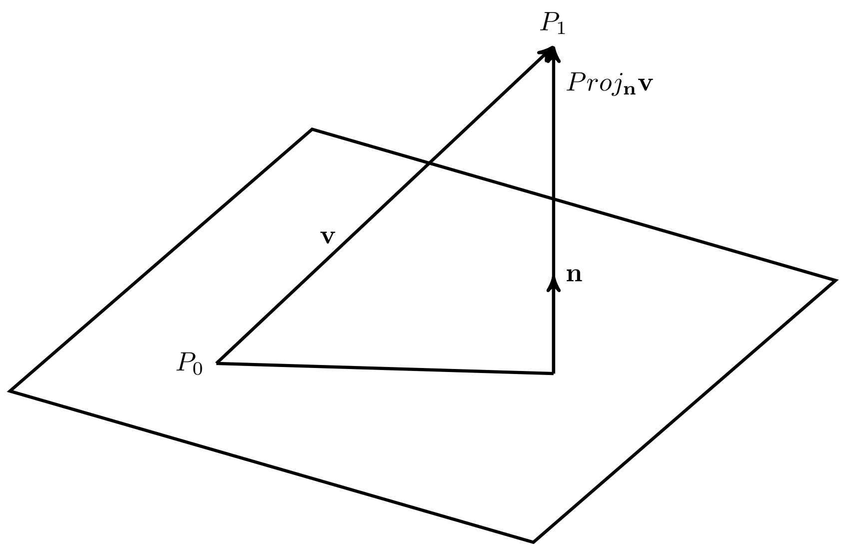
Find the angle between the planes in the previous example.

We have $\mathbf{n}_1 = \langle 1, 1, 1 \rangle$ and $\mathbf{n}_2 = \langle 1, 2, 2 \rangle$. Then

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = 5 \quad \text{and} \quad |\mathbf{n}_1| = \sqrt{3} \quad \text{and} \quad |\mathbf{n}_2| = 3$$

$$\theta = \cos^{-1}\left[\frac{5}{3\sqrt{3}}\right] \approx 15.8^\circ$$

Shortest Distance Between Plane and Point



Let $P_1(x_1, y_1, z_1)$ be the point of interest and $P_0(x_0, y_0, z_0)$ be any point on the plane. The shortest distance from P_1 to the plane will be along the normal vector $\mathbf{n}$. If we know the vector $\mathbf{v} = \overrightarrow{P_0P_1}$ then the vector $Proj_{\mathbf{n}}\mathbf{v}$ will have the same magnitude as the distance from P_1 to the plane. We have

$\mathbf{v} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$ then

$$D = Proj_{\mathbf{n}}\mathbf{v} = \left| \frac{\mathbf{v} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}} \mathbf{n} \right| = \frac{|\mathbf{v} \cdot \mathbf{n}|}{|\mathbf{n}|^2} |\mathbf{n}| = \frac{|\mathbf{v} \cdot \mathbf{n}|}{|\mathbf{n}|}$$

If we let $\mathbf{n} = \langle a, b, c \rangle$ be the normal vector then we have

$$D = \frac{|\mathbf{v} \cdot \mathbf{n}|}{|\mathbf{n}|} = \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}} =$$

$$\frac{|ax_1 + by_1 + cz_1 - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

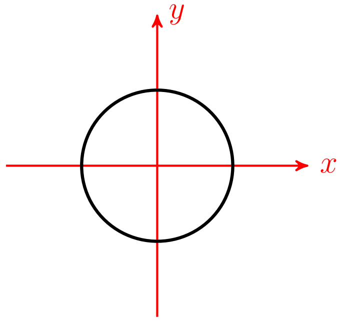
Example 45

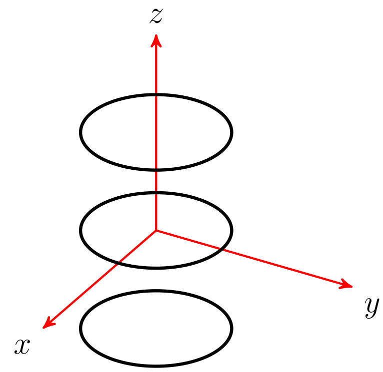
Find the distance between the point $P(1, 2, 3)$ plane $3x + 4y + 5z = 1$.

$$D = \frac{|3(1) + 4(2) + 5(3) - 1|}{\sqrt{3^2 + 4^2 + 5^2}} = \frac{25}{\sqrt{50}} = \frac{5}{\sqrt{2}}$$

10.6 Cylinders and Quadratic Surfaces

Consider the usual cylinder of the form $x^2 + y^2 = a^2$:



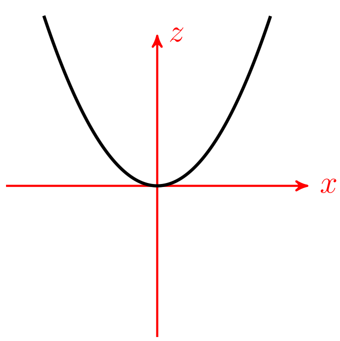


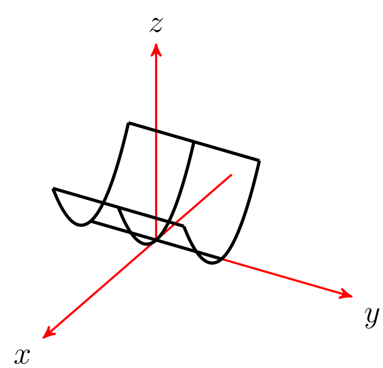
Note that the curve intersecting any plane $z = c$ looks like a cirlce.

We can do this with any shape we want. The curve that generates the cylinder is called the **generating curve**. The cylinder can be oriented perpendicular to any of the three coordinate planes.

$f(x, y) = c$	perpendicular to the xy-plane
$g(x, z) = c$	perpendicular to the xz-plane
$h(y, z) = c$	perpendicular to the yz-plane

Example Plot the cylinder $z = x^2$ in three dimensions.





Quadratic Surfaces

In general $Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Jz + K = 0$ where $A, B, C, \ldots$ are constants.

Example: We've already seen a few quadratic surfaces with spheres and the previous example.

In this section we'll consider ellipsoids (sorta spheres) parabloids, cones, and hyperboloids.

The Ellipsoid

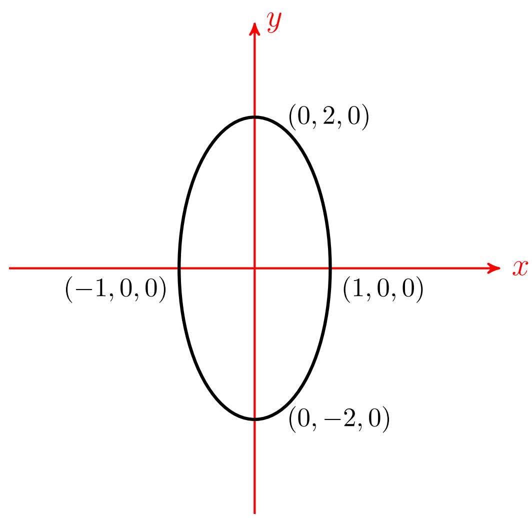
Ellipsoids have the general form $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

Example: Let's look at the particular case of the ellipsoid when $a = 1$, $b = 2$, and $c = 3$. Then we have

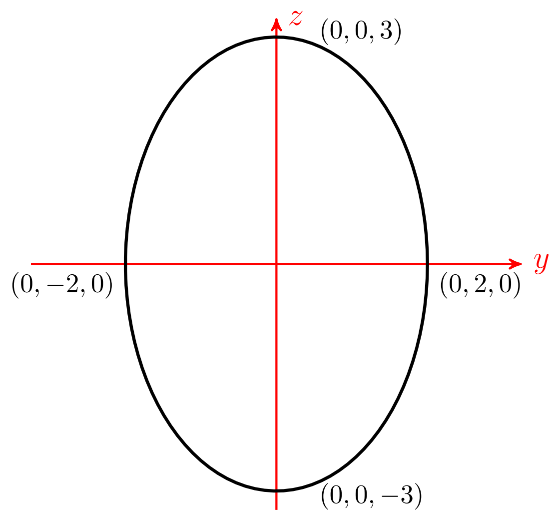
$$\frac{x^2}{1} + \frac{y^2}{4} + \frac{z^2}{9} = 1$$

What does the surface look like if we restrict it to the three coordinate planes?

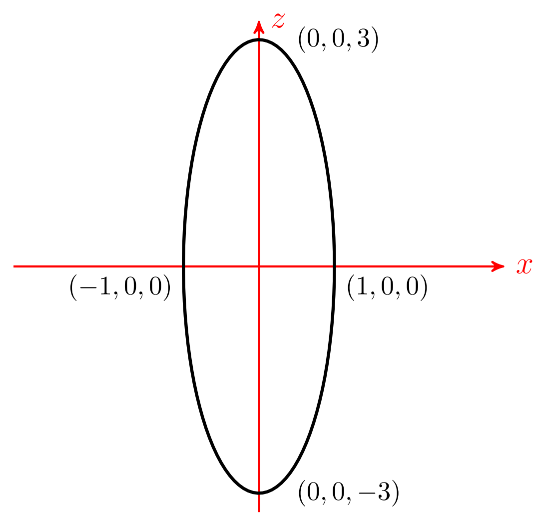
$$z = 0 \quad \Rightarrow \quad \frac{x^2}{1} + \frac{y^2}{4} = 1$$



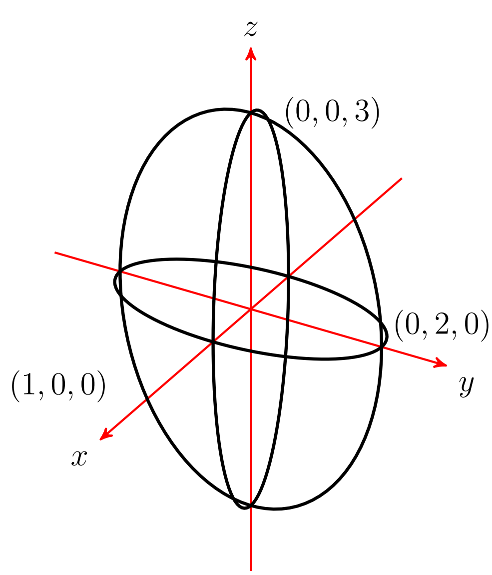
$$x = 0 \quad \Rightarrow \quad \frac{y^2}{4} + \frac{z^2}{9} = 1$$



$$y = 0 \quad \Rightarrow \quad \frac{x^2}{1} + \frac{z^2}{9} = 1$$



Now we can plot all three ellipses in the coordinate planes and get a general idea of the shape.



Now we have a general outline of the shape. Sometimes this is good enough to finish the plot, but other times we'd like more information than just the intersection with the coordinate planes. In these cases it's helpful to look at cross-sections in planes that are parallel to the coordinate planes. For example, we could look at cross-sections formed by cutting the surface with planes of the form $z = k$, where here k is just a constant. Note that this is a plane that is parallel to the xy-plane at height $z = k$.

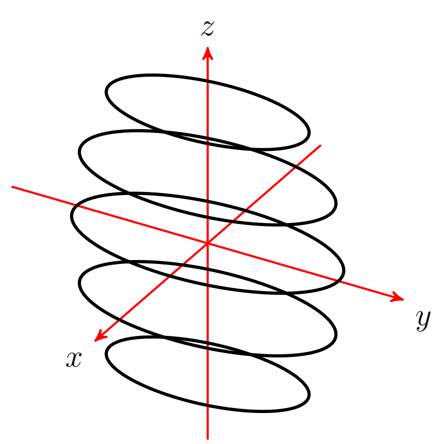
First note that we can't choose a k-value that is bigger than $c = 3$ or smaller than $-c = -3$, because the ellipsoid does not exist there. We can see this from the equation if we try to choose something like $z = 6$ because we get

$$\frac{x^2}{1} + \frac{y^2}{4} + \frac{6^2}{9} = 1 \quad \Rightarrow \quad \frac{x^2}{1} + \frac{y^2}{4} = -3$$

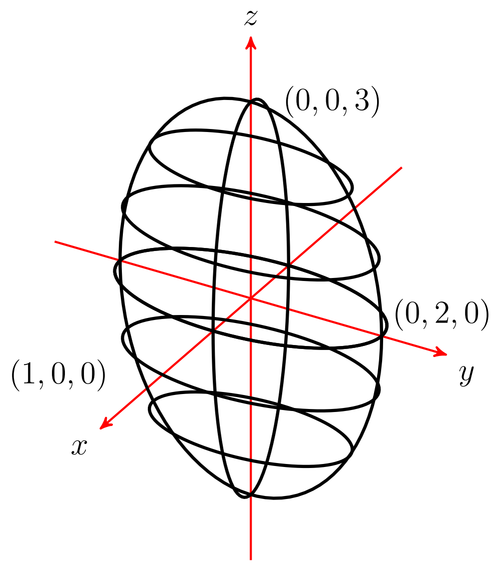
which can't possibly have a solution because everything on the left side of the equals sign is positive and the right side is negative. Then, for any $z = k$ for $-3 \leq k \leq 3$ the part of the surface lying in the $z = k$ plane is an ellipse

$$\frac{x^2}{1} + \frac{y^2}{4} + \frac{k^2}{9} = 1 \quad \Rightarrow \quad \frac{x^2}{1} + \frac{y^2}{4} = 1 - \frac{k^2}{9} \quad \Rightarrow \frac{x^2}{1\left(1 - \frac{k^2}{9}\right)} + \frac{y^2}{4\left(1 - \frac{k^2}{9}\right)} = 1$$

Now, remember that the size of the quantity in the denominator of each term determines how long the ellipse is along that axis. For different values of k we see that the denominators can be bigger or smaller. This means that the elliptical cross-sections will get bigger or smaller depending on which $z = k$ plane we're in. For instance, the biggest that the two denominators can be occurs when $k = 0$. Or in other words, the largest ellitpical cross-section occurs in the xy-plane. As we get farther away from the xy-plane the elliptical cross sections get smaller. Let's plot a few cross sections in the three dimensions corresponding to $z = 0$, $z = \pm 1$ and $z = \pm 2$.



Now we have an even better idea of what the ellipsoid looks like. If we put them all together in one plot, we have the final form of the ellipsoid.



The Elliptic Paraboloid

Of course, sometimes we're not actually handed the equation and told to draw the surface. Consider the following example.

Example 46

Find the equation of the surface consisting of all points that are equidistance from the point $(0, 0, 1)$ and the plane $z = -1$.

We need to express the fact that the distance between an arbitrary point (x, y, z) and the point (0,0,1) is the same as the distance between (x, y, z) and the plane $z = -1$. We have

$$d_{point} = \sqrt{x^2 + y^2 + (z-1)^2} \quad \text{and} \quad d_{plane} = |z - (-1)| = |z + 1|$$

Setting them equal and squaring both sides we have

$$x^2 + y^2 + (z-1)^2 = (z+1)^2 \quad \Rightarrow \quad x^2 + y^2 - 4z = 0$$

Rearranging the expression above we have $z = \frac{1}{4}\left(x^2 + y^2\right)$.

Let's try to figure out what this surface looks like. If we look at it's cross-section in the xy-plane we have

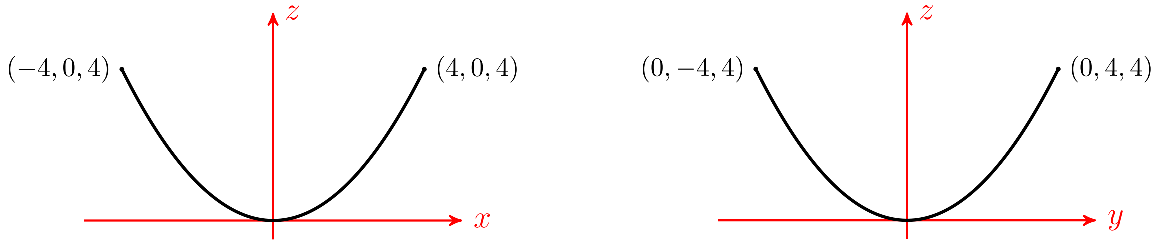
$$0 = \frac{1}{4}\left(x^2 + y^2\right)$$

Clearly the only xy-pair that satisfies this equation is the point $(0, 0)$

More interesting are the cross-sections in the xz- and yz-planes. For these we have, respectively,

$$z = \frac{x^2}{4} \qquad \text{and} \qquad z = \frac{y^2}{4}$$

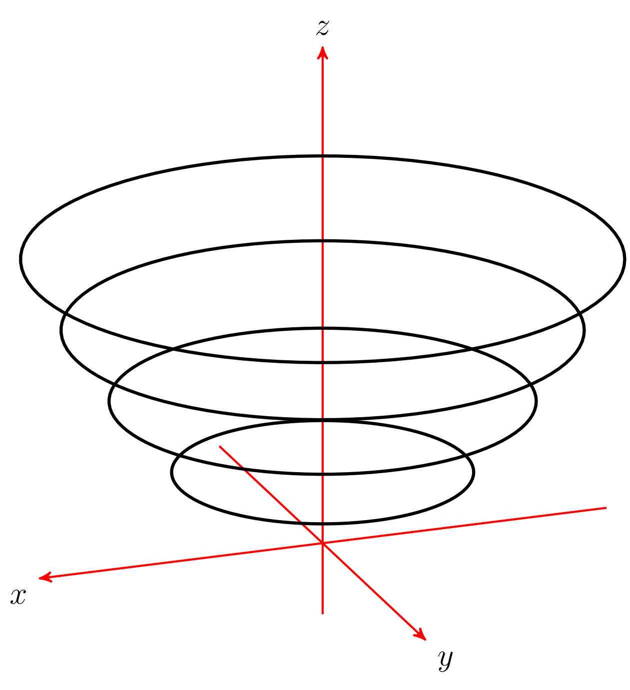
These are parabolas. If we graph them we have



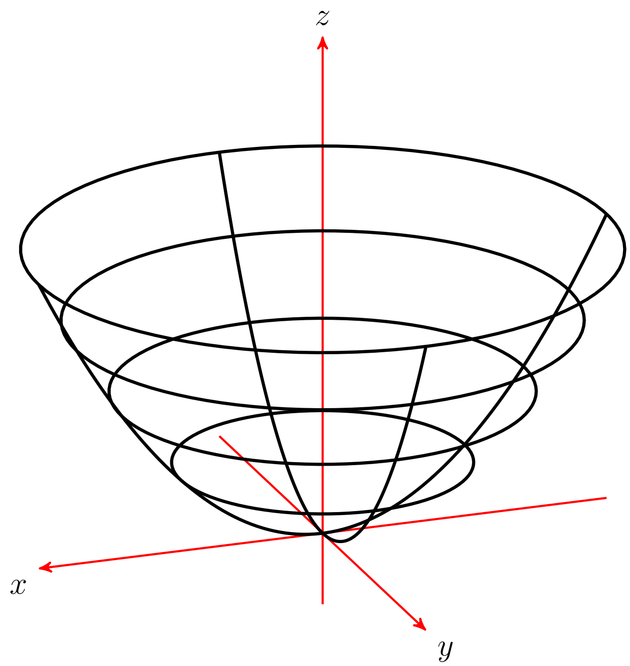
Finally, we can look at the cross-sections in various planes of parallel to the xy-plane of the form $z = k$. We have

$$x^2 + y^2 = 4k$$

From this expression we can tell various things. First, the cross-sections in the $z = k$ planes are all circles of radius $2\sqrt{k}$. Furthermore, we can tell that the surface does not exist below the xy-plane, because for values of $k < 0$ there are no values of x and y that satisfy the equation. Plotting several of these circles in three dimensions we have



Finally, we can put it all together by plotting the parabolic cross-sections in the xz- and yz-planes. We have



This surface is a *paraboloid*. Technically it's called an *elliptic paraboloid* because the cross-sections could be ellipses instead of circles if the coefficients in front of the x^2 and y^2 terms were different.

The Cone

Cones in three dimensional space have general equations of the form

$$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

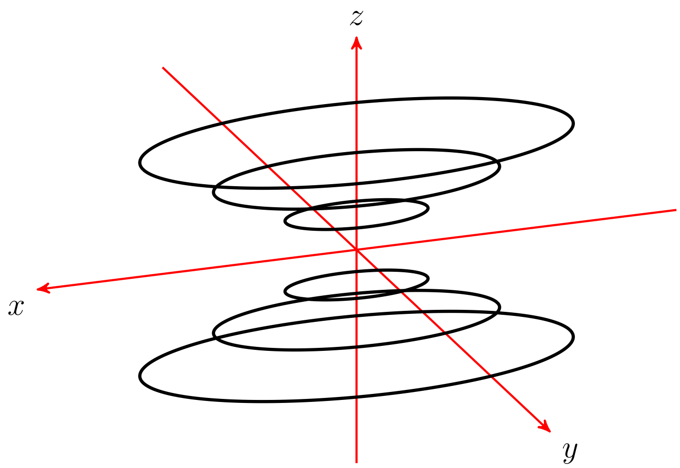
Let's look at the following specific case:

$$z^2 = \frac{x^2}{4} + y^2$$

The cross-sections in the $z = k$ planes look as follows:

$$\frac{x^2}{4} + y^2 = k^2 \quad \Rightarrow \quad \frac{x^2}{(2k)^2} + \frac{y^2}{k^2} = 1$$

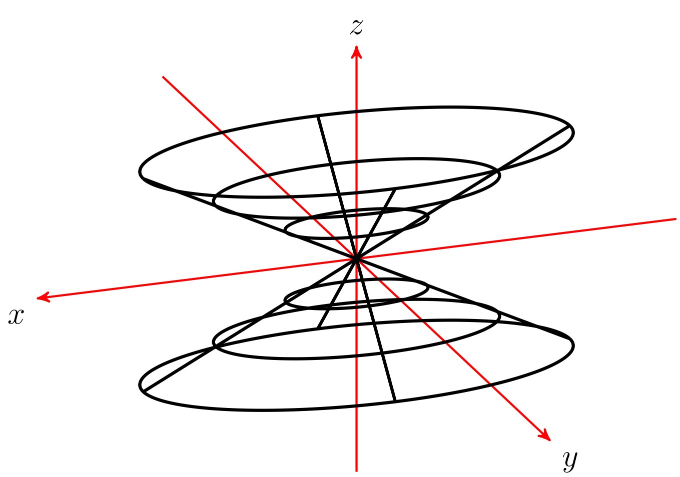
Note that these are ellipses that are 2 times wider in the x-direction than the y-direction. Notice also that because the z term appears everywhere as a square, the cone exists both above and below the xy-plane. Plotting several of these elliptic cross-sections we have



To get the cross-sections in the xz- and yz-planes we set $y = 0$ and $x = 0$ in the equation, respectivley. This gives

$$z^2 = \frac{x^2}{4} \quad \Rightarrow \quad z = \pm\frac{x}{2} \quad \text{and} \quad z^2 = y^2 \quad \Rightarrow \quad z = \pm y$$

These are just sets of criss-crossing lines that go through the origin. In the xz-plane the lines have slopes of $\pm 1/2$ and in the yz-plane the lines have slopes of ± 1. Adding these to the picture gives

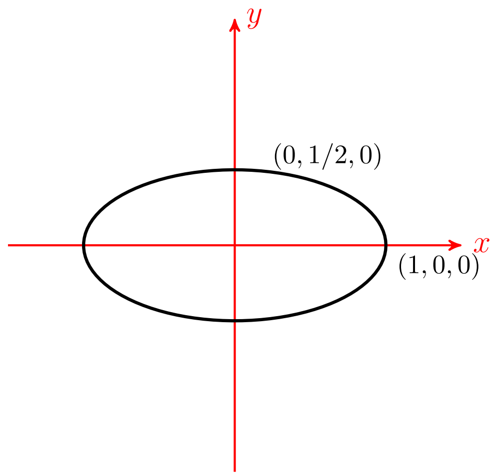


The Hyperboloid of One Sheet

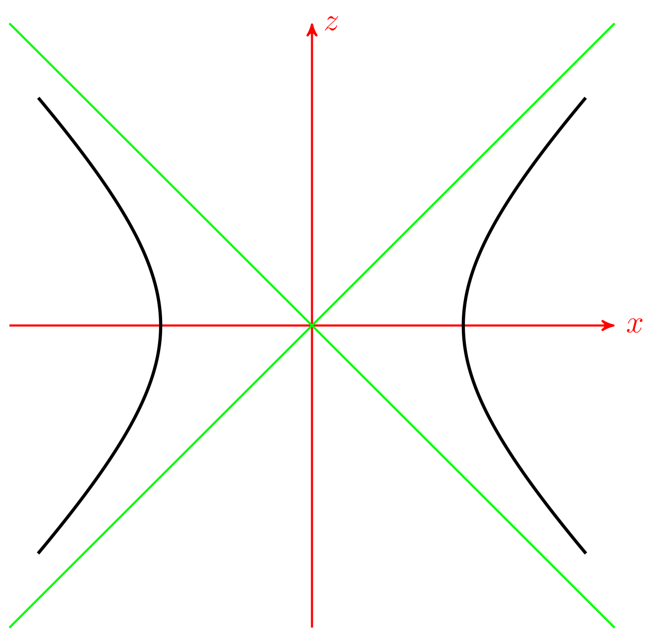
The so-called hyperboloid of one sheet has the general form $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$. One example of this is $x^2 + 4y^2 - z^2 = 1$.

We first plot the cross-sections of the quadratic surface in each of the coordinate planes. We have:

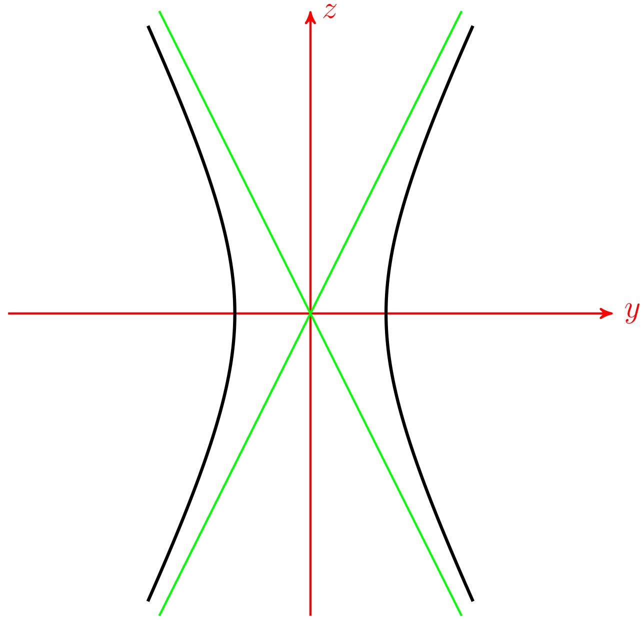
For the xy-plane $\quad \Rightarrow \quad z = 0 \quad \Rightarrow \quad x^2 + 4y^2 = 1 \quad \Rightarrow \quad x^2 + \dfrac{y^2}{(1/2)^2} = 1$ which is an ellipse centered at the origin with x-major axis length 1 and y-major axis length 1/2. It looks like



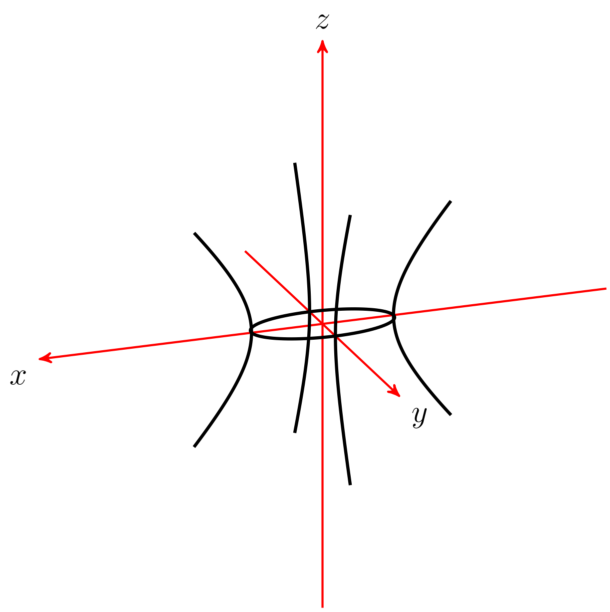
For the xz-plane $\quad \Rightarrow \quad y = 0 \quad \Rightarrow \quad x^2 - z^2 = 1$ which is a hyperbola. The hyperbola has asymptotes $z = \pm x$ and turns around at $x = \pm 1$.



For the yz-plane $\quad \Rightarrow \quad x = 0 \quad \Rightarrow \quad 4y^2 - z^2 = 1$ which is also a hyperbola. This hyperbola has asymptotes $z = \pm 2y$ and turns around at $y = \pm 1/2$.



Plotting these together on the 3D axis we have



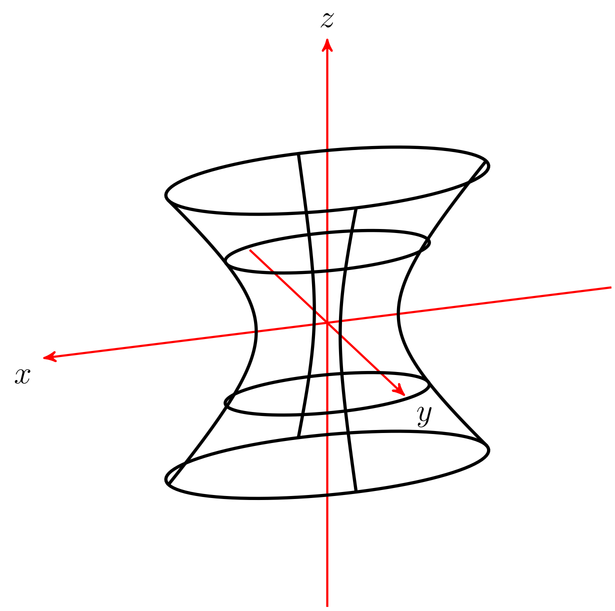
From this we can almost tell what the graph looks like, but we better plot some more z cross-sections just to be sure. Notice that if we set $z = k$ then the equation of the hyperboloid becomes

$$x^2 + 4y^2 = 1 + k^2 \quad \Rightarrow \quad \frac{x^2}{1+k^2} + \frac{x^2}{4\left(1+k^2\right)} = 1$$

These cross-sections are ellipses whose size depends on the size of k. Let's determine the cross-sections in the $z = \pm 1$ planes

$$z = \pm 1 \quad \Rightarrow \quad \frac{x^2}{2} + \frac{x^2}{8} = 1$$

which are both ellipses with x-major axis length $\sqrt{2}$ and y-major axis length $2\sqrt{2}$. Plotting these we have



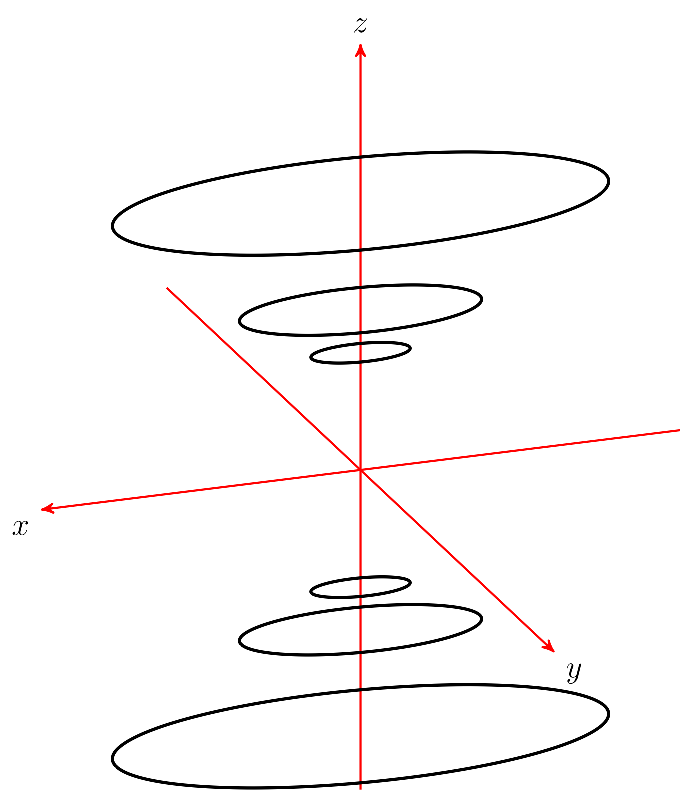
The Hyperboloid of Two Sheets

By tweaking the coefficients in the surface equation for the hyperboloid of one sheet, we obtain the hyperboloid of two sheets. Consider the example $z^2 - x^2 - 4y^2 = 1$.

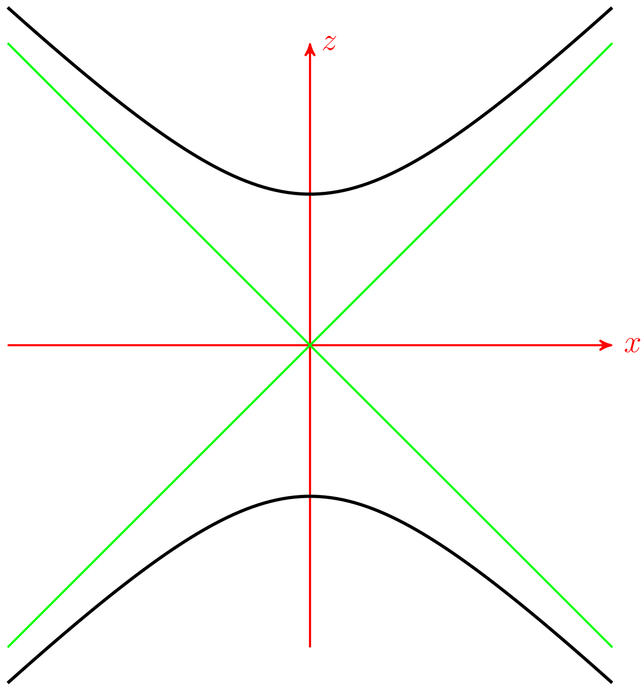
Let's first look at the cross-sections in the $z = k$ planes. We have

$$x^2 + 4y^2 = z^2 - 1$$

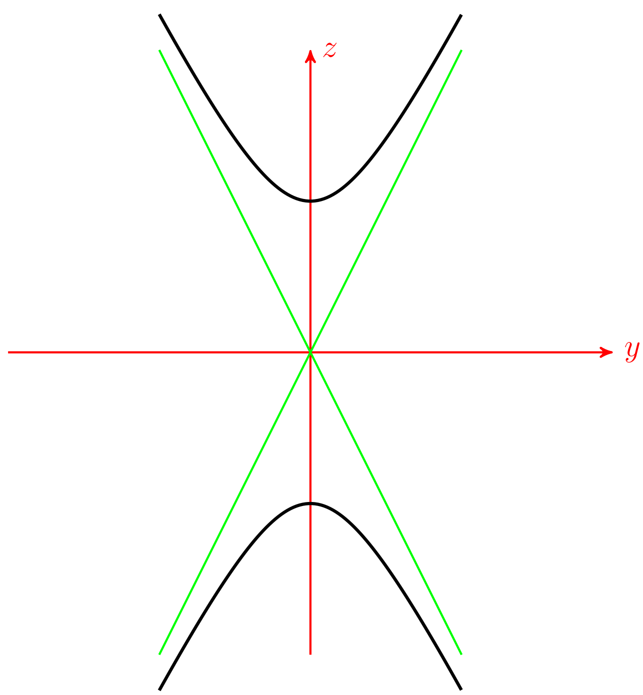
These are ellipes, but notice that the equation becomes undefined for z-values between -1 and 1 (noninclusize). This means that the surface only exists for z larger than 1 and smaller than -1. In other words, the surface is really two surfaces with a big gap in the middle. If we pick several values of z larger than 1 and smaller than -1 they look as follows:



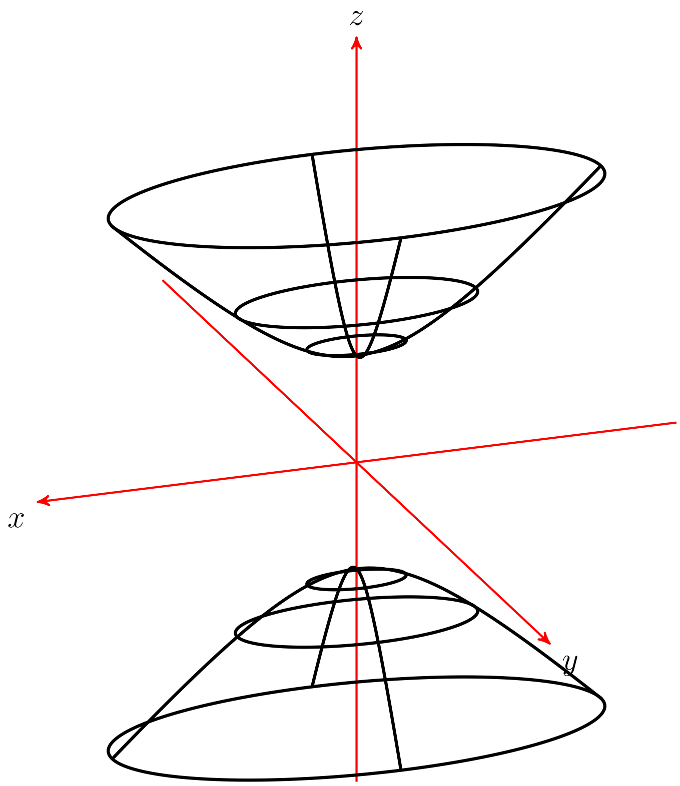
For the xz-plane $\quad \Rightarrow \quad y = 0 \quad \Rightarrow \quad z^2 - x^2 = 1$ which *almost* the same hyperbola we saw in the previous example, but with the roles of x and z swapped. The hyperbola again has asymptotes $z = \pm x$ and turns around at $z = \pm 1$.



For the yz-plane $\quad\Rightarrow\quad x = 0 \quad\Rightarrow\quad z^2 - 4y^2 = 1$ which is also a hyperbola. This hyperbola has asymptotes $z = \pm 2y$ and turns around at $z = \pm 1$.



Plotting these together with the $z = k$ cross-sections on the 3D axis we have



You can find (crude) animations of the drawing of each of the six quadratic surfaces discussed in the book at the following links:

- **Ellipsoid**: `https://youtu.be/gg6Wr33HmWg`
- **Elliptic Paraboloid**: `https://youtu.be/rk2cFG_PNiw`
- **Hyperbolic Paraboloid**: `https://youtu.be/uTNaNoLnbW0`
- **Cone**: `https://youtu.be/k8p4SY9ScVE`
- **Hyperboloid of One Sheet**: `https://youtu.be/IoUtzaK25Lc`
- **Hyperboloid of Two Sheets**: `https://youtu.be/GZXVxop_A2Y`

10.7 Vector Functions and Space Curves

Most functions that you've seen in Calculus have been what we call **scalar functions**. They take a point or a set of real numbers in 1D or 2D space and return a scalar back. For example, a surface is described in 3D by a function $f(x, y) = c$ which $\mathbb{R}^2 \mapsto \mathbb{R}$

Example 47

$f(x) = y = x^2$ Domain: all real #s $\mathbb{R} \mapsto \mathbb{R}$ $f(2) = 4$

$g(x, y) = z = x^2 + y^2$ Domain: all $(x, y) \in \mathbb{R}^2$ $\mathbb{R}^2 \mapsto \mathbb{R}$ $g(2, 1) = 5$

In this course we are mostly concerned with so-called **vector-valued** functions. These take a set of inputs and return a vector. If, for instance, the function takes a set of size m and returns a vector of size n we say $f : \mathbb{R}^m \mapsto \mathbb{R}^n$.

Example 48

$\mathbf{r}(t) = \left\langle t^3, \sqrt{t}, \ln(5 - t) \right\rangle$ Domain: $0 \le t \le 5$ or $[0, 5)$ $\mathbf{r} : \mathbb{R} \mapsto \mathbb{R}^3$

In general we have $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$

The individual functions in the vector-valued function $\mathbf{r}(t)$ are called the **component functions**. If we think of $\mathbf{r}(t)$ as a position vector of a point in three-space, then for a given t the component functions tell you the x, y, and z components of the point, respectively.

Example 49

We've already seen a vector function that defines a line through the point $P(2, 1, 0)$ and parallel to the vector $\mathbf{v} = \langle 1, -2, 3 \rangle$.

$$\mathbf{r}(t) = \langle 2 + t, 1 - 2t, 3t \rangle$$

So this was one parametrization of a curve – specifically a line. But we can parameterize all kinds of curves!

Parameterizing 2D Function

If we want to parametrize a standard function in 2D with the form $y = f(x)$ then the process is extremely straightforward. You simply set the component function associated with the x variable to t and the y variable to $f(t)$.

Example 50

Parameterize the curve $f(x) = x^2$.

We have $x = t$ and $y = f(t) = t^2$. Then the vector-valued function which parameterizes the curve is given by

$$\mathbf{r}\,(t) = \left\langle t, t^2 \right\rangle = t\mathbf{i} + t^2\mathbf{j}$$

Example 51

Parameterize the curve $g(x) = \sqrt{x}$

$$\mathbf{r}\,(t) = \left\langle t, \sqrt{t} \right\rangle = t\mathbf{i} + \sqrt{t}\mathbf{j} \quad \text{for} \quad t \geq 0.$$

Example 52

What shape in 2D does the following vector-valued function parameterize?

$$\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j}$$

From the parameterization, we know that for a given t value, the position vector $\mathbf{r}\,(t)$ points at the point $(x = 2\cos t, y = 2\sin t)$. So now we can start playing games with this to see if we can make it look like a function in terms of x and y that we are familiar with. How about if we square both of them and add?

$$(2\cos t)^2 + (2\sin t)^2 = 4\cos^2 t + 4\sin^2 t = 4\left(\cos^2 t + \sin^2 t\right) = 4$$

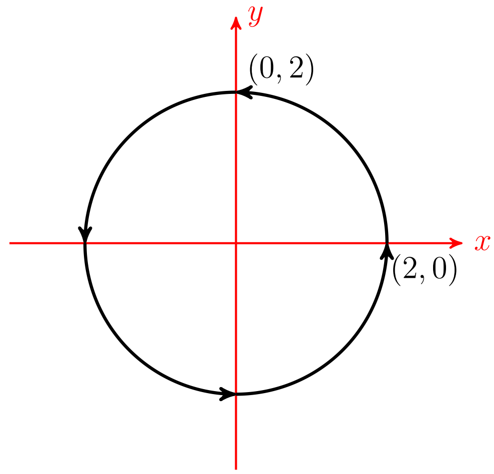
So the component functions satisfy the equation $x^2 + y^2 = 4$ which is a circle of radius 2 centered at the origin.

OK, so suppose that we let t vary from 0 to 2π. Where does the curve start? In what direction does it traverse the circle?

Let's start plugging in points...

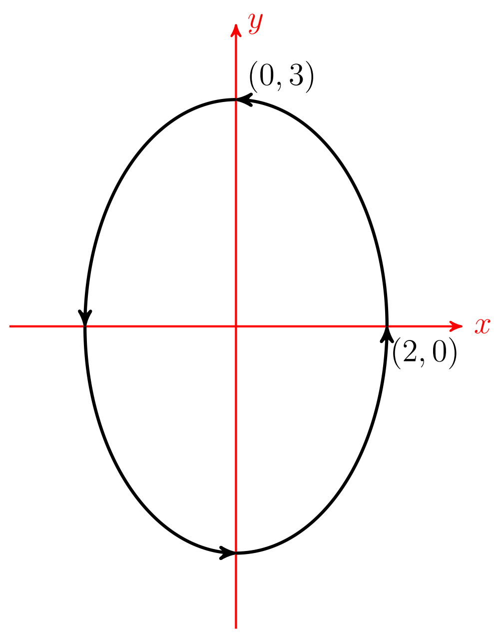
t	$\mathbf{r}\,(t)$
0	$(2, 0)$
$\pi/2$	$(0, 2)$
π	$(-2, 0)$
$3\pi/2$	$(0, -2)$

It is clear from looking at the data that the curve starts at the point $(2, 0)$ and traverses the circle in the clockwise direction.



Example 53

Find the parameterization of the curve that traverses clockwise the ellipse centered at the origin with x-major axis length 2 and y-major axis length 3.



which is described in cartesian coordinates by $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$.

Now we want to find functions of t to represent each of the x and y components, that when I plug them into the equation for the ellipse I get a true statement. Looking at the demonimators we see that one choice that works is

$$x = f(t) = 2\cos t \quad \text{and} \quad y = g(t) = 3\sin t$$

So the parameterization of the ellipse is given by $\mathbf{r}\,(t) = \langle 2\cos t, 3\sin t\rangle = 2\cos t\mathbf{i} + 3\sin t\mathbf{j}$. We can then plug in points of increasing t and confirm that this parameterization does in fact traverse the ellipse in the counterclockwise direction.

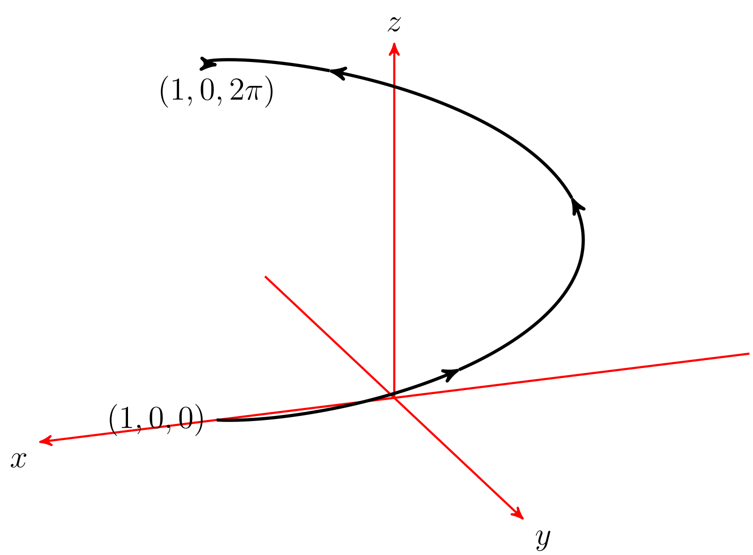
It is also useful to parameterize curves in three dimensional space.

Example 54

What curve does the following vector-valued function represent: $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$?

If we look at just the x and y components of the vector, it is clear that from the top looking down the curve is a circle with radius 1 centered at the origin. What does the z component of the vector do? It gives the circle some height! As t increases the circle creeps off the ground to become a **helix**!

Notice that when $t = 0$ we're at the point $(1, 0, 0)$ and at $t = 2\pi$ we're at $(1, 0, 2\pi)$. So as we traverse the helix one revolution in the counterclockwise direction the helix rises to a height of 2π.



OK, let's try to use what we know about parameterizing curves to solve a more interesting problem.

Example 55

Find a parameteric equation for the curve of intersection of the cylinder $x^2 + y^2 = 4$ and the plane $x + y + z = 1$.

We want to find a vector-valued function $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ where all values of $\mathbf{r}(t)$ satisfy both the cyclinder and the plane. With these it's usually easiest to try chipping away at the answer by finding a few of the components and then solving for the last one.

Since the curve necessarily must lie on the cylinder, we can immediately see that the x and y components of the parameterization must lie on the circle $x^2 + y^2 = 4$. From the previous example we know that this means that

$$x = 2\cos t \quad \text{and} \quad y = 2\sin t$$

Notice that as long as x and y are described as above, the curve will lie on the cylinder no matter what the parameterization of z is. Because of this, the cylinder can provide us no information about what z is. Now we turn to the plane. In order for the curve to be a curve of intersection it must satisfy the equation of the plane. Since we already know what x and y are it's very easy to plug them into the equation of the plane, and determine what z must be in order to be on the plane.

$$x + y + z = 1 \quad \Rightarrow \quad 2\cos t + 2\sin t + z = 1 \quad \Rightarrow \quad z = 1 - 2\cos t - 2\sin t$$

Thus the curve of intersection is given by $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, 1 - 2\cos t - 2\sin t\rangle$

Calculus with Vector-Valued Functions

The limit of a vector-valued function $\mathbf{r}(t)$ is simply the limit of its components, i.e.

$$\lim_{t\to a} \mathbf{r}(t) = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \right\rangle$$

Since derivatives of functions are defined in terms of limits, it's believable that since limits of vector-valued functions are computed componentwise, derivatives are computed componentwise as well. This is of course true and we have

$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} = \langle f'(t), g'(t), h'(t)\rangle$$

Example 56

Find $\mathbf{r}'(t)$ for $\mathbf{r}(t) = \langle t^3, e^t, \sin t\rangle$

We have $\mathbf{r}'(t) = \langle 3t^2, e^t, \cos t\rangle$

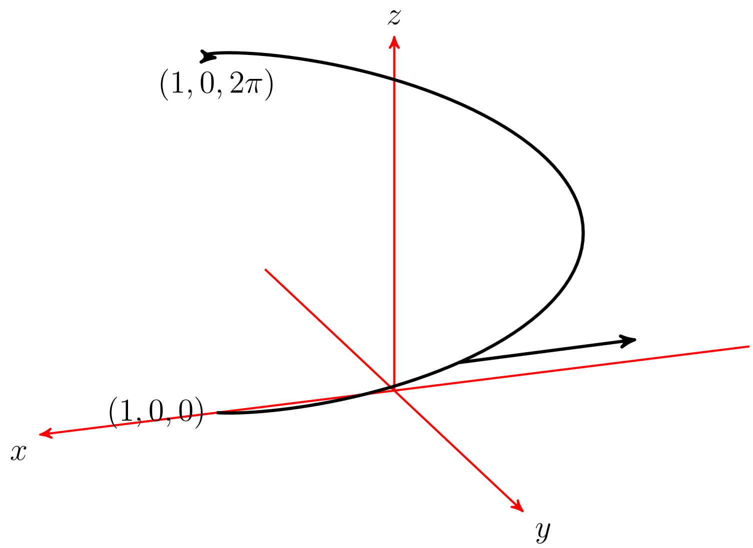
Recall that in Calculus I you learned that the derivative of a function at a point gives the slope of the tangent vector to the function at that point. Can you guess what the derivative of a parameterized curve $\mathbf{r}(t)$ gives you?

If you guessed that it's the tangent vector then you are correct!

Example 57

Find the vector tangent to the helix $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$ at the point $t = \pi/2$.

We have $\mathbf{r}'(t) = \langle -\sin t, \cos t, 1\rangle$ which gives a tangent vector of $\mathbf{r}'(\pi/2) = \langle -1, 0, 1\rangle$ at $t = \pi/2$.



Often times it's useful to express the vector tangent to a curve as a unit vector. When this happens we denote it by $\mathbf{T}(t)$ and call it the **unit tangent vector**. It is defined by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

For the previous example we have $\mathbf{T}(t) = \dfrac{\langle -1, 0, 1\rangle}{\sqrt{2}}$

Differentiation Rules for Vector-Valued Functions

1. $\dfrac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$

2. $\dfrac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$

3. $\dfrac{d}{dt}[f(t)\,\mathbf{u}(t)] = f'(t)\,\mathbf{u}(t) + f(t)\,\mathbf{u}'(t)$

4. $\dfrac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$

5. $\dfrac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$

6. $\dfrac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\,\mathbf{u}'(f(t))$

Integration of Vector-Valued Functions

Shockingly, integration of a vector-valued function is also performed componentwise, both for definite and indefinite integrals.

$$\int \mathbf{r}\,(t)\ dt = \left(\int f(t)\ dt\right)\mathbf{i} + \left(\int g(t)\ dt\right)\mathbf{j} + \left(\int h(t)\ dt\right)\mathbf{k}$$

Example 58

Compute the following integral: $\displaystyle\int_a^b \left(t\mathbf{i} - t^3\mathbf{j} + 3t^5\mathbf{k}\right)\ dt$

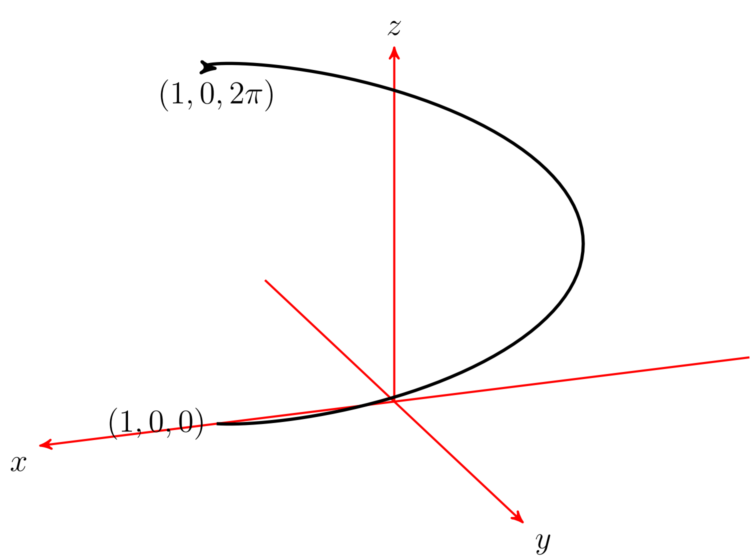
$$\int_a^b \left(t\mathbf{i} - t^3\mathbf{j} + 3t^5\mathbf{k}\right)\ dt = \left.\frac{t^2}{2}\right|_0^2 \mathbf{i} + \left.\frac{t^4}{4}\right|_0^2 \mathbf{j} + \left.\frac{t^6}{2}\right|_0^2 \mathbf{k} = 2\mathbf{i} + 4\mathbf{j} + 32\mathbf{k}$$

10.8 Arc Length and Curvature

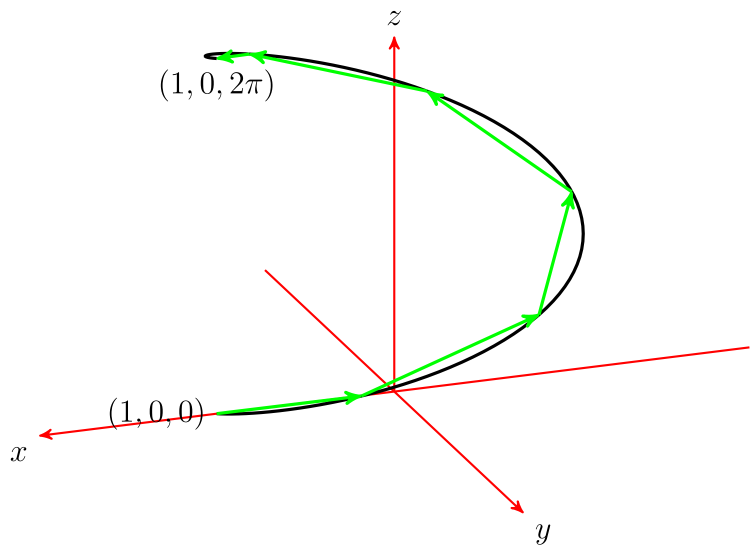
Last time we talked about how to parameterize all kinds of curves in three dimensionsal space. Our favorite example was the helix described by

$$\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$$

which, for $0 \le t \le 2\pi$ looks like



Now suppose we want to find the length arc made by the helix in one revolution from $t = 0$ to $t = 2\pi$. You've seen things like this before in both Calc I and Calc II but let's go over it again. As with all integration, we start by breaking the thing we're trying to measure into discrete chunks.



Now, the length of the helix is approximated by the sum of the length of each one of the little chunks, which we'll refer to as Δs_k. Let's look at one arbitrary Δs_k which starts at time t and ends at time $t + \Delta t$. If we represent the start and endpoints of the chunk by position vectors we see that the first point is at $\mathbf{r}(t)$ and the second point is at $\mathbf{r}(t + \Delta t)$. Then the length of a little chunk is given by

$$\Delta s_k = |\mathbf{r}(t + \Delta t) - \mathbf{r}(t)|$$

Another way to write this is

$$\Delta s_k = \left| \frac{\mathbf{r}(t_k + \Delta t) - \mathbf{r}(t)}{\Delta t} \right| \Delta t$$

Then, if we say there are n of these little chunks and we add them up, we get

$$L_n = \sum_{k=1}^{n} \left| \frac{\mathbf{r}(t_k + \Delta t) - \mathbf{r}(t)}{\Delta t} \right| \Delta t$$

Now, the approximation gets better as we increase the number of chunks we divide the curve up into. Taking this to the extreme we let the number of little chunks go to infinity, and we get

$$L = \lim_{n \to \infty} \sum_{k=1}^{n} \left| \frac{\mathbf{r}(t_k + \Delta t) - \mathbf{r}(t)}{\Delta t} \right| \Delta t = \int_a^b |\mathbf{r}'(t)|\, dt$$

$$= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt = \int_a^b \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2 + \left[\frac{dz}{dt}\right]^2} + dt$$

Example 59

Find the arc length of the helix $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$.

We have

$$\mathbf{r}'(t) = \langle -\sin t, \cos t, 1 \rangle \quad \Rightarrow \quad |\mathbf{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

Then

$$L = \int_0^{2\pi} \sqrt{2}\, dt = 2\sqrt{2}\pi$$

Parameterization by Arc Length

Recall that we noticed last time that there are numerous ways that we can parameterize the same curve. For instance, another parameterization of the same helix is given by

$$\mathbf{r}_2(t) = \langle \cos(2t), \sin(2t), 2t \rangle \quad \text{for} \quad 0 \leq t \leq \pi$$

Now, with all these different parameterizations lying around, it would be nice if we one that was standardized. One way to do this is to parameterize by arc length. We can define arc length as a function of the parameter t by

$$s(t) = \int_a^t |\mathbf{r}'(u)|\, du$$

Notice that this expression says that, given a starting position at time $t = a$, the function $s(t)$ returns the distance along the arc from time a to time t.

Example 60

Let's compute the arc length function for the helix starting at time $t = 0$. We have

$$s(t) = \int_0^t \sqrt{\sin^2 u + \cos^2 u + 1}\, du = \int_0^t \sqrt{2}\, du = \sqrt{2}t$$

This tells us that, for the original parameterization, when $t = 1$ we have traveled $s(1) = \sqrt{2}$ units along the arc. Now, solving for t in terms of $s(t)$ we can reparameterize the curve in terms of arc length

$$s = \sqrt{2}t \quad \Rightarrow \quad t = s/\sqrt{2} \quad \Rightarrow \quad \mathbf{r}(s(t)) = \left\langle \cos\left(s/\sqrt{2}\right), \cos\left(s/\sqrt{2}\right), s/\sqrt{2} \right\rangle$$

So, with the original paramterization, we chose a parameter t and that told us which point on the helix we were at. Now, we can choose an arc length s, interpreted as the distance traveled along the arc, to tell us which point on the curve we're at.

The arc length function also gives us our first physical interpretation of the parameterization $\mathbf{r}(t)$. Notice, by applying the Fundamental Theorem of Calculus to the arc length function we obtain

$$\frac{ds}{dt} = |\mathbf{r}'(t)|$$

The quantity ds/dt is the rate at which the distance traveled along the curve is changing, also known as **speed**. So, the magnitude of the derivative of the position vector tells us the speed at which the particle is traveling. This will become more apparent next time when we talk about velocity and acceleration.

Curvature

Another important geometric measurement for a space curve is its curvature. Recall that the vector $\mathbf{r}'(t)$ gives us a vector tangent to the curve $\mathbf{r}(t)$ at time t. Then to get the unit vector that is tangent to the curve we compute

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

We define the curvature of a curve $\mathbf{r}(t)$ as the magnitude of the rate of change of the unit tangent vector with respect to arc length. Or

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right|$$

Now, with the arc length term in there, it looks like we might have to do a lot of work to compute curvature. But we can play some games and reduce this to something that is more manageable. Notice that

$$\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}}{dt}\frac{dt}{ds} = \frac{d\mathbf{T}/dt}{ds/dt} \qquad \Rightarrow \qquad \kappa = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$$

Example 61

Let's compute the curvature of a circle of radius a in the xy-plane. The curve is parameterized by $\mathbf{r}(t) = \langle a\cos t, a\sin t\rangle$. Then we have

$$\mathbf{r}'(t) = \langle -a\sin t, a\cos t\rangle \qquad \Rightarrow \qquad |\mathbf{r}'(t)| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = a$$

$$\Rightarrow \quad \mathbf{T} = \frac{\langle a\cos t, a\sin t\rangle}{a} = \langle \cos t, \sin t\rangle \quad \Rightarrow \quad \mathbf{T}'(t) = \langle -\sin t, \cos t\rangle \quad \Rightarrow \quad |\mathbf{T}'(t)| = 1$$

Then

$$\kappa = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{1}{a}$$

Notice that in this case the curvature of the circle is constant for all points on the circle and is equal to the reciprocal of the radius. This should make intuitive sense because the smaller the radius of the circle the more the curve is curving.

There are several other formulas for curvature that are equivalent. We will state them here without proof, but the proofs can be found in your textbook.

$$\kappa = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

In the special case that the curve is a planar curve described by $y = f(x)$ we can also use

$$\kappa = \frac{f''(x)}{\left[1 + (f'(x))^2\right]^{3/2}}$$

The TNB Frame

For general space it's extremely convenient to have a framework that decomposes three dimensional space into orthogonal vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. But when studying particle motion, it's also convenient to have a set of three orthogonal vectors that move with the particle reference frame. We already have one of those vectors, namely the unit tangent vector. The next orthogonal vector is can be found using a consequence of the following fact:

Claim: A vector of constant length is always orthogonal to its derivative.

Proof: Assume that $\mathbf{u}$ has constant length. Then

$$0 = \frac{d}{ds}\left(|\mathbf{u}|^2\right) = \frac{d}{ds}\mathbf{u} \cdot \mathbf{u} = \frac{d\mathbf{u}}{ds} \cdot \mathbf{u} + \mathbf{u} \cdot \frac{d\mathbf{u}}{ds} = 2\left(\mathbf{u} \cdot \frac{d\mathbf{u}}{ds}\right) \quad \Rightarrow \mathbf{u} \perp \frac{d\mathbf{u}}{ds}$$

Since $\mathbf{T}(t)$ is a unit vector it has constant length. Then, to get a vector orthogonal to $\mathbf{T}$ we can take it's derivative with respect to arc length. However, the resultant vector won't be a unit vector. To make it a unit vector we can scale it by the reciprocal of the curvature. The resultant vector is what we call the **unit normal vector** to the curve.

$$\mathbf{N}(t) = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}$$

Of course this is cumbersome to compute because we'd have to parameterize $\mathbf{T}$ in terms of arc length. But again we can play some games to get $\mathbf{N}$ into a simpler form. We have

$$\mathbf{N}(t) = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{(d\mathbf{T}/dt)\,(dt/ds)}{|(d\mathbf{T}/dt)\,(dt/ds)|} = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

Where here the two dt/ds terms cancel because we've defined ds/dt to be position.

Example 62

Compute the unit tangent vector **T** and the unit normal vector **N** for the curve $\mathbf{r}(t) = \langle 3\sin t, 3\cos t\rangle$.

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\langle 3\cos t, -3\sin t\rangle}{|\langle 3\cos t, -3\sin t\rangle|} = \langle \cos t, -\sin t\rangle$$

Then we have

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{\langle -\sin t, \cos t\rangle}{|\langle -\sin t, \cos t\rangle|} = \langle -\sin t, \cos t\rangle$$

The final unit vector that makes up the orthogonal framework for the paricle's motion is called the **binormal vector**. It is the unit vector that is orthogonal to both **T** and **N**. This is easily computed using the cross-product:

$$\mathbf{B}(t) = \mathbf{T} \times \mathbf{N}$$

Note that the order here is chosen by convention.

Example 63

Calculate the binormal vector for the curve in the previous example.

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = (\cos t\mathbf{i} + \sin t\mathbf{j}) \times (-\sin t\mathbf{i} + \cos t\mathbf{j}) =$$

$$\cos t \sin t\,(\mathbf{i} \times \mathbf{i}) + \cos^2 t\,(\mathbf{i} \times \mathbf{j}) - \sin^2 (\mathbf{j} \times \mathbf{i}) + \sin t \cos t\,(\mathbf{j} \times \mathbf{j}) =$$

$$\cos^2 t\mathbf{k} + \sin^2 t\mathbf{k} = \mathbf{k}$$

The TNB frame has a particular interpretation with respect to the particle's motion. The unit tangent **T** points in the direction the particle is traveling, the unit normal **N** points in the direction the particle is turning, and the unit binormal vector points directly up from the particle's perspective.

We can derive two important planes from the TNB frame. The first is called the **oscillating plane**, and it is the plane in which all turning happens. It is the plane formed by the vectors $\mathbf{T}$ and $\mathbf{N}$. Of course, if we want to write down an equation for the plane, we need a vector normal to the plane. This is conveniently provided by the binormal vector $\mathbf{B}$. The second plane is formed by the $\mathbf{N}$ and $\mathbf{B}$ vectors. It is called the **normal plane**, and it's the plane in which all twisting happens. The vector normal to the normal plane is the unit tangent vector $\mathbf{T}$.

10.9 Motion in Space: Velocity and Acceleration

We've already surmized that the derivative of the position vector $\mathbf{r}(t)$ with respect to time is the velocity of a particle. It should not be a surprise then that the acceleration of the particle is given by the second derivative of position.

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$$

Example 64

Find a particles velocity, speed, and acceleration if its position vector is given by $\mathbf{r}(t) = \left\langle \sqrt{2}t, e^t, e^{-t} \right\rangle$.

We have the velocity is given by $\mathbf{v}(t) = \mathbf{r}'(t) = \left\langle \sqrt{2}, e^t, -e^{-t} \right\rangle$

and acceleration by $\mathbf{a}(t) = \mathbf{r}''(t) = \langle 0, e^t, e^{-t} \rangle$.

The speed is given by $\nu = |\mathbf{v}(t)| = \sqrt{2 + e^{2t} + e^{-2t}} = \sqrt{\left(e^t + e^{-t}\right)^2} = e^t + e^{-t}$

Example 65

Find the position and velocity vectors of a particle with $\mathbf{a}(t) = \langle 2t, \sin t, \cos 2t \rangle$ with $\mathbf{v}\,(0) = \langle 1, 0, 0 \rangle$ and $\mathbf{r}\,(0) = \langle 0, 1, 0 \rangle$.

We integrate the acceleration vector to obtain the velocity vector

$$\mathbf{v}(t) = \int \mathbf{a}(t)\ dt = \int \langle 2t, \sin t, \cos 2t \rangle\, dt = \left\langle t^2, -\cos t, \frac{\sin 2t}{2} \right\rangle + \mathbf{C_1}$$

where here $\mathbf{C_1}$ is a vector of (possibly different) constants. To determine those constants we can use the initial condition on the velocity.

$$\mathbf{v}(0) = \langle 1, 0, 0 \rangle = \langle 0, -1, 0 \rangle + \mathbf{C}_1 \quad \Rightarrow \quad \mathbf{C}_1 = \langle 1, 1, 0 \rangle$$

Then we have $\mathbf{v}(t) = \langle t^2+1, -\cos t+1, \frac{\sin 2t}{2}\rangle$

We can then get the position vector by integrating the velocity vector

$$\mathbf{r}(t) = \int \mathbf{v}(t)\ dt = \int \left\langle t^2+1, -\cos t+1, \frac{\sin 2t}{2}\right\rangle\ dt = \left\langle \frac{t^3}{3}+t, t-\sin t, \frac{-\cos 2t}{4}\right\rangle + \mathbf{C_2}$$

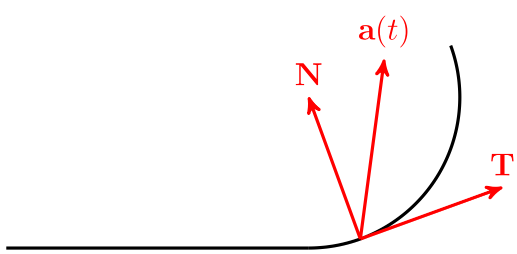
Then, using the initial condition on the position we have

$$\mathbf{r}(0) = \langle 0,1,0\rangle = \left\langle 0,0,-\frac{1}{4}\right\rangle + \mathbf{C}_2 \quad \Rightarrow \quad \mathbf{C}_2 = \left\langle 0,1,\frac{1}{4}\right\rangle$$

which gives us the final position vector as $\mathbf{r}(t) = \left\langle \frac{t^3}{3}+t, t-\sin t+1, \frac{1-\cos 2t}{4}\right\rangle$

Tangential and Normal Components of Acceleration

Imagine you're driving in a car and you step on the gas while exiting a turn. The resulting acceleration has two components. One component comes from you stepping on the gas, and occurs in the tangential direction to the curve. The second comes from the centripital acceleration due to the turn and points towards the center of the oscillating circle of the curve.



We decompose the acceleration vector into its tangential and normal components.

$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$$

The tangential component of $\mathbf{a}$, namely a_T is the rate of change of the **length** of $\mathbf{v}$. The normal component is the rate of change of the **direction** of $\mathbf{v}$. To see this

$$\begin{aligned}
\mathbf{a} &= \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left(\mathbf{T}\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\frac{d\mathbf{T}}{dt} \\
&= \nu'\mathbf{T} + \nu\left(\frac{d\mathbf{T}}{ds}\frac{ds}{dt}\right) = \nu'\,\mathbf{T} + \nu^2\frac{d\mathbf{T}}{ds} \\
&= \nu'\,\mathbf{T} + \kappa\nu^2\mathbf{N}
\end{aligned}$$

So $\mathbf{a}(t) = a_T\mathbf{T} + a_N\mathbf{N}$ where $a_T = \nu'$ and $a_N = \kappa\nu^2$.

Example 66

Find the tangential and normal components of the acceleration of the helix $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$.

We have $\mathbf{v} = \langle -\sin t, \cos t, 1\rangle \quad \Rightarrow \quad \nu = |\mathbf{v}| = \sqrt{2}$.

To compute the curvature we need the unit tangent vector $\mathbf{T} = \frac{1}{\sqrt{2}}\langle -\sin t, \cos t, 1\rangle$. Then

$$\kappa = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{\frac{1}{\sqrt{2}}\,|\langle -\cos t, -\sin t, 0\rangle|}{\sqrt{2}} = \frac{1}{2}$$

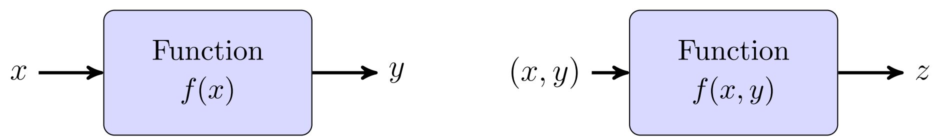
Then $a_T = 0$ and $a_N = \frac{1}{2}\left(\sqrt{2}\right)^2 = 1$ or we could write $\mathbf{a}(t) = \mathbf{N}$.

Chapter 11: Partial Derivatives

11.1 Functions of Several Variables

One Dimension

Two Dimensions



Domain: $\forall x$ vals
Range: $\forall y$ vals

Domain: $\forall\,(x, y)$ pairs
Range: $\forall z$ vals

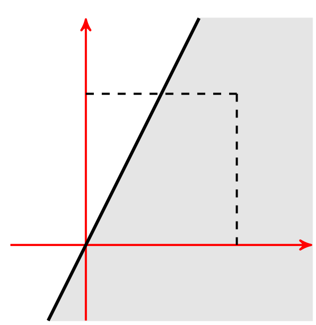
In the function $z = f(x, y)$ the x and y variables are called the **independent** variables and the z variable is the **dependent** variable.

Example 1

Consider the function $f(x, y) = \sqrt{2x - y}$.

The function is well-defined provided that $2x - y \geq 0$.

We write the domain as $D = \{(x, y) \mid 2x - y \geq 0\}$.



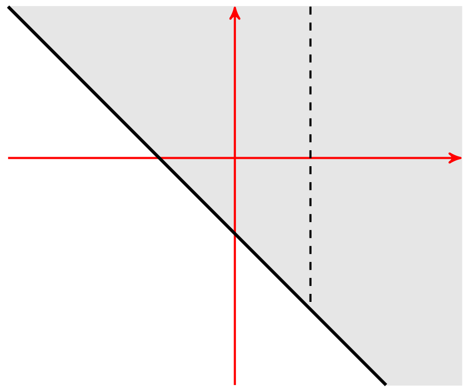
Chris Ketelsen
APPM 2350
Chapter 11
November 19, 2015

Example 2

Consider the function $f(x, y) = \dfrac{\sqrt{x+y+1}}{x-1}$.

The function is well-defined provided that $x + y + 1 \geq 0$ and $x \neq 1$.

We write the domain as $D = \{(x, y) \mid x + y + 1 \geq 0 \text{ and } x \neq 1\}$.



Graphs

Def: If f is a function of two variables with domain D, then the **graph** of f is a set of all points $(x, y, z) \in \mathbb{R}^3$ such that $z = f(x, y)$ and $(x, y) \in D$.

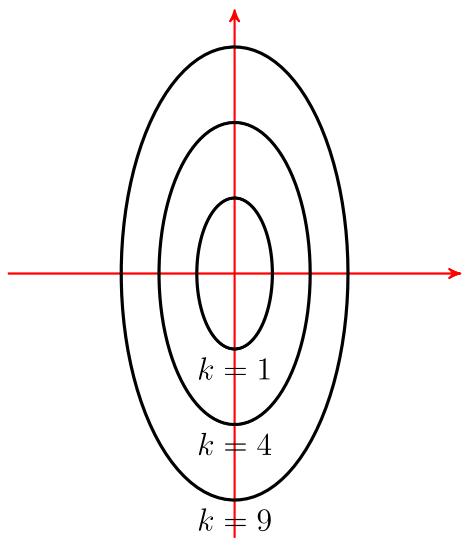
We've already seen graphs of some functions of two variables, namely, quadratic surfaces. We've already practiced drawing quadratic surfaces in 3D. Another helpful method for visualizing functions of two variables is with the use of level curves.

Def: The **level curves** of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant (in the range of f).

Example 3

Plot the level curves of of the function $z = f(x,y) = 4x^2 + y^2$. We have

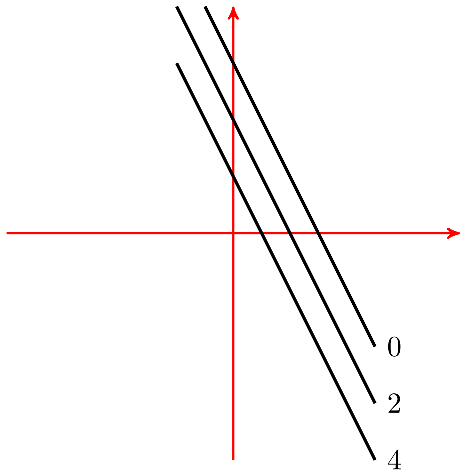
$$\begin{array}{lll} k=1 & 4x^2+y^2=1 & \Rightarrow \quad \dfrac{x^2}{1/4}+\dfrac{y^2}{1}=1 \\ k=4 & 4x^2+y^2=4 & \Rightarrow \quad \dfrac{x^2}{1}+\dfrac{y^2}{4}=1 \\ k=9 & 4x^2+y^2=9 & \Rightarrow \quad \dfrac{x^2}{9/4}+\dfrac{y^2}{9}=1 \end{array}$$

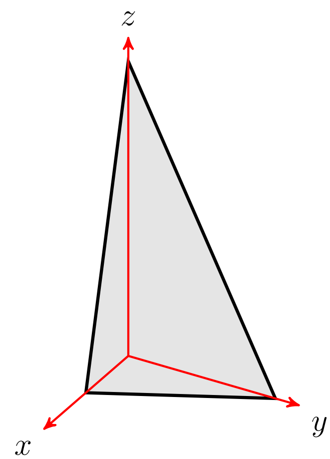


Example 4

Plot the level curves of the plane given by $4x + 2y + z = 6 \quad \Leftrightarrow \quad f(x,y) = 6 - 4x - 2y$

$$\begin{array}{llll} k=0 & 6-4x-2y=0 & \Rightarrow & y=3-2x \\ k=2 & 6-4x-2y=2 & \Rightarrow & y=2-2x \\ k=0 & 6-4x-2y=4 & \Rightarrow & y=1-2x \end{array}$$

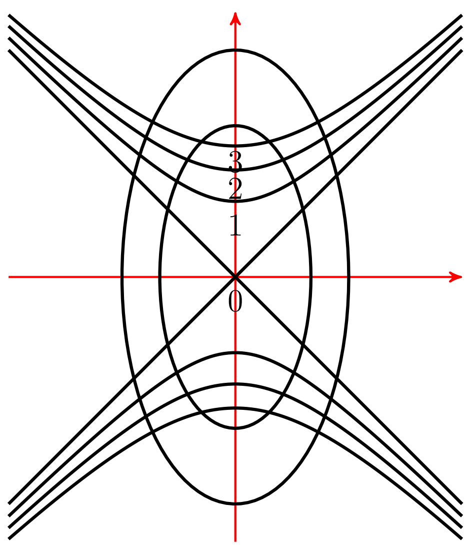




Example 5

Plot the level curves of of the function $z = f(x, y) = \sqrt{y^2 - x^2}$. We have

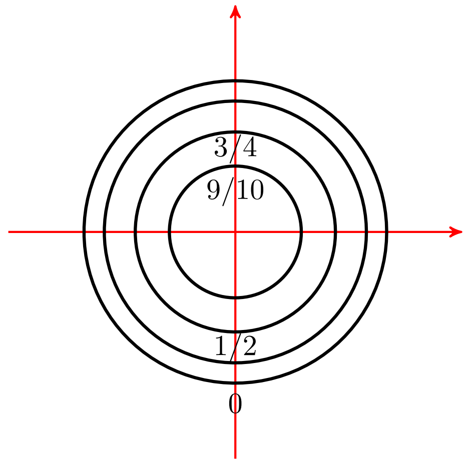
$$\begin{array}{llll} k=0 & \sqrt{y^2-x^2}=0 & \Rightarrow & y=\pm x \\ k=1 & \sqrt{y^2-x^2}=1 & \Rightarrow & y^2-x^2=1 \\ k=2 & \sqrt{y^2-x^2}=2 & \Rightarrow & y^2-x^2=2 \\ k=3 & \sqrt{y^2-x^2}=2 & \Rightarrow & y^2-x^2=3 \end{array}$$



Note: Recall that we test to see if the graph of a function in one dimension represents a function using the **vertical line test**. For graphs of functions of two variables we can also apply the vertical line test. In this case though the vertical line runs parallel to the z-axis.

Example 6

The graph of the sphere $x^2 + y^2 + z^2 = 1$ does not represent a function since it does not pass the vertical line test. On the other hand, the top half of the sphere does pass the vertical line test and the function is represented by $z = f(x,y) = \sqrt{1 - x^2 - y^2}$. It's level curves are concentric circles centered at the origin.



Functions of Three Variables

We can also have functions with three independent variables. Consider

$$h(x,y,z) = x^2 + y^2 + z^2$$

These functions are impossible to draw in three dimensional space, since technically they live in four dimensional space. But, we can represent them using a method similar to level curves. In this case, setting the function equal to a constant gives

$$h(x,y,z) = x^2 + y^2 + z^2 = k$$

which form **level surfaces**, e.g.

$$\begin{aligned} x^2 + y^2 + z^2 = 0 &\Rightarrow \text{ The point } (0,0,0) \\ x^2 + y^2 + z^2 = 1 &\Rightarrow \text{ The sphere with radius 1} \\ x^2 + y^2 + z^2 = 4 &\Rightarrow \text{ The sphere with radius 2} \end{aligned}$$

11.2 Limits and Continuity

We want to compute limits of functions of two variables, which look like

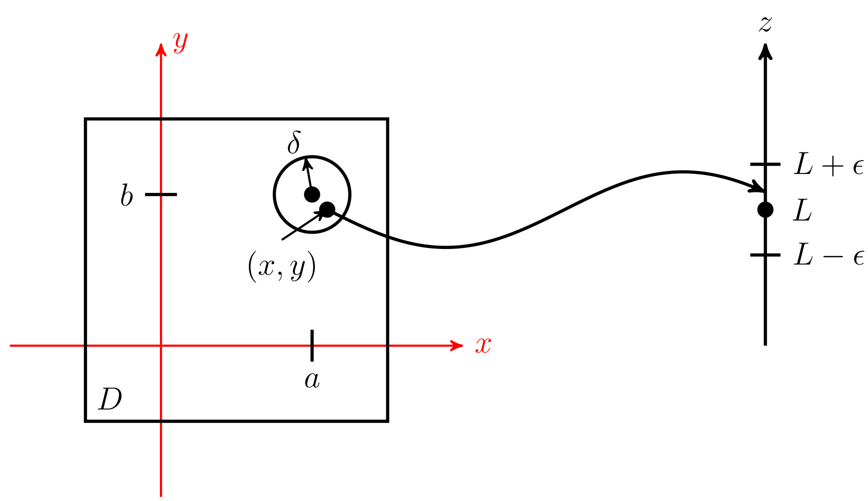
$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

This is very similar to the typical limit of a function of one variable. The only difference is that where for a function $f(x)$ we check to see that the values of $f(x)$ as $x \to a$ approaches some number L as x goes to a from both the **left** and the **right**, we now need to check that values of $f(x,y)$ apprach L as (x,y) approaches (a,b) along **any** possible path in the domain.

Def: Let f be a function of 2 variables whose domain D includes points arbitrarily close to (a,b). Then we say that the limit of $f(x,y)$ as (x,y) approaches (a,b) is L and we write

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if for ever $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that if $(x,y) \in D$ and $\sqrt{(x-a)^2 + (y-b)^2} < \delta$ then $|f(x,y) - L| \leq \epsilon$.



In other words, given some small positive number ϵ we must be able to find some δ such that if (x,y) is in a ball of radius δ centered at (a,b) then $f(x,y)$ must fall in the interval $(L-\epsilon, L+\epsilon)$.

Recall that in $1D$ we had to check to see if the limit was the same from the **left** and the **right** of the point. We concluded that if

$$\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$$

then the limit did not exist. In 2D we can take $(x,y) \to (a,b)$ along **any path**. If the limit isn't the same along **all** paths then we say the limit does not exist.

Example 7

Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$ does not exis.

Path 1: Along x-axis $\quad \Rightarrow \quad y=0 \quad \Rightarrow \quad f(x,0)=\dfrac{x^2}{x^2}=1.$

We say that $f(x,y)\to 1$ as $(x,y)\to(0,0)$ along the x-axis.

Path 2: Along y-axis $\quad \Rightarrow \quad x=0 \quad \Rightarrow \quad f(0,y)=\dfrac{-y^2}{y^2}=-1.$

We say that $f(x,y)\to -1$ as $(x,y)\to(0,0)$ along the y-axis.

Since these limits along two paths do not match we say that the limit does not exist.

Example 8

Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ does not exis.

$\underline{\boldsymbol{x}\textbf{-axis}}$: $\quad f(x,0)=\dfrac{0}{y^2}=0$

$\underline{\boldsymbol{y}\textbf{-axis}}$: $\quad f(0,x)=\dfrac{0}{x^2}=0$

$\underline{\mathbf{y}=mx}$: $\quad f(x,mx)=\dfrac{x\,(mx)}{x^2+(mx)^2}=\dfrac{mx^2}{x^2+m^2x^2}=\dfrac{m}{1+m}$

Since the limits along any line of the form $y=mx$ do not agree we conclude that the limit does not exist.

So, how do we show that a limit **does** exist?

1. Use the $\epsilon-\delta$ definition of a limit
2. Use the Squeeze Theorem
3. If $(x,y)\to(0,0)$ use the polar coordinate trick

Squeeze Theorem

We want to find functions $\ell\,(x,y)$ and $u\,(x,y)$ such that

$$\ell\,(x,y)\le f(x,y)\le u\,(x,y)$$

and both $\ell\,(x,y)\to L$ and $u\,(x,y)\to L$ as $(x,y)\to(a,b)$.

Example 9

Find the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$

Note that $\displaystyle\frac{x^2}{x^2+y^2} \leq 1$ because $x^2 \leq x^2+y^2$

Then $\displaystyle -3|y| \leq \frac{3x^2y}{x^2+y^2} \leq 3|y|$ and $\displaystyle\lim_{(x,y)\to(0,0)} -3|y| = \lim_{(x,y)\to(0,0)} 3|y| = 0$

So we conclude by the Squeeze Theorem that

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$$

Polar Coordinates Trick

Since here we are taking the limit $(x,y) \to (0,0)$ it is helpful to convert the limit to polar coordinates. Let $x = r\cos\theta$ and $y = r\sin\theta$ and then take the limit as $r \to 0^+$. By doing this we are really taking the limit of all possible paths that lead to the origin.

Example 10

Find the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$

$$= \lim_{r\to 0^+} \frac{3r^2\cos^2\theta r\sin\theta}{r^2\sin^2\theta + r^2\cos^2\theta} = \lim_{r\to 0^+} \frac{3r^3\cos^3\theta\sin\theta}{r^2} = \lim_{r\to 0^+} 3r\cos^2\theta\sin\theta = 0$$

Note again that this works because without fixing θ we cover every path as $(x,y) \to (0,0)$ and the r-ball shrinks to the origin. We can also use this trick to show that a limit does not exist.

Example 11

Find the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$

$$= \lim_{r\to 0^+} \frac{r^2\cos^2\theta - r^2\sin^2\theta}{r^2\sin^2\theta + r^2\cos^2\theta} = \lim_{r\to 0^+} \frac{r^2\left(\cos^2\theta - \sin^2\theta\right)}{r^2} = \cos^2\theta - \sin^2\theta$$

Notice that the resulting limit depends on the choice of θ along the path, and so the limit does not exist.

Continuity

Def: A function of two variables f is called **continuous** at (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

We say that f is **continuous on** D if it is continuous at every point in D.

Some Special Cases

1. A polynomial in two variables is continuous everywhere.
2. A rational function is continuous everywhere on its domain.

Recall that if you know that a function is continuous at a point that you're taking a limit to, you can use direct substitution to figure out the limit

Example 12

$\lim_{(x,y)\to(1,2)} x^2 + xy + y^3 = 1 + 2 + 8 = 11$

Example 13

$\lim_{(x,y)\to(1,0)} \dfrac{x^2 - y^2}{x^2 + y^2} = 1$

Remark: Everything that we've discussed for limits of functions of two variables is naturally extended to functions of three or more variables.

11.3 Partial Derivatives

Let $f(x, y)$ be a function of two variables and suppose we want to know how f varies with x and y at a particular point. Consider the case when we want to know how f is varying w.r.t. x at the point $(x, y) = (a, b)$. Since we're only letting f vary with x, we can think of the y variable as held constant at $y = b$. Then, if we define a new function

$$g(x) = f(x, b)$$

If the derivative of $g(x)$ exists at $x = a$ then we write

$$g'(a) = f_x(a, b)$$

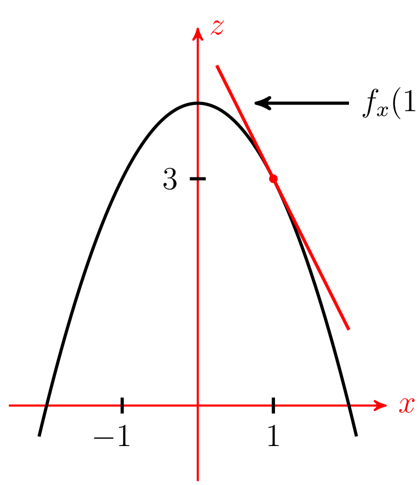
and call $f_x(a, b)$ the partial derivative with respect to x at the point (a, b). By the definition of a derivative of a function of a single variable we have

$$g'(a) = \lim_{h \to 0} \frac{g(a+h) - g(a)}{h} = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h} = f_x(a, b)$$

Example 14

Suppose we want to find the partial derivative of the function $f(x, y) = 5 - x^2 - y^2$ with respect to x at the point $(1, 1)$.

Since we're keeping y fixed at $y = 1$, we have $g(x) = 5 - x^2 - 1 = 4 - x^2$. If we plot this cross-section in the xz-plane we have



Then we have that the change if f w.r.t. x at $x = 1$ is given by

$$g'(1) = f_x(1, 1) = -2\,(1) = -2$$

This value is precisely the slope of the tangent line to the $y = 1$ cross-section $g(x) = 4 - x^2$ at the point $x = 1$.

We can over course do this for a general point (x, y). The method is essentially the same. To take the partial derivative w.r.t. x we pretend the y-value is fixed and take the derivative of $f(x, y)$ with respect to x. For our example problem we have

$$f_x(x, y) = -2x$$

Similarly, to take the partial derivative w.r.t. y we pretend the x-value is fixed and take the derivative of $f(x, y)$ w.r.t. y.

$$f_y(x, y) = -2y$$

Example 15

Find the first partial derivatives of the function $g(x, y) = y^5 - 3xy$ and determine the rate of change of g w.r.t. x and y at the point $(2, 1)$.

$$g_x(x, y) = -3y \qquad g_y(x, y) = 5y^4 - 3x$$

$$g_x(2, 1) = -3\,(1) = -3 \qquad g_y(2, 1) = 5\,(1)^4 - 3\,(2) = -1$$

So, at the point $(2, 1)$ the function $g(x, y)$ is decreasing with a rate of change of -3 and -1 in the x and y-directions, respectively.

Example 16

Find the first partial derivatives of $h(x, y) = x \ln y + \sin(xy)$

$$h_x(x, y) = \ln y + y \sin(xy) \qquad h_y(x, y) = \frac{x}{y} + x \sin(xy)$$

We can of course take higher-order partial derivatives. For instance f_{xx} is the second partial derivative of f w.r.t. x and is obtained by differentiating f_x w.r.t. x. Similarly we can take the second-partial derivatives f_{yy}, f_{xy} and f_{yx}.

Example 17

Find all second partial derivatives of the function $h(x, y)$ from the previous example.

$$\begin{aligned}
h_{xx} &= y^2 \sin xy \\
h_{xy} &= \frac{1}{y} + \sin xy + xy \sin xy \\
h_{yx} &= \frac{1}{y} + \sin xy + xy \sin xy \\
h_{yy} &= -\frac{x}{y^2} + x^2 \sin xy
\end{aligned}$$

Notice that we got the same thing for h_{xy} and h_{yx}. It turns out that this will always be the case so long as h_{xy} and h_{yx} are continuous. This result is known as **Clairaut's Theorem**.

Just like functions of one variable, there are several different ways to denote partial derivatives. The subscript notation introduced above is the partial derivative equivalent of using primes to denote derivatives. The Leibniz notation for partial differentiation is similar to the single-variable case, except we use a script ∂ instead of a regular d. We have

$$f_x = \frac{\partial f}{\partial x} \qquad f_{yy} = \frac{\partial^2 f}{\partial y^2} \qquad f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$$

Just like functions of one variable, we sometimes want to take partial derivatives of a function z which is defined implicitly in terms of x and y, i.e. $z = z(x, y)$.

Example 18

Use implicit differentiation to find $\partial z/\partial x$ if z is a function of x and y and satisfies

$$x^2 + 2y^2 + 3z^2 + xyz = 1$$

Taking the partial derivative of both sides with respect to x we have

$$\frac{\partial}{\partial x}\left(x^2 + 2y^2 + 3z^2 + xyz\right) = \frac{\partial}{\partial x}(1)$$

$$\Rightarrow \quad 2x + 6z\frac{\partial z}{\partial x} + yz + xy\frac{\partial z}{\partial x} = 0 \quad \Rightarrow \quad (6z + xy)\frac{\partial z}{\partial x} = -2x - yz$$

$$\Rightarrow \quad \frac{\partial z}{\partial x} = -\frac{2x + yz}{6z + xy}$$

Partial Differential Equations

Just like we could use equations involving derivatives to model certain phenomenon in Calc 1 in one dimension, we can use partial derivatives to model physical phenomenon in two dimensions. The following **partial differential equation** or **PDE** is ubiquitous in science and engineering as a way to model diffusion of heat, contaminant transportation in a fluid, and electrical potential. It is called **Laplace's Equation**

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Example 19

Show that the function $u(x, y) = \sin(\pi x)\cosh(\pi y)$ is a solution to Laplace's Equation.

We have

$$\begin{aligned} u_x &= \pi\cos(\pi x)\cosh(\pi y) \quad \Rightarrow \quad u_{xx} = -\pi^2\sin(\pi x)\cosh(\pi y) \\ u_y &= \pi\sin(\pi x)\sinh(\pi y) \quad \Rightarrow \quad u_{yy} = \pi^2\sin(\pi x)\cosh(\pi y) \end{aligned}$$

which clearly satisfies $u_{xx} + u_{yy} = 0$.

Another partial differential equation that appears literally everywhere is called the wave equation. Here $u(x, t)$ describes the displacement of something like a water wave or a guitar string at position x and time t. It is given by

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

Example 20

Show that the function $u(x, t) = \cos(x - at)$ satisfies the wave equation

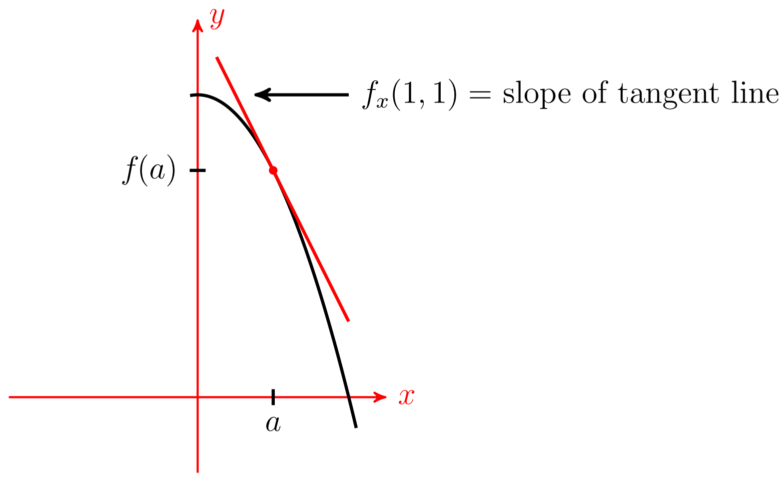
We have

$$\begin{aligned} u_t &= a\sin(x - at) \quad \Rightarrow \quad u_{tt} = -a^2\cos(x - at) \\ u_x &= -\sin(x - at) \quad \Rightarrow \quad u_{xx} = -\cos(x - at) \end{aligned}$$

It is easy to see that this $u(x, t)$ satisfies the wave equation.

11.4 Tangent Planes and Linear Approximations

Recall that for functions of one variable, i.e. $y = f(x)$ we can use the tangent line at a point $x = a$ to approximate the function for values of x very near a.



This function is often called the **linearization** of $f(x)$ around a and is just the tangent line of $f(x)$ at $x = a$.

$$f(x) \approx L(x) = f(a) + f'(a)\,(x - a)$$

Notice that the derivative of f at $x = a$ tells us the slope of the tangent line / linearization.

For functions of two variables we're concerned with working with surfaces. The same idea applies here. Around some point (x_0, y_0) the surface can be approximated well by a linear function. A linaer function of two variables is just a plane. We call this the linearization of $f(x, y)$ around the point (x_0, y_0) or just the **tangent plane** of f at (x_0, y_0).

Let the surface S be described by $z = f(x, y)$ where f has continuous first partial derivatives in a neighborhood around the point (x_0, y_0). Let $P\,(x_0, y_0, z_0)$ be a point on the surface.

Suppose we take cross-sections of the surface in the $x = x_0$ and $y = y_0$ planes, which intersect at point P. The intersection of each plane with surface S traces out a curve. Call them C_1 and C_2.

$$C_1 : z = f(x_0, y) \qquad \text{and} \qquad C_2 : z = f(x, y_0)$$

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be tangent lines to C_1 and C_2 at (x_0, y_0). Then the tangelt plane to S at point P is the plane which contains $P\,(x_0, y_0, z_0)$ and contains $\mathbf{v}_1$ and $\mathbf{v}_2$.

Note that we picked $\mathbf{v}_1$ and $\mathbf{v}_2$ for convenience. Any curve C that lies on S and passes through P will have a tangent line $\mathbf{v}$ that lies in the tangent plane.

We know that in general the plane containing $P(x_0, y_0, z_9)$ has the form

$$A\,(x-x_0)+B\,(y-y_0)+C\,(z-z_0)=0$$

Solving for z we have

$$z=z_0-\frac{A}{C}\,(x-x_0)-\frac{B}{C}\,(y-y_0)$$

Think about the tangent lines $\mathbf{v}_1$ and $\mathbf{v}_2$.

$$\mathbf{v}_1: \quad y=y_0 \quad \Rightarrow z=z_0+a\,(x-x_0)$$

But we know from last time that this line has the slope $f_x\,(x_0,y_0) \quad \Rightarrow \quad a=f_x(x_0,y_0)$

and

$$\mathbf{v}_2: \quad x=x_0 \quad \Rightarrow z=z_0+b\,(y-y_0)$$

This line has the slope $f_y\,(x_0,y_0) \quad \Rightarrow \quad b=f_y(x_0,y_0)$

So the tangent plane is given by

$$z=z_0+f_x\,(x_0,y_0)\,(x-x_0)+f_y\,(x_0,y_0)\,(y-y_0)$$

Example 21

Find the tangent plane to the function $z=f(x,y)=2x^2+y^2$ at the point $(1,1,3)$.

We have

$$\begin{aligned} f_x &= 4x \quad \Rightarrow f_x\,(1,1)=4 \\ f_y &= 2y \quad \Rightarrow f_y\,(1,1)=2 \end{aligned}$$

$$z=3+4\,(x-1)+2\,(y-1) \quad \Rightarrow \quad z=4x+2y-3$$

Recall that the linearization of $f(x,y)$ at $P\,(1,1,3)$ is a good approximation to the function near the point $(1,1,3)$. Suppose we want to use the linearization derived above to approximate the function $f(x,y)$ at the point $(1.1,0.95)$. We have

$$f(1.1,0.95)\approx L\,(1.1,0.95)=4\,(1.1)+2\,(0.95)-3=3.3$$

while the actual value is given by $f(1.1,0.95)=3.3225$.

Away from the point $(1,1,3)$ the linearization is not a good approximation to $f(x,y)$

$$L(2,3) = 11 \quad \text{vs} \quad f(2,3) = 17$$

Another way that the linearization is commonly written is when approximating $f(x,y)$ near the point (a,b) is

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

The linearization is only good if $f_x(a,b)$ and $f_y(a,b)$ exist near (a,b) and are continuous at (a,b). When this is true we say that f is differentiable at (a,b).

Example 22

Show that $f(x,y) = xe^{xy}$ is differentiable at $(1,0)$ and find it's linearization there

$$\begin{aligned} f_x &= e^{xy} + xye^{xy} \quad \Rightarrow f_x(1,0) = 1 \\ f_y &= x^2 e^{xy} \quad \Rightarrow f_y(1,1) = 1 \end{aligned}$$

Note that both partial derivatives are continuous (everywhere) so f is differentiable (everywhere).

$$L(x,y) = f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0) = 1 + (x-1) + y \quad \Rightarrow \quad L(x,y) = x + y$$

Differentials

Let's go back to functions of one variable for a minute. Let $y = f(x)$ and consider the small change in y that is caused by a small change in x

$$\Delta y = f(x + \Delta x) - f(x)$$

With our linearization we can approximate this change via

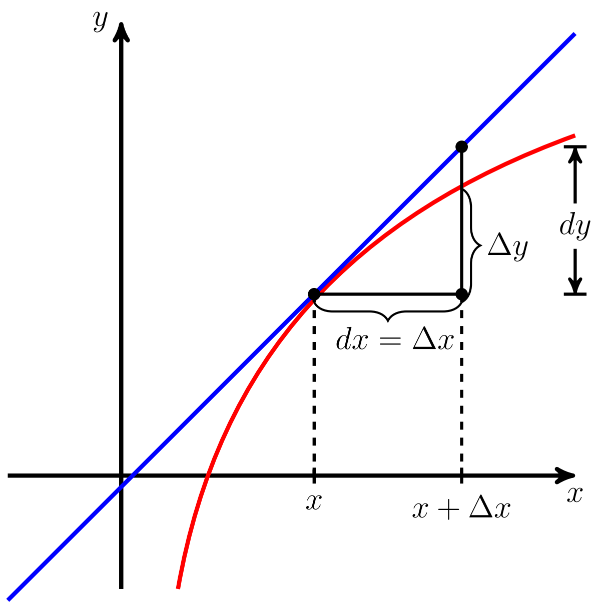
$$f(x + \Delta x) \approx f(x) + f'(x)\left[(x + \Delta x) - x\right] = f(x) + f'(x)\,\Delta x$$

Then we can write

$$\Delta y \approx f'(x)\,\Delta x$$

When we actually make this approximation, we approximate the actual changes by so-called **differentials**.

$$dx = \Delta x \qquad \text{and} \qquad dy = f'(x)\,dx$$



Example 23

Suppose you measure the side length of a cube as $x = 2$cm with a potential overestimate of 0.02 cm. Use differentials to estimate the maximum error in the calculated volume of the cube.

The formula relating the side length of a cube and it's volume is given by

$$V = x^3$$

Since we know that $dV = f'(x)\,dx$ we have

$$dV = 3x^2 dx$$

If $dx = 0.02$ (Note: this is an over-measurement) then we have

$$dV = 3 \cdot 4 \cdot 0.02 = 0.24\text{m}^3$$

We can easily generalize this concept to functions of two or more variables. Let $z = f(x, y)$ and consider the small change in z that is caused by small changes in x and y. We have

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$$

But from our linearization of $f(x, y)$ we have

$$\begin{aligned} f(x + \Delta x, y + \Delta y) &\approx f(x, y) + f_x(x, y)\,[(x + \Delta x) - x] + f_y(x, y)\,[(y + \Delta y) - y] \\ &= f(x, y) + f_x\,\Delta x + f_y\,\Delta y \end{aligned}$$

Subtracting $f(x, y)$ from both sides we have

$$\Delta z = \Delta f \approx f_x\,\Delta x + f_y\,\Delta y$$

Now, letting $dx = \Delta x$ and $dy = \Delta y$, and $dz \approx \Delta z$ we have the equation representing differentials

$$dz = df = f_x\,dx + f_y\,dy$$

When f is a function of multiple variables we call df the **total differential** of f. Note that dx and dy are independent variables representing the exact change in x and y, respectively, while dz or df is a dependent variable which represents an **approximation** of the change in f.

Example 24

Let $z = f(x, y) = x^2 + 3xy - y^2$. Approximate the change in z if x changes from 2 to 2.05 and y changes from 3 to 2.96.

We want to compute the approximate change dz. We have

$$dz = f_x\, dx + f_y\, dy = (2x + 3y)\ dx + (3x - 2y)\ dy$$

The partial derivatives are computed at the center point $(2, 3)$ and the deltas are given by $dx = 2.05 - 2 = 0.05$ and $dy = 2.96 - 3 = -.04$.

$$dz = 13\,(0.05) + 0\,(-0.4) = 0.65$$

If we compute the **exact** change in dz we have

$$f(2.05, 2.96) - f(2, 3) = 0.6449$$

Remark: The differential approximation is **exactly** the same approximation we would get if we approximated $f(2.05, 2.96)$ using the tangent plane approximation and subtracted $f(2, 3)$. Differentials are just a more convenient way to formulate the problem if you're interested in computing **changles** instead of values of the function.

Example 25

The side lengths of a closed box with square base are measured as 10cm for the side of the base and 20cm for the height with possible errors in measurement of ± 0.1cm each. Use differentials to estimate the maximum error in calculating the surface area of the box. Which measurement is more sensitive to error?

We have

$$A = 2x^2 + 4xy \quad \Rightarrow \quad dA = (4x + 4y)\ dx + 4x\, dy$$

Substituting $(10, 20)$ for (x, y) and 0.1 for the differentials we have

$$dA = [4\,(10) + 4\,(20)]\ (0.1) + 4\,(10)\ (0.1) = 120\,(0.1) + 40\,(0.1) = 16\text{cm}^3$$

Since the f_x term was larger than the f_y term, the x measurement is more sensitive to error. In other words, errors in the x measurement cause a larger overall error in the estimation of the surface area than errors in the y measurement.

Example 26

While working a summer job in the university arts department, you are tasked with transporting a new piece of conical metal artwork (all the rage right now) from a local gallery to the campus. The cones exterior measurements are taken to be 4m for the radius of the base and 6m for the height. Thankfully, you're told that piece is hollow with the thickness of the metal being approximately 1cm. Estimate the weight of the piece assuming that copper has density 9 g/cm^3.

We'll estimate the volume and approximate the weight at the end. We can estimate the volume of the metal by considering the differential of the volume when the radius and height are decreased by 1cm. The volume of a right circlular cone is given by

$$V = \frac{1}{3}\pi r^2 h \quad \Rightarrow \quad dV = \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial h} dh = \frac{2}{3}\pi r h\, dr + \frac{1}{3}\pi r^2\, dh$$

$$dV = \frac{2}{3}\pi\,(4)\,(6)\,(-0.01) + \frac{1}{3}\pi\,(4)^2\,(-0.01) \approx 0.67m^3 = 6.7 \times 10^5\ \text{cm}^3$$

You estimate that the artwork has weight (670000×9) grams or 6030 KGs at which point you point out to your boss that this is above your pay grade.

Differentials with functions of three variables is analagous.

Example 27

Suppose you measure the sides of a rectangular box to be 50cm, 40cm, and 60cm respectively. Suppose your measurements of the side lengths were subject to overmeasurement by 1cm for the first two measurements and undermeasurement by 2cm in the last dimension (because your ruler broke and you decided to use your shoe). Estimate the error in computing the volume.

$$V = xyz \quad \Rightarrow \quad dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz = yz\, dx + xz\, dy + xy\, dz$$

$$dV = (40)\,(60)\,(1) + (50)\,(60)\,(1) + (50)\,(40)\,(-2) = 340\text{cm}^3$$

11.5 The Chain Rule

Example: Suppose the function $T(x, y, z)$ describes the temperature in a room and we want to know how the temperature is changing along a curve

$$\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$$

That is, $x(t) = f(t)$, $y(t) = g(t)$, and $z(t) = h(t)$.

We could substitute the functions for x, y, and z in terms of t into the termperature function and compute dT/dt, but this could be hard if the component functions are at all complicated. Instead we use the **Chain Rule** for functions of multiple variables.

Single Variable Case: Say we have temperature as a function of just x, i.e. $T(x)$, and $x = f(t)$. Then

$$\frac{dT}{dt} = \frac{dT}{dx}\frac{dx}{dt}$$

Two Variable Case: For $T(x, y)$ with $x = f(t)$ and $y = g(t)$

$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt}$$

Note that when differentiating a function of multiple variables we use the ∂ notation, and when differentiating a function of a single variable we use the d notation.

This should be believable based on what we know about linearizations and tangent planes. We have that

$$\Delta T \approx \frac{\partial T}{\partial x}\Delta x + \frac{\partial T}{\partial y}\Delta y \quad \Rightarrow \quad \frac{\Delta T}{\Delta t} \approx \frac{\partial T}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial T}{\partial y}\frac{\Delta y}{\Delta t}$$

Then, taking the limit as $\Delta t \to 0$ we obtain the result.

Example 28

Compute dw/dt for $w = xy$ with $x(t) = \cos t$ and $y(t) = \sin t$

Note that dw/dt gives the rate of change of the function $w = xy$ as we move around a circle of radius 1 in the xy-plane.

$$\begin{aligned}
\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} \\
&= y\,(-\sin t) + x\,(\cos t) \\
&= (\sin t)\,(-\sin t) + (\cos t)\,(\cos t) \\
&= -\sin^2 t + \cos^2 t \\
&= \cos\,(2t)
\end{aligned}$$

So when $t = \pi/2$ we have $\dfrac{dw}{dt}\left(\frac{\pi}{2}\right) = \cos\,(\pi) = -1$

EFY: Check that this is what you would get if you substituted x and y in terms of t and took the derivative.

One helpful way for determining the form of the chain rule is to use a **tree diagram**

$$\begin{tikzcd}
& w \arrow[dl, "\frac{\partial w}{\partial x}"'] \arrow[dr, "\frac{\partial w}{\partial y}"] & \\
x \arrow[d, "\frac{dx}{dt}"'] & & y \arrow[d, "\frac{dy}{dt}"] \\
t & & t
\end{tikzcd}$$

Then we sum all paths that start at w and end at t.

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

What if if $w = f(x, y, z)$ with $x = g(t)$, $y = h(t)$, and $z = k(t)$?

$$\begin{tikzcd}
 & w \arrow[dl] \arrow[d] \arrow[dr] & \\
x \arrow[d] & y \arrow[d] & z \arrow[d] \\
t & t & t
\end{tikzcd}$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

Example 29

Suppose that the temperature in a room is given by $T(x, y, z) = x + yz$ and we want to know the rate that the temperature changes with respect to time along the helix $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$.

$$\begin{aligned}
\frac{dT}{dt} &= \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt} \\
&= 1\,(-\sin t) + z\,(\cos t) + y\,(1) \\
&= -\sin t + t\cos t + \sin t \\
&= t\cos t
\end{aligned}$$

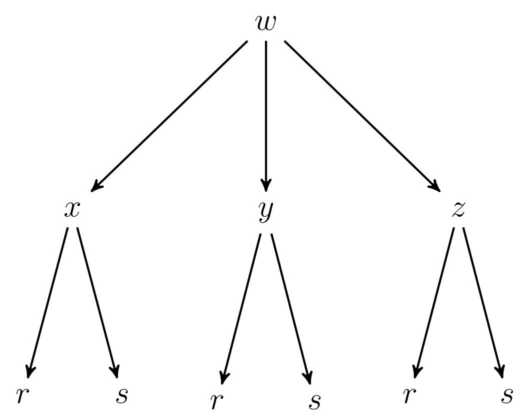
Then we could find, for instance, that at time $t = \pi$ the temperature is changing at a rate of

$$\frac{dT}{dt}(\pi) = \pi\cos(\pi) = -\pi$$

What if the spatial variables depend on two independent variables instead of one? Suppose $w = f(x, y, z)$ where $x = g(r, s)$, $y = h(r, s)$ and $z = k(r, s)$. Then

$$w = f(g(r, s), h(r, s), k(r, s))$$

Now suppose we want to know how w changes with respect to one of the independent variables, say r. We again draw a tree.



Following all of the branches that lead to an r we find

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

Example 30

Suppose $T = \ln(x^2 + y^2)$ where x and y are defined in terms of polar coordinates, i.e. $x = r\cos\theta$ and $y = r\sin\theta$. Compute $\partial w/\partial\theta$.

$$\begin{aligned}
\frac{\partial w}{\partial \theta} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta} \\
&= \frac{2x}{x^2+y^2}(-r\sin\theta) + \frac{2y}{x^2+y^2}(r\cos\theta) \\
&= \frac{2r\cos\theta}{r^2}(-r\sin\theta) + \frac{2r\sin\theta}{r^2}(r\cos\theta) \\
&= 0
\end{aligned}$$

This makes sense. The temperature depends on the square of the distances from the origin. If we hold r fixed and let θ vary we should see no change in temperature.

EFY: Compute the change in temperature with respect to radius.

Implicit Differentiation Trick:

Example 31

Find dy/dx for y an implicit function of x related by $x^3 - 2y^2 + xy = 0$.

$$\frac{d}{dx}\left(x^3 - 2y^2 + xy\right) = 0 \;\Leftrightarrow\; 3x^2 - 4y\frac{dy}{dx} + y + x\frac{dy}{dx} = 0 \;\Leftrightarrow\; \frac{dy}{dx} = \frac{3x^2+y}{4y-x}$$

This is kind of a pain. We can use the chain rule to find a shortcut. Define $w = F(x, y) = 0$ where here $F(x, y) = F(x, y(x))$ is the function of two variables that describes the implicit relationship between y and x. If we draw a tree for F we have

$$\begin{tikzcd} & F \arrow[dl] \arrow[dr] & \\ x & & y \arrow[d] \\ & & x \end{tikzcd}$$

Now if we differentiate F w.r.t. x (just like we normally do in implicit differentiation) we have by the chain rule

$$\frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx}$$

Then

$$F(x, y) = 0 \iff \frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

Solving for dy/dx we have

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

With $F(x, y) = x^3 - 2y^2 + xy$

$$\frac{dy}{dx} = -\frac{3x^2 + y}{-4y + x} = \frac{3x^2 + y}{4y - x}$$

EFY: Look in the book for implicit differentiation with three variables.

11.6 Directional Derivatives and the Gradient Vector

Recall that $\partial f/\partial x$ and $\partial f/\partial y$ tell us the instantaneous rate at which the function $f(x, y)$ is changing in the x- and y-directions, respectively.

What if I want to know how f is changing in another direction?

Suppose we want to know how $z = f(x, y)$ is changing in the direction of a unit vector $\mathbf{u}$ at the point $P_0(x_0, y_0, z_0)$. Let's parameterize a line through P_0 that is parallel to the vector $\mathbf{u} = \langle u_1, u_2 \rangle$. Notice that since u is a unit vector we can parameterize using arc length.

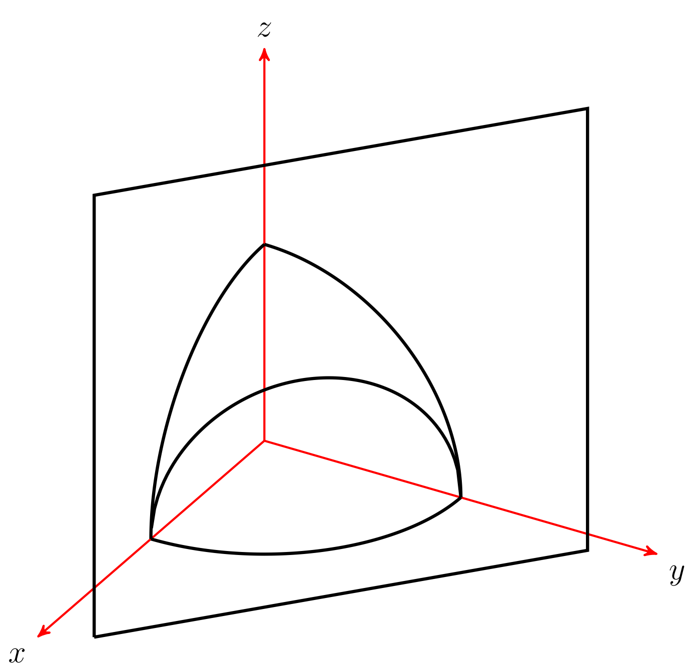
$$\mathbf{r}(s) = \langle x_0 + su_1, y_0 + su_2 \rangle \quad \text{or} \quad x = x_0 + su_1 \text{ and } y = y_0 + su_2$$

Then the derivative of f at P_0 in the direction of $\mathbf{u}$ is

$$\left(\frac{df}{ds}\right)_{\mathbf{u}, P_0} = \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s} = D_{\mathbf{u}} f(x_0, y_0)$$

This is called the **directional derivative** of f in the direction $\mathbf{u}$. Sometimes we write $(D_{\mathbf{u}} f)_{P_0}$

Geometric Interpretation: Let $z = f(x, y)$ be a surface. The point (x_0, y_0, z_0) is on the surface. Recall that we interpretted $f_x(x_0, y_0)$ by drawing a vertical plane at $y = y_0$ and noticed that the partial derivative was the slope of the tangent line of the curve of intersection. For directional derivatives we can do the same thing, only this time the vertical plane is not aligned with the x or y axes. This time the plane is vertical but in the direction of $\mathbf{u}$.



OK, but we don't want to compute these things using the limit definition. That would suck! Instead we use the chain rule.

$$(D_{\mathbf{u}}f)_{P_0} = \left(\frac{df}{ds}\right)_{\mathbf{u},P_0} = \left(\frac{df}{dx}\right)_{P_0}\frac{dx}{ds} + \left(\frac{df}{dy}\right)_{P_0}\frac{dy}{ds}$$

But remember that along $\mathbf{u}$ and through P_0 we have $x = x_0 + su_1$ and $y = y_0 + su_2$, so

$$= \left(\frac{df}{dx}\right)_{P_0} u_1 + \left(\frac{df}{dy}\right)_{P_0} u_2$$

So we can compute the directional derivative of f in direction $\mathbf{u}$ with

$$(D_{\mathbf{u}}f)_{P_0} = f_x(x_0, y_0)\, u_1 + f_y(x_0, y_0)\, u_2$$

That's all nice, but there is an even better way to do it that makes life even easier. Notice that we can write this expression as a dot product of two vectors.

$$(D_{\mathbf{u}}f)_{P_0} = \left[\left(\frac{\partial f}{\partial x}\right)_{P_0}\mathbf{i} + \left(\frac{\partial f}{\partial y}\right)_{P_0}\mathbf{j}\right] \cdot [u_1\mathbf{i} + u_2\mathbf{j}] = (\nabla f)_{P_0} \cdot \mathbf{u}$$

This new vector is so useful that we give it a name. It's called the **gradient vector**.

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} \quad \text{or in 3 variables} \quad \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

Example 32

Compute ∇f at $P_0\,(1, 1, 1)$ for $f(x, y, z) = x^2 + y^2 - 2z^2 + z\ln x$

$$\frac{\partial f}{\partial x} = 2x + \frac{z}{x} \quad \frac{\partial f}{\partial y} = 2y \quad \frac{\partial f}{\partial z} = -4z + \ln x$$

$$\nabla f = \left\langle 2x + \frac{z}{x}, 2y, -4z + \ln x\right\rangle \quad \Rightarrow \quad (\nabla f)_{P_0} = \langle 3, 2, -4\rangle$$

Example 33

Compute the directional derivative of f at the point P_0 in the direction $\mathbf{u} = \left\langle \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\rangle$

$$(D_{\mathbf{u}}f)_{(1,1,1)} = (\nabla f)_{(1,1,1)} \cdot \mathbf{u} = \langle 3, 2, -4\rangle \cdot \left\langle \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\rangle$$

$$3\frac{3}{\sqrt{3}} - 2\frac{3}{\sqrt{3}} - 4\frac{3}{\sqrt{3}} = \frac{-3}{\sqrt{3}}$$

So at the point $(1,1,1)$ the function f is changing in the $\mathbf{u}$ direction at an instantaneous rate of $-3\sqrt{3}$.

What happens if we take the directional derivative of f in directions along the x and y axes?

$$\begin{aligned} \mathbf{u}=\mathbf{i}: \quad D_{\mathbf{u}} f &= \nabla f \cdot \mathbf{i} = f_x \\ \mathbf{u}=\mathbf{j}: \quad D_{\mathbf{u}} f &= \nabla f \cdot \mathbf{j} = f_y \end{aligned}$$

So directional derivatives are just generalizations of partial derivatives.

Properties of $D_{\mathbf{u}} f$.

Writing the directional derivative as a dot product allows us to say some interesting things about it. Recall that $\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}|\,|\mathbf{B}| \cos\theta$ where θ is the angle between $\mathbf{A}$ and $\mathbf{B}$. Applying this to the directional derivative we have

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = |\nabla f|\,|\nabla \mathbf{u}| \cos\theta = |\nabla f| \cos\theta$$

In what direction does f increase the fastest? We see that the expression above is maximized if $\cos\theta = 1$ which happens when $\theta = 0$. In other words, f increases the fastest in the direction of it's gradient.

Following similar logic we see that f **decreases** the fastest in the direction of $-\nabla f$.

OK, so how much is this increase/decrease?

Recall that the direction vector in the expression for the directional derivative needs to be a unit vector. So we have $\mathbf{u} = \nabla f / |\nabla f|$ and

$$D_{\mathbf{u}} f = \nabla f \cdot \frac{\nabla f}{|\nabla f|} = \frac{|\nabla f|^2}{|\nabla f|} = |\nabla f|$$

Similarly, the greatest decrease happens in the direction of $-\nabla f$, so we have

$$D_{\mathbf{u}} f = \nabla f \cdot -\frac{\nabla f}{|\nabla f|} = -\frac{|\nabla f|^2}{|\nabla f|} = -|\nabla f|$$

We can also use the gradient vector to determine a direction to move which will not change f at all. We do this by picking a direction **orthogonal** to the gradient vector.

Example 34

Find the direction of greatest increase/decrease and no change for $f(x,y) = xy + y^2$ at the point $(3,2)$.

Example 35

Two velociraptors are hunting you in a 4 by 4 room which has a door at the point $(-2,-1)$. You are standing at the origin. They always hunt in pairs, one to distract you by standing right in front of you, and one to attack from the side. If the raptors are standing at $(0,1)$ $(2,0)$ then the likelihood of you getting eaten is given by the function $P(x,y) = \left[(x-1)^2 + y^2\right]^{-1} = \left[x^2 + (y-2)^2\right]^{-1}$.

1. If you run towards the door, how is your chance of being eaten changing w.r.t. **distance**?
2. In what direction should you go to minimize your chance of being devoured?
3. If you move at a speed of 0.5 m/s in this direction, how is your chance of being eaten changing w.r.t. **time**?

For all of these we'll need the gradient of P. We have

$$\frac{\partial P}{\partial x} = \frac{-2\,(x-1)}{\left[(x-1)^2+y^2\right]^2} + \frac{-2x}{\left[x^2+(y-2)^2\right]^2} \qquad \frac{\partial P}{\partial y} = \frac{-2y}{\left[(x-1)^2+y^2\right]^2} + \frac{-2\,(y-2)}{\left[x^2+(y-2)^2\right]^2}$$

Since the question only asks about what happens when you're standing at the origin, we will only need the gradient vector evaluated at that point.

$$\nabla P\,(0,0) = \left\langle 2, \frac{1}{4} \right\rangle$$

1. The rate of change with respect to distance in the direction of some unit vector $\mathbf{u}$ is given by

   $$\frac{dP}{ds} = \nabla P \cdot \mathbf{u}$$

   Since we're running towards the door, we do so along the vector $\mathbf{u} = \frac{1}{\sqrt{5}}\langle -2,-1\rangle$ so we have

   $$\frac{dP}{ds} = \nabla P \cdot \mathbf{u} = \left\langle 2, \frac{1}{4}\right\rangle \cdot \langle -2,-1\rangle \, \frac{1}{\sqrt{5}} = -\frac{15}{4\sqrt{5}}$$

2. To minimize our chance of being devoured, we want to move in the direction of $-\nabla P$, but we first need to express this direction as a unit vector. We have

   $$\mathbf{u} = \frac{-\nabla P}{|\nabla P|} = \frac{-\langle 2, 1/4\rangle}{|\langle 2, 1/4\rangle|} = \frac{\langle -8,-1\rangle}{\sqrt{65}}$$

   in which the chance of being eaten is changing w.r.t. distance at a rate of $-|\nabla P| = -\sqrt{65}/4$

3. We want to compute dP/dt if we move with speed 0.5 in the direction of greatest decrease in P. We have

$$\frac{dP}{dt} = \frac{dP}{ds}\frac{ds}{dt} = -\frac{\sqrt{65}}{4}\frac{1}{2} = -\frac{\sqrt{65}}{8}$$

Example 36

Bonnie the honey bee is flying along the path $\mathbf{r}(t)$ in Suluclcac Swamp looking for sweet nectar. The temperature distribution that morning in the swamp is given by $T(x, y)$. At some instance in time t^*, you know that $\mathbf{r}(t^*) = \mathbf{i} + 3\mathbf{j}$, $\mathbf{v}(t^*) = 2\mathbf{i} + \mathbf{j}$, and $\mathbf{a}(t^*) = 3\mathbf{i} + 2\mathbf{j}$. Furthermore, you know that $\nabla T|_{(1,3)} = 2\mathbf{i} + 5\mathbf{j}$, and $T\,(1, 3) = 10$.

1. As Bonnie flies past the location $\mathbf{r}(t^*)$, at what rate is the temperature T changing with respect to **time**.

$$\frac{dT}{dt} = \nabla T \cdot \mathbf{v} = \langle 2, 5\rangle \cdot \langle 2, 1\rangle = 9$$

2. As she flies past the location $\mathbf{r}(t^*)$ at what rate is the temperature T changing with respect to **distance**.

$$\frac{dT}{ds} = \nabla T \cdot \mathbf{u} = \langle 2, 5\rangle \cdot \frac{\langle 2, 1\rangle}{\sqrt{5}} = \frac{9}{\sqrt{5}}$$

3. If Bonnie continues on her original path $\mathbf{r}(t)$ for a short interval of time $\Delta t = 0.1$, by approximately how much does the temperature change?

$$\Delta t \approx \frac{dT}{dt}\Delta t = 9\,(0.1) = 0.9$$

4. Onthe other hand, suppose at time t^* Bonnie suddenly sees her favorite flower, and starts to fly towards it in a direction that happens to be the direction of greatest increase of T. Assuming Bonnie maintains her same speed, by approximately how much does the temperature change after she flies for $\Delta t = 0.1$?

   First we need to find dT/dt in the direction of greatest increase.

$$\frac{dT}{dt} = \nabla T \cdot (|\mathbf{v}|\,\frac{\nabla T}{|\nabla T|}) = |\mathbf{v}|\,\frac{|\nabla T|^2}{|\nabla T|} = |\mathbf{v}|\,|\nabla T| = \sqrt{5}\sqrt{29}$$

Significance of the Gradient Vector

Consider the level curves of a function $z = f(x, y)$. That is, consider a curve of the form $f(x, y) = k$ for some constant k. Suppose we have some parameterization of this curve given by $\mathbf{r}(t) = \langle x(t)\,, y(t)\rangle$. Now, along this curve the function is described by

$$f(x(t)\,, y(t)) = k$$

Taking the derivative w.r.t. time using the chain rule we have

$$\frac{d}{dt}f(x(t)\,,y(t)) = \frac{d}{dt}k \quad \Leftrightarrow \quad \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = 0 \quad \Leftrightarrow \quad \nabla f \cdot \mathbf{r}'(t) = 0$$

This expression tells us that along the level curve, the gradient vector and the velocity vector are always orthogonal. This makes perfect sense if you think of a topographic map.

Example 37

Consider the function $f(x,y) = y - x^2$. Then the level curves have the form

$$f(x,y) = y - x^2 = k \quad \Rightarrow \quad y = x^2 + k$$

So the level curves are just shifted parabolas. Consider the point $(2,4)$ on the surface. The gradient at this point is given by

$$\nabla f = \langle -2x, 1\rangle \quad \Rightarrow \quad \nabla f\,(2,4) = \langle -4, 1\rangle$$

From calc 1 techniques we know that the slope of the tangent line to the level curve that contains the point $(2,4)$ is given by

$$y = x^2 + 0 \quad \Rightarrow \quad y' = 2x \quad \Rightarrow \quad m = 4$$

from which we can see that ∇f is orthogonal to the tangent line.

Relation to Tangent Plane

Consider the surface described by $z = f(x,y)$ that contains the point $P(x_0, y_0, z_0)$. Letting $F(x,y) = f(x,y) - z$ and taking the gradient at the point P we have

$$\nabla F\,(P) = \left\langle \frac{\partial f}{\partial x}(P)\,, \frac{\partial f}{\partial y}(P)\,, -1 \right\rangle = \langle f_x(x_0,y_0)\,, f_y(x_0,y_0)\,, -1\rangle$$

Now, if we take this vector and let it be the normal vector of a plane that contains the point (x_0, y_0, z_0) we get

$$0 = -1\,(z - z_0) + f_x(x_0,y_0)\,(x - x_0) + f_y(x_0,y_0)\,(y - y_0) \quad \Leftrightarrow$$

$$z = z_0 + f_x(x_0,y_0)\,(x - x_0) + f_y(x_0,y_0)\,(y - y_0)$$

Does this plane look familiar? It turns out this is exactly the tangent plane to the surface at the point P!

We can also do this for level surfaces. Consider the function Fx, y, z. It's level surfaces are of the form $Fx, y, z = k$ for some constant k. Consider the level surface that contains the point $P(x_0, y_0, z_0)$. Then the tangent plane to the level surface containing the point P is given by

$$0 = F_x\left(x_0, y_0, z_0\right)\left(x - x_0\right) + F_y\left(x_0, y_0, z_0\right)\left(y - y_0\right) + F_z\left(x_0, y_0, z_0\right)\left(z - z_0\right)$$

which has ∇F as it's normal vector. This again makes sense. Clearly if we want F to increase (or descrease) we should move in a direction normal to the level surface. If we move along the level surface we should expect no change in F.

Example 38

Find the equation of the tangent plane and the normal line to the function $x + y + z = e^{xyz}$ at the point $(0, 0, 1)$. The given relation is a level surface to the function

$$F(x, y, z) = x + y + z - e^{xyz}$$

We have

$$\nabla F = \langle 1 - yze^{xyz}, 1 - xze^{xyz}, 1 - xye^{xyz} \rangle \quad \Rightarrow \quad \nabla F(0, 0, 1) = \langle 1, 1, 1 \rangle$$

The tangent plane is then given by

$$1\,(x - 0) + 1\,(y - 0) + 1\,(z - 1) = 0 \quad \Rightarrow \quad x + y + (z - 1) = 0$$

The **normal line** is the line through the point $(0, 0, 1)$ in the direction of the gradient $\langle 1, 1, 1 \rangle$. So we have $L(t) = \langle t, t, 1 + t \rangle$

Example 39

Reconstructing the Gradient: Suppose you have some function $f(x, y)$ and you're told that at the point $(1, 1)$ the directional derivative of $f(x, y)$ along the vector $\mathbf{v}_l = \langle 1, 1 \rangle$ is $2\sqrt{2}$, and the directional derivative along the vector $\mathbf{v}_2 = \langle 1, 1 \rangle$ is $-3\sqrt{2}$. Find ∇f at the point $(1, 1)$.

Let $\nabla f = \langle a, b \rangle$. We need to find unit vectors in the direction of $\mathbf{v}_1$ and $\mathbf{v}_2$. We have

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{|\mathbf{v}_1|} = \frac{\langle 1, 1 \rangle}{\sqrt{2}} \quad \text{and} \quad \mathbf{u}_2 = \frac{\mathbf{v}_2}{|\mathbf{v}_2|} = \frac{\langle 1, -1 \rangle}{\sqrt{2}}$$

$$\begin{aligned} D_{\mathbf{u}_1} f = \nabla f \cdot \mathbf{u}_1 &= \langle a, b \rangle \cdot \frac{\langle 1, 1 \rangle}{\sqrt{2}} = 2\sqrt{2} \\ D_{\mathbf{u}_2} f = \nabla f \cdot \mathbf{u}_2 &= \langle a, b \rangle \cdot \frac{\langle 1, -1 \rangle}{\sqrt{2}} = -3\sqrt{2} \end{aligned}$$

From this we have

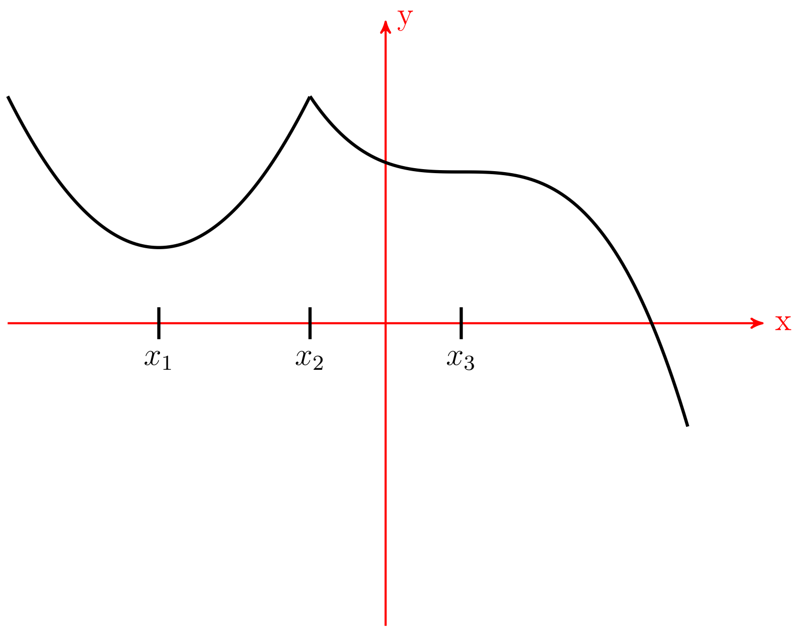
$$\begin{aligned} a+b &= 4 \\ a-b &= -6 \end{aligned}$$

Adding the equations together we find $a = -1$ and $b = 5$. So $\nabla f\,(1,1) = \langle -1, 5 \rangle$.

Under what circumstances would you **NOT** be able to reconstruct the gradient given two directional derivatives?

11.7 Maximum and Minimum Values

Motivation: Functions of One Variable: Consider $y = f(x)$



x_1 has a **local min** because for any small for any small open region around x_1, $f(x_1) \leq f(x)$.

x_2 has a **local max** because for any small for any small open region around x_2, $f(x_2) \geq f(x)$.

x_3 has a **saddle** because for any small for any small open region around x_3, we have functiona values above and below $f(x_3)$.

So, if we don't have a picture, how do we know that something may or may not be happening at these points?

First we notice that something might be happening at x_2 because $f'(x_2) = \text{DNE}$.

Then we notice that something might be happening at x_1 and x_3 because $f'(x_1) = f'(x_3) = 0$. So the slope of the tangent line at thise points is zero.

So how do we know that x_1 has a local min? We do the **Second Derivative Test**!

Second Derivative Test: If $f'(x)$ is continuous at x_c then

1. $f''(x_c) > 0 \quad \Rightarrow \quad$ graph is concave up $\quad \Rightarrow \quad$ Local Minimum
2. $f''(x_c) < 0 \quad \Rightarrow \quad$ graph is concave down $\quad \Rightarrow \quad$ Local Maximum
3. $f''(x_c) = 0 \quad \Rightarrow \quad$ graph has no extreme point $\quad \Rightarrow \quad$ Saddle

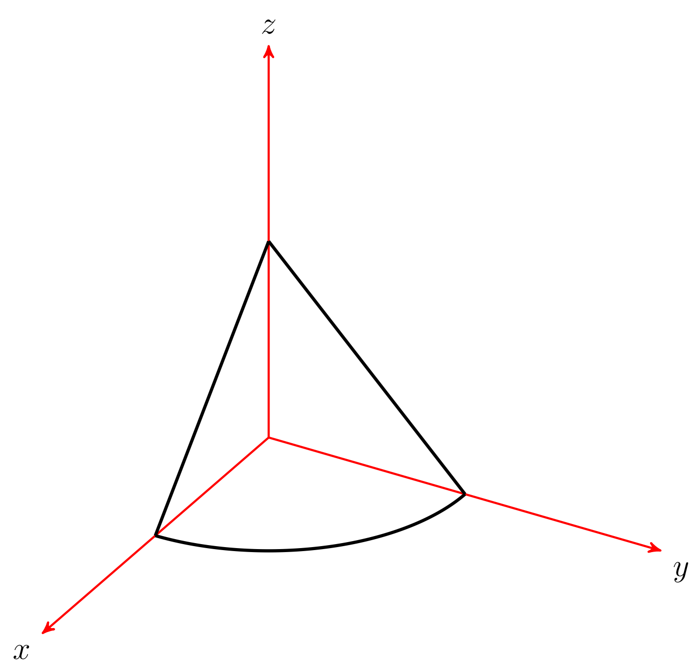
So how do we extend this to surfaces?

We need away to look for critical points of a function $z = f(x, y)$. Let's look at a couple of examples.

Example 40

Consider $f(x, y) = 1 - \sqrt{x^2 + y^2}$.

This is a cone that intersects the z-axis at the point $(0, 0, 1)$.



This guy has a local max at the point $(0, 0)$. Notice that at that point there is a sharp corner in the surface. We can recognize this by taking partial derivatives

$$f_x = \frac{-x}{\sqrt{x^2 + y^2}} \qquad \text{and} \qquad f_y = \frac{-y}{\sqrt{x^2 + y^2}}$$

Then, if we plug in the alleged point where something is happening, i.e. $(0, 0)$, we get

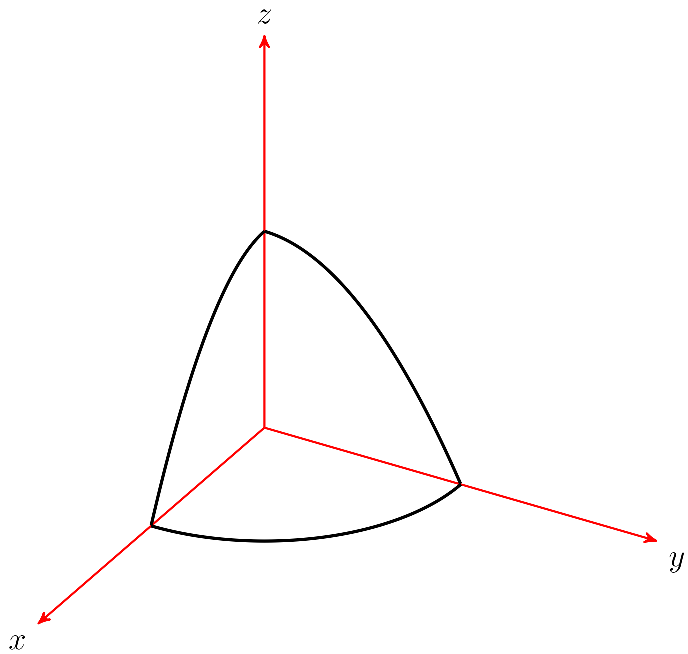
$$f_x(0, 0) = \text{DNE} \qquad \text{and} \qquad f_y(0, 0) = \text{DNE}$$

So if one or both of the partial derivatives does not exist then there might be something going on.

Example 41

Consider the function $f(x, y) = 1 - x^2 - y^2$

This is an upside down paraboloid with it's top at the point $(0, 0, 1)$.



Again we see that there is a local maximum at the point $(0, 0, 1)$. What's happening with the partial derivatives at this point?

$$f_x = -2x \qquad \text{and} \qquad f_y = -2y$$

Then, if we plug in the alleged point where something is happening, i.e. $(0, 0)$, we get

$$f_x(0,0) = 0 \qquad \text{and} \qquad f_y(0,0) = 0$$

Notice that this means that a tangent plane to the surface at the point $(0, 0)$ is perfectly flat.

Definition: A **critical point** of $f(x, y)$ is a point where the partial derivatives are zero or at least one of the partial derivatives does not exist.

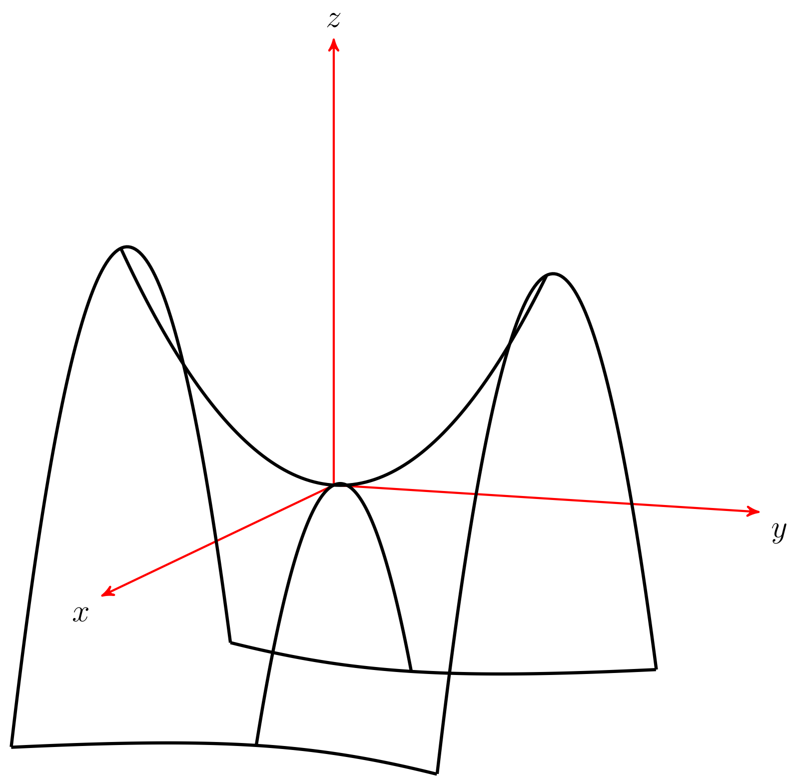
So, we find critical points by taking partial derivatives and setting them equal to zero, or finding points where one of them does not exist.

If we find a critical point does it guarantee that there is a local min or max at that point?

Example 42

Consider $f(x, y) = y^2 - x^2$.

This is a hyperbolic paraboloid.



Here we have

$$f_x = -2x \quad \text{and} f_y = 2y$$

which has a critical point at $(0, 0)$ but no corresponding min or max. It's a saddle!

So, suppose we found a critical point. How do we classify it as a local min or local max? If the critical point (x_c, y_c) is such that the partial derivatives are zero, then we have a version of the second derivative test for functions of two variables.

Method: Second Derivative Test: Suppose $f(x, y)$ has continuous first and second derivatives around (x_c, y_c), then

1. If $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ then f has a local min at (x_c, y_c)
2. If $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ then f has a local max at (x_c, y_c)
3. If $f_{xx}f_{yy} - f_{xy}^2 < 0$ then f has a saddle at (x_c, y_c)
4. If $f_{xx}f_{yy} - f_{xy}^2 = 0$ then f the test is indeterminant

Example 43

Consider the upside-down paraboloid $f(x,y) = 1 - x^2 - y^2$

$$f_x = -2x \quad \Rightarrow \quad f_x = 0 \text{ when } x = 0 \qquad f_y = -2y \quad \Rightarrow \quad f_x = 0 \text{ when } y = 0$$

So the only critical point is $(0,0)$. We also have

$$f_{xx} = -2 \qquad f_{yy} = -2 \qquad f_{xy} = 0$$

So $D = f_{xx}f_{yy} - f_{xy}^2 = (-2)(-2) - 0 = 4 > 0$

Since $f_{xx} = -2 < 0$ we have that the point is a local minimum.

Example 44

Consider the hyperbolic paraboloid $f(x,y) = y^2 - x^2$.

$$f_x = -2x \quad \Rightarrow \quad f_x = 0 \text{ when } x = 0 \qquad f_y = 2y \quad \Rightarrow \quad f_x = 0 \text{ when } y = 0$$

So the only critical point is $(0,0)$. We also have

$$f_{xx} = -2 \qquad f_{yy} = 2 \qquad f_{xy} = 0$$

So $D = f_{xx}f_{yy} - f_{xy}^2 = (-2)(2) - 0 = -4 < 0$

Since $D < 0$ we have that the cricical point $(0,0)$ is a saddle point.

Example 45

Find the shortest distance from the point $(2,0,-3)$ to the plane $x + y + z = 1$

We want to find the point (x,y,z) on the plane such that the distance from (x,y,z) to $(2,0,-3)$ is minimized. In other words, for general point (x,y,z) we want to minimize

$$\text{distance} = \sqrt{(x-2)^2 + y^2 + (z+3)^2}$$

Since minimizing the distance is the same as miniizing the square of the distance we can instead ask to minimize the the expression

$$\text{squared distance} = (x-2)^2 + y^2 + (z+3)^2$$

Now, we somehow need to enforce the fact that (x, y, z) is on the plane. We can do this by solving the equation of the plane for z and substituting this into the squared distance expression. Then, any value of x and y that we pick, the substituted expression for z guarantees that the point is on the plane. We have

$$z = 1 - x - y \qquad \Rightarrow \qquad f(x, y) = (x-2)^2 + y^2 + (4 - x - y)^2$$

We find the critical points by taking the partial derivatives and setting them equal to zero. Then

$$f_x = 2\,(x-2) - 2\,(4-x-y) = 4x + 2y - 12 \quad \text{and} \quad f_y = 2y - 2\,(4-x-y) = 2x + 4y - 8$$

So we need to find x and y that simultaneously satisfy

$$\begin{aligned} 4x + 2y - 12 &= 0 \qquad (1) \\ 2x + 4y - 8 &= 0 \qquad (2) \end{aligned}$$

Subtracting 2 times (2) from (1) gives

$$-6y + 4 = 0 \quad \Rightarrow \quad y = \frac{2}{3}$$

Then

$$2x + 4\left(\frac{2}{3}\right) = 8 \quad \Rightarrow \quad x = \frac{8}{3}$$

So the one critical point is $(8/3, 2/3)$. To check that this is really a minimum we do the second derivative test.

Computing the second partial derivatives we have

$$f_{xx} = 4 \qquad f_{xy} = 2 \qquad f_{yy} = 4 \qquad \Rightarrow \qquad f_{xx}f_{yy} - f_{xy}^2 = 16 - 4 = 12 > 0$$

Since $f_{xx} = 4 > 0$ and $f_{xx}f_{yy} - f_{xy}^2 = 16 - 4 = 12 > 0$ we conclude by the second derivative test that the critical point is a local minimum. Then the critical point $(8/3, 2/3)$ gives the x and y values of the closest point on the plane to $(2, 0, -3)$. Since we know that this point is on the plane, we can compute the z value using

$$z = 1 - x - y = 1 - \frac{8}{3} - \frac{2}{3} = -\frac{7}{3}$$

The mimimum distance from the point to the plane is then give by

$$D = \sqrt{\left(2 - \frac{8}{3}\right)^2 + \left(0 - \frac{2}{3}\right)^2 + \left(-3 + \frac{7}{3}\right)^2} = \frac{2\sqrt{3}}{3}$$

You can of course check that this is the same distance we'd find if we used the formula for point-to-plane distance from Chapter 10.

Example 46

Consider the function $F(a,b) = \int_a^b (-1)(x-1)(x-2)\,dx$ where $a, b > 0$ and $b > a$. Find intervals on which this integral is maximized.

We have

$$F_a = \frac{\partial}{\partial a}\int_b^a (+1)(1-x)(2-x)\,dx = (a-1)(a-2) \quad \text{and}$$

$$F_b = \frac{\partial}{\partial b}\int_a^b (-1)(1-x)(2-x)\,dx = -(b-1)(b-2)$$

Then

$$F_a(a,b) = 0 \quad \Rightarrow \quad a = 1 \text{ or } 2 \quad \text{and} \quad F_b(a,b) = 0 \quad \Rightarrow \quad b = 1 \text{ or } 2$$

So we have the four critical points $(1,1)$, $(1,2)$, $(2,1)$, and $(2,2)$. Now, the problem descriptions says that we need only consider intervals $[a,b]$ for which $b > a$. So technically we don't need to check the points $(1,1)$, $(2,1)$, and $(2,2)$, but we'll do so anyway because it's instructive.

To classify these points we need to do the second derivative test. The descriminant is $D = F_{aa}F_{bb} - F_{ab}^2$ where

$$F_{aa} = a - 2 + a - 1 = 2a - 3 \quad F_{bb} = -(b - 2 + b - 1) = 3 - 2b \quad F_{ab} = 0 \text{ so}$$

$$D(a,b) = (2a-3)(3-2b) - 0 = (2a-3)(3-2b)$$

Setting up a table for our second derivative test, we have

(a,b)	D	F_{aa}	Classification
$(1,1)$	-	-	saddle
$(1,2)$	$+$	-	max
$(2,1)$	$+$	$+$	min
$(2,2)$	-	$+$	indeterminant

Not surprisingly, the interval on which the integral is maximized is $[1,2]$. The intervals $[1,1]$ and $[2,2]$ will make the integral zero. The backwards interval $(2,1)$ will give the negative of the integral over $[1,2]$.

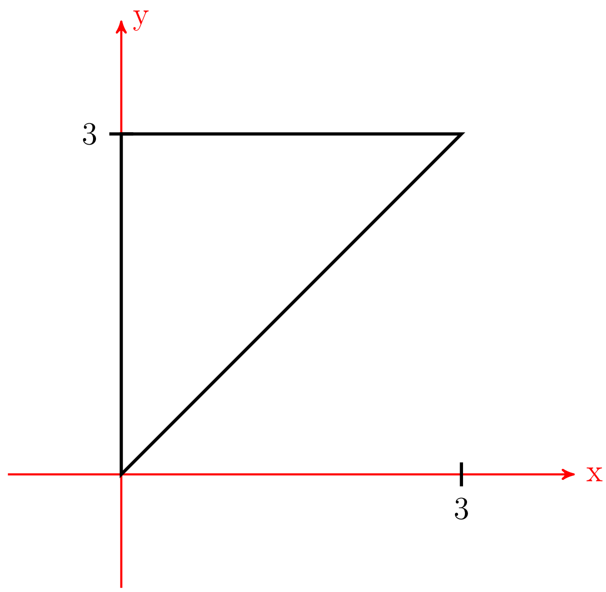
What if we want to know about the absolute extrema of $f(x, y)$ over some bounded region?

Extreme Value Theorem: If $f(x, y)$ is continuous on some bounded region B then f has an extreme min and an extreme max on the region B and these extrema occur either at critical poitns or on the boundary of the domain.

Example 47

Find the absolute mins and max's of $f(x, y) = 2x^2 - 4x + y^2 - 4y + 1$ on the region R bounded by the lines $x = 0$, $y = 3$, and $y = x$.

The region R looks as follows



The process for finding the absolute minimum and absolute maximum values of f on the region R is as follows

1. Find the values of f at the critical points on the interior of the region
2. Find the extreme values of f on the boundary
3. The largest value of f from these points is the absolute maximum. The smallest value is the absolute minimum.

Interior Critical Points

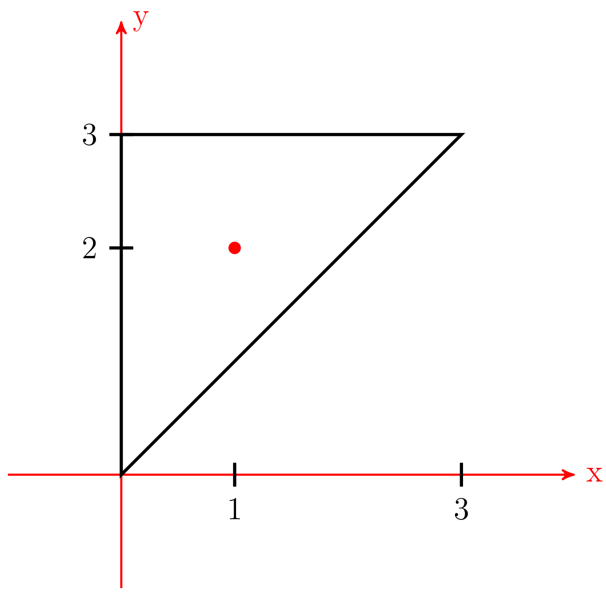
We find the critical values of f on the interior by taking partial derivatives and seeing where they're both zero or at least one does not exist.

$$f_x = 4x - 4 \qquad \text{and} \qquad f_y = 2y - 4$$

Setting these equal to zero we have

$$\begin{aligned} 4x - 4 &= 0 \quad \Rightarrow \quad x = 1 \\ 2y - 4 &= 0 \quad \Rightarrow \quad y = 2 \end{aligned}$$

Therefore the point $(1, 2)$ is a critical point. We also need to check that the critical point is in R, because if it's not then there is no point in considering it as an absolute extrema. In this case the point $(1, 2)$ is in R.



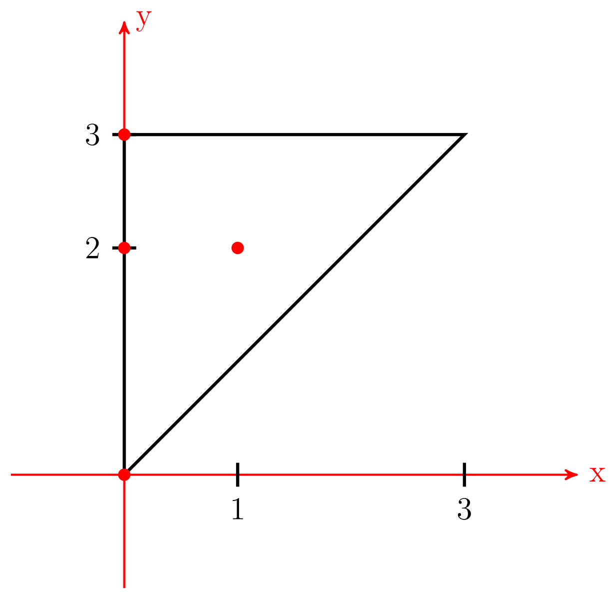
Next we find the extreme values on the boundaries. Since the boundary of R is made up of three different lines, we need to check each one.

Boundary: x=0

On the line $x = 0$ we have $f(0, y) = y^2 - 4y + 1$. Now the problem has been reduced to finding the extreme values of the function of one variable $g(y) = y^2 - 4y + 1$ on the interval $0 \le y \le 3$. Extreme values of $g(y)$ can occur at critical points of $g(y)$ and the endpoints of the interval. Taking the derivative and setting it equal to zero we have

$$g'(y) = 2y - 4 = 0 \quad \Rightarrow \quad y = 2 \quad \Rightarrow \quad (0, 2)$$

We also need to check the endpoints $y = 0$ and $y = 3$, which on the line $x = 0$ are the points $(0, 0)$ and $(0, 3)$. Updating our picture we have (so far) the following points to check

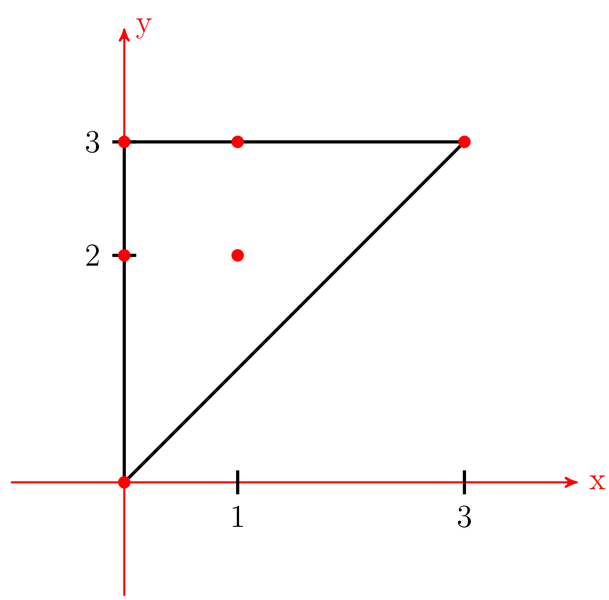


Boundary: y=3

On the line $y = 3$ we have $f(x, 3) = 2x^2 - 4x - 2$. Now the problem has been reduced to finding the extreme values of the function of one variable $h(x) = 2x^2 - 4x - 2$ on the interval $0 \leq x \leq 3$. Extreme values of $h(x)$ can occur at critical points of $h(x)$ and the endpoints of the interval. Taking the derivative and setting it equal to zero we have

$$h'(x) = 4x - 4 = 0 \quad \Rightarrow \quad x = 1 \quad \Rightarrow \quad (1, 3)$$

We also need to check the endpoints $x = 0$ and $x = 3$, which on the line $y = 3$ are the points $(0, 3)$ and $(3, 3)$. Updating our picture again we have (so far) the following points to check

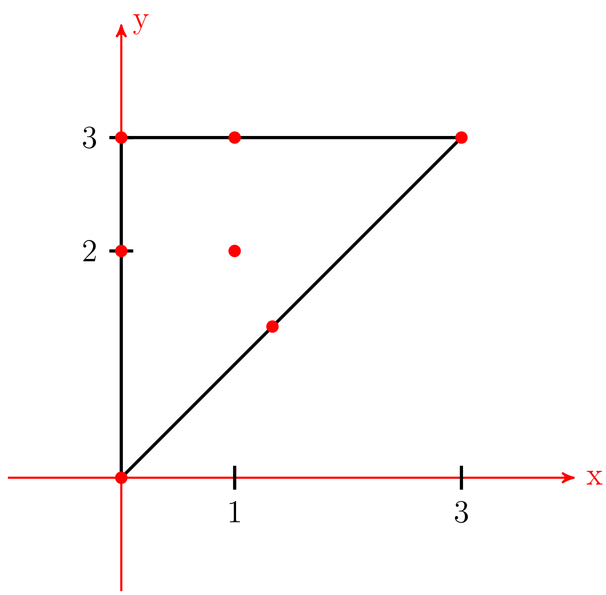


Boundary: y=x

On the line $y = x$ we have $f(x,x) = 3x^2 - 8x + 1$. Now the problem has been reduced to finding the extreme values of the function of one variable $q(x) = 3x^2 - 8x + 1$ on the interval $0 \le x \le 3$. Extreme values of $q(x)$ can occur at critical points of $q(x)$ and the endpoints of the interval. Taking the derivative and setting it equal to zero we have

$$q'(x) = 6x - 8 = 0 \quad \Rightarrow \quad x = \frac{4}{3} \quad \Rightarrow \quad \left(\frac{4}{3}, \frac{4}{3}\right)$$

Technically we need to check the endpoints, but we've already put them in the list from the analysis of the other two boundaries. So we now have the following points to check



We now take all of the potential points and compute their function values. We summarize these in a table

(x_c, y_c)	$f(x_c, y_c)$	Classification
$(1,2)$	-5	absolute min
$(0,0)$	1	
$(0,2)$	-3	
$(0,3)$	-2	
$(1,3)$	-4	
$(3,3)$	4	absolute max
$(4/3, 4/3)$	$-13/3$	

So the absolute maximum of the function on the region R is 4 which occurs at the point $(3,3)$. The absolute minimum is -5 which occurs at the point $(1,2)$.

Example 48

Find the absolute extrema of the function $f(x, y) = 2x^2 - y^2 + 6y$ on the region $R = \{(x, y) \,|\, x^2 + y^2 \leq 16\}$.

The region R is the circle of radius four centered at the origin. To find the abslute extrema we will find the critical points of f on the interior of R and then find the extreme values of f on the boundary of R.

Interior Points

Taking partial derivatives and setting them equal to zero we have

$$f_x = 2x = 0 \quad \Rightarrow \quad x = 0 \qquad \text{and} \qquad f_y = 6 - 2y = 0 \quad \Rightarrow \quad y = 3$$

So the single interior critical point is at $(0, 3)$ which is inside the region R.

Boundary

To look for extrema on the boundary we need to look at all values of $f(x, y)$ such that (x, y) satisfy $x^2 + y^2 = 16$. Solving for x^2 we have

$$x^2 = 16 - y^2 \qquad \text{for} \qquad -4 \leq y \leq 4$$

Remark: There are actually many ways to find the extreme values on the boundary. We could solve $x^2 + y^2 = 16$ for y to get $y = \pm\sqrt{16 - x^2}$ for $-4 \leq x \leq 4$. This represents the top and bottom halfs of the circle. Similarly we could solve the boundary expression for x to get the left and right halfs of the circle. But, in this case, since there is a sole x^2 in f it is convenient to solve directly for x^2.

Substituing the expression describing the circle into $f(x, y)$ we have

$$g(y) = 2\left(16 - y^2\right) - y^2 + 6y = -3y^2 + 6y + 32$$

Taking the derivative w.r.t. y and setting it equal to zero we have

$$g'(y) = -6y + 6 = 0 \quad \Rightarrow \quad y = 1 \quad \text{which gives the points on the circle } \left(\pm\sqrt{15}, 1\right)$$

We also need to check the endpoints $y = \pm 4$. On the circle these are the points $(0, \pm 4)$.

Making a table of function values we have

(x_c, y_c)	$f(x_c, y_c)$	Classification
$(0, 3)$	9	
$(0, 4)$	8	
$(0, -4)$	-40	absolute min
$\left(\pm\sqrt{15}, 1\right)$	35	absolute max

So the absolute minimum value of -40 occurs at the point $(0, -4)$. The absolute maximum of 35 occurs at the two points $\left(\pm\sqrt{15}, 1\right)$.

11.8 Lagrange Multipliers

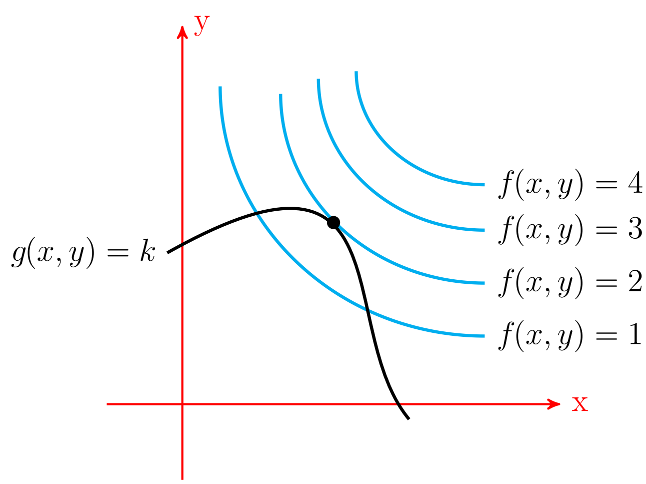
Last time we saw how we could find the extrema of a function $f(x, y)$ globally or inside some bounded domain. This is called **unconstrained optimization**. There are other instances when we want to know specifically the largest and smallest values the function $f(x, y)$ takes on subject to some constraint. This is called **constrained optimization**.

Suppose the constraint is described by the level curve $g(x, y) = k$. The mathematically this is described by

$$\begin{array}{rll} \text{Maximize/Minimize} & f(x,y) & \\ \text{Subject to} & g(x,y) & = \quad k \end{array}$$

Here $f(x, y)$ is called the **objective function** and $g(x, y) = k$ is called the **constraint** or the **constraint equation**.

Let's think about the problem visually by plotting level curves of $f(x, y)$ and the constraint $g(x, y) = k$ on the same axes. Suppose we have



From the picture, it's clear that the point where the maximum of f occurs on the constraint is the point (x, y) where the constraint curve and a level curve just barely touch. Recalling that the gradient vector of a function always points normal to the level curve, we see that this is precisly the point where the gradient vectors of f and g are parallel to each other.

One way to express this mathematically is that there exists some constant λ such that at the point (x, y) we have

$$\nabla f = \lambda \nabla g$$

The particular scalar λ that makes this statement true is called the **Lagrange Multiplier**.

Then, we can find the points on the constraint curve where the objective function is maximized/minimized by finding all points on the constraint curve where the gradient vectors of f and g are parallel. Then, we can reformulate the mathematical problem as finding all points (x, y) and multipliers λ such that simultaneously satisfy

$$\begin{aligned} \nabla f(x,y) &= \lambda \nabla g(x,y) \\ g(x,y) &= k \end{aligned}$$

Note that in solving for (x, y) we may need to also find λ along the way, but the value λ has no bearing on the solution.

Example 49

Find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

We have

$$\begin{aligned} \nabla \left(x^2+2y^2\right) &= \lambda \nabla \left(x^2+y^2\right) \\ x^2+y^2 &= 1 \end{aligned}$$

which gives

$$\begin{aligned} 2x &= \lambda 2x \\ 4y &= \lambda 2y \\ x^2+y^2 &= 1 \end{aligned}$$

From the first equation we know that either $x = 0$ or $\lambda = 1$.

If we assume $x = 0$ then the constraint equation gives us $y = \pm 1$. So we need to check the points $(0, \pm 1)$.

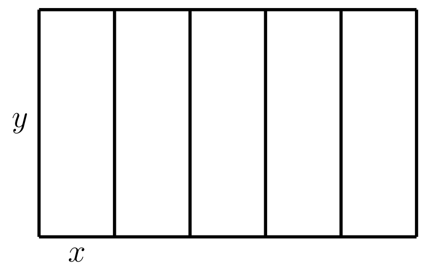
If $\lambda = 1$ then the second equation gives us $y = 0$ and the constraint equation gives us $x = \pm 1$. So we need to check the points $(\pm 1, 0)$.

Plugging these into the function we have

(x, y)	$f(x, y)$	Classification
$(0, \pm 1)$	2	absolute max
$(\pm 1, 0)$	1	absolute min

Example 50

Suppose you have 420 ft of fencing and decide to make a kennel by building 5 identical adjacent rectangular runs. Find the dimension of each run that maximizes its area.



Solution: Let x,y be the dimensions of a run and let A be its area. Then we want to maximize the objective function $A(x, y) = xy$ subject to the constraint that we use exactly 420 ft of fence. The constraint equation is

$$g(x, y) = 10x + 6y = 420$$

The equations we need to solve for the Lagrange Multiplier problem are

$$\begin{aligned} \nabla A &= \lambda \nabla g \\ g(x, y) &= 420 \end{aligned}$$

Which becomes

$$\begin{aligned} y &= 10\lambda \\ x &= 6\lambda \\ 10x + 6y &= 420 \end{aligned}$$

First we check if $\lambda = 0$ yields a solution. In this case it clearly does not because $\lambda = 0$ implies that $(x, y) = (0, 0)$ which yields a run of zero area. Instead, we solve for λ in terms of y in the first equation and substitute it into the second. This then yields the reduced system

$$\begin{aligned} x &= \frac{3}{5}y \\ 10x + 6y &= 420 \end{aligned}$$

Plugging this into the constraint equation yields

$$6y + 6y = 420 \quad \Rightarrow \quad y = 35 \quad \Rightarrow \quad x = 21$$

This gives a maximum area of a run of $A(21, 35) = 735$ sq ft

Example 51

The moon's orbit around the earth is well-approximated by the curve

$$x^2 + y^2 = \left(3 \times 10^5 - 0.05y\right)^2$$

where distance is in miles and the earth is located at the origin. How close is the moon to the earth at its closest point? What is the farthest the moon ever is from the earth?

Solution: The objective function that we are trying to both maximize and minimize is the distance from the point (x, y) to the origin. Of course, minimizing the distance is the same as minimizing the square of the distance, so we take as our objective function

$$f(x, y) = x^2 + y^2$$

The constraint is that the point is on the moon's orbit, so we have

$$g(x, y) = x^2 + y^2 - \left(3 \times 10^5 - 0.05y\right)^2 = 0$$

Then, the system of equations for the Lagrange Multiplier problem is given by

$$\begin{aligned} \nabla f &= \lambda \nabla g \\ g(x, y) &= 0 \end{aligned}$$

which becomes

$$\begin{aligned} 2x &= \lambda 2x & (3) \\ 2y &= \lambda \left[2y - 2\left(3 \times 10^5 - 0.05y\right)(-0.05)\right] & (4) \\ 0 &= x^2 + y^2 - \left(3 \times 10^5 - 0.05y\right)^2 & (5) \end{aligned}$$

First we check to see if $\lambda = 0$ gives us a valid solution. From the first two equations we have

$$\lambda = 0 \quad \Rightarrow \quad (x, y) = 0$$

which is not on the constraint curve. Since $\lambda \neq 0$ we are free to eliminate λ from the equation by dividing equation (2) by equation (1). Cross multiplying then gives

$$2x \left[2y - 2\left(3 \times 10^5 - 0.05y\right)(-0.05)\right] = 4xy$$

which simplifies to

$$x\left(3\times 10^5-0.05y\right)=0$$

Solving this equation for x and y gives $x=0$ or $y=6\times 10^6$.

If $y=6\times 10^6$ then the constraint equation gives

$$x^2+6\times 10^6=0$$

which has no real solution. On the other hand if we choose $x=0$ the constraint equation gives

$$y^2-\left(3\times 10^5-0.05y\right)^2=0$$

Using the quadratic formula we find that two possible solutions are

$$(x,y)=(0,-315790)\quad \text{and}\quad (x,y)=(0,285710)$$

Plugging these into the distance formula we find that for the first point the moon is 315790 miles from the earth and the second point is 285710 miles from the earth. Clearly the first point is the location at which the moon is farthest from the earth and the second is when it's closest to the earth.

Example 52

Many airlines require that carry-on baggage have a linear distance (length + width + height) of no more than 45 inches with an additional requirement of being able to slide under the seat in front of you. If we assume that the carry-on is roughly the shape of a rectangular box and one dimension is no more than half of one of the other dimensions (to insure that it can slide under the seat) then what dimensions of the carry-on lead to maximum volume?

Solution: If we let x, y and z denote length, width, and height, respectively, then our goal is maximize the volume $V(x,y,z)=xyz$ subject to the constraints

$$g(x,y)=x+y+z=45\quad\text{and}\quad h(x,y)=y-2x=0$$

Note that this is an optimization problem with two constraints. The Lagrange Multipliers system looks like

$$\begin{aligned}\nabla V &= \lambda\nabla g+\mu\nabla h\\ g &= 45\\ h &= 0\end{aligned}$$

Taking partial derivatives we obtain the following system

$$
\begin{aligned}
yz &= \lambda - 2\mu \\
xz &= \lambda + \mu \\
xy &= \lambda \\
x + y + z &= 45 \\
y - 2x &= 0
\end{aligned}
$$

Eliminating λ yields the updated system

$$
\begin{aligned}
yz &= xy - 2\mu \\
xz &= xy + \mu \\
x + y + z &= 45 \\
y &= 2x
\end{aligned}
$$

Solving for μ in the second equation and substituting it into the first gives

$$
\begin{aligned}
yz &= xy - 2\left(xz - xy\right) \\
x + y + z &= 45 \\
y &= 2x
\end{aligned}
$$

which simplifies to

$$
\begin{aligned}
yz &= 3xy - 2xz \\
x + y + z &= 45 \\
y &= 2x
\end{aligned}
$$

Substituting in for y in terms of x gives

$$
\begin{aligned}
4xz &= 6x^2 \\
3x + z &= 45
\end{aligned}
$$

Then, eliminating z gives

$$
4x\left(45 - 3x\right) = 3x^2 \quad \Rightarrow \quad 90x - 6x^2 = 3x^2 \quad \Rightarrow \quad x\left(90 - 9x\right) = 0
$$

Since $x \neq 0$ we must have that $x = 10$ which implies that $y = 20$ and $z = 15$.

Taylor's Formula

Consider the function $f(x, y)$. Recall that we can approximate $f(x, y)$ with a linear function in x and y:

$$f(x, y) \approx f(a, b) + f_x(a, b)\,(x - a) + f_y(a, b)\,(y - b)$$

Notice that again this is just a linear polynomial in two-variables that does a good job of approximating f near the point $(x, y) = (a, b)$. It's also exactly the equation of the tangent plane to the surface f at the point (a, b).

Example 53

Find the linear approximation to $f(x, y) = xe^y$ at the point $(0, 0)$.

We need evaluate the function and its first partial derivatives at the point $(0, 0)$. We have

$$\begin{array}{ll} f = xe^y & f\,(0, 0) = 0 \\ f_x = e^y & f_x\,(0, 0) = 1 \\ f_y = xe^y & f_y\,(0, 0) = 0 \end{array}$$

Then the linear approximation is

$$f(x, y) \quad \approx \quad 0 + (1 \cdot (x - 0) + 0 \cdot (y - 0)) = x = L(x, y)$$

Example 54

Use $L(x, y)$ to approximate $f(x, y) = xe^y$ at the point $(0.05, 0.05)$ and find the error in the approximation.

$$L(0.05, 0.05) = 0.05 \qquad f(0.05, 0.05) = 0.05e^{0.05} = 0.052564$$

$$|E(0.05, 0.05)| = |L(0.05.0.05) - f(0.05, 0.05)| = 2.5 \times 10^{-3}$$

OK, that's pretty good. But what if we need to do better? The linearization is the best approximation by a linear polynomial of $f(x, y)$ near the point $(0, 0)$. It's natural to ask if we can get a better approximation if we use a quadratic polynomial.

It turns out that we can. The quadratic approximation of $f(x, y)$ near the general point (a, b) is given by

$$\begin{aligned} f(x,y) &\approx f(a,b) + f_x(a,b)\,(x-a) + f_y(a,b)\,(y-b) + \\ &\frac{1}{2}\left[f_{xx}(a,b)\,(x-a)^2 + 2f_{xy}(a,b)\,(x-a)\,(y-b) + f_{yy}(a,b)\,(y-b)^2\right] \end{aligned}$$

Notice that the first three terms in the approximation are just the linearization of $f(x, y)$ about the point (a, b). The additional terms are quadratic in x and y and involve the second partial derivatives of f evaluated at the point (a, b).

Example 55

Find a quadratic approximation to $f(x, y) = xe^y$ at the point $(0, 0)$.

We already computed the value of the function and it's first partial derivatives at the point $(0, 0)$ when computing the linearization in the previous example. Now we need the second partials.

$$\begin{array}{ll} f_{xx} = 0 & f_{xx}\,(0,0) = 0 \\ f_{xy} = e^y & f_{xy}\,(0,0) = 1 \\ f_{yy} = xe^y & f_{yy}\,(0,0) = 0 \end{array}$$

Then the quadratic approximation is

$$\begin{aligned} f(x,y) &\approx L\,(x,y) + \frac{1}{2}\left(0\cdot(x-0)^2 + 2\cdot 1\cdot(x-0)\,(y-0) + 0\cdot(y-0)^2\right) \\ &\approx x + xy = Q(x,y) \end{aligned}$$

Example 56

Use $Q(x, y)$ to approximate $f(x, y) = xe^y$ at the point $(0.05, 0.05)$ and find the error in the approximation.

$$Q(0.05, 0.05) = 0.05 + (0.05)^2 = 0.0525 \qquad f(0.05, 0.05) = 0.05e^{0.05} = 0.052564$$

$$|E(0.05, 0.05)| = |Q(0.05.0.05) - f(0.05, 0.05)| = 6.4 \times 10^{-5}$$

Notice that, not surprisingly, the quadratic approximation has a smaller error than the linear approximation.

OK, so we've found a linear approximation and quadratic approximation to $f(x, y)$ near the point (a, b). It turns out that we can come up with a polynomial approximation of any degree we like that approximates $f(x, y)$ near the point (a, b). This result is called **Taylor's Theorem** for functions of two variables. Of course, we've seen this before.

Recall that in Calc II we used Taylor's Formula to approximate a function $f(x)$ near a point $x = a$ by a sequence of polynomials.

Theorem: If f has $n+1$ continuous partial derivatives in an open interval I around $x = a$, then

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \\ + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{(n+1)}$$

for some $c \in I$.

Notice that if we take $n = 1$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(c)}{2}(x-a)^2$$

then the first two terms are exactly the equation of the tangent line to f at the point $x = a$, which in turn is exactly the linearization of f about the point $x = a$.

The remainder term is just the next term in the Taylor Series. Notice that the second derivative in the remainder term is evaluated at some point $x = c$ instead of $x = a$. It turns out that for some value c between x and a this expression is exact. The hitch is that we don't know exactly what c is. The remainder term is useful because it can be used to get an upper bound on the error incurred by using the linear approximation to approximate values of f near $x = a$.

Of course, the power of Taylor's Formula is that we can use it to obtain higher-order polynomial approximations to f near $x = a$ for any degree polynomial that we like. If we want to approximate f using a quadratic polynomial then we use

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(c)}{3!}(x-a)^3$$

Taylor's Theorem for Functions of Two Variables

OK, so how do we do this for functions of two variables? It turns out it's pretty straightforward and very similar to Taylor's Theorem for functions of one variable. But to do this we need to introduce some new notation. First, let $\Delta x = (x-a)$ and $\Delta y = (y-b)$. Then we define a special **operator** as follows

$$\left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)f\bigg|_{(a,b)} = \Delta x f_x(a,b)+\Delta y f_y(a,b) = (x-a)\,f_x(a,b)+(y-b)\,f_y(a,b)$$

Notice that the operator is a rule for applying this particular sum of partial derivatives to the function f and then evaluating them at the point (a,b). Notice also that this is exactly the linear part of the linearization $L(x,y)$.

To get the quadratic term for the quadratic approximation we do this twice. Note that when taking derivatives, we treat Δx and Δy as constants.

$$\begin{aligned}
\left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)^2 f\bigg|_{(a,b)} &= \left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)\left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)f\bigg|_{(a,b)} \\
&= \left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)(\Delta x f_x+\Delta y f_y)\bigg|_{(a,b)} \\
&= \left(\Delta x^2 f_{xx}+2\Delta x\Delta y f_{xy}+\Delta y^2 f_{yy}\right)\bigg|_{(a,b)} \\
&= f_{xx}\,(a,b)\,(x-a)^2+2f_{xy}\,(a,b)\,(x-a)\,(y-b)+f_{yy}\,(a,b)\,(y-b)^2
\end{aligned}$$

So the quadratic approximation to $f(x,y)$ at the point (a,b) can be written as

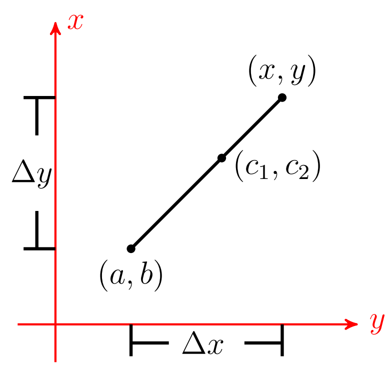
$$f(x,y)\approx Q(x,y)=f(a,b)+\left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)f\bigg|_{(a,b)}+\frac{1}{2}\left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)^2 f\bigg|_{(a,b)}$$

OK, so how do we generalize to an n-degree polynomial approximation of $f(x,y)$, and what about that remainder term? It turns out that it exactly follows the same pattern of Taylor's Theorem for functions of one variable, but the regular derivatives are replaced by powers of the operator described above. We have the following

Taylor's Theorem. Suppose $f(x,y)$ has $n+1$ continuous partial derivatives in an open region R near $(x,y)=(a,b)$, then for $\Delta x=(x-a)$ and $\Delta y=(y-b)$ we have

$$\begin{aligned} f(x,y) &= f(a,b) + \left(\Delta x\frac{\partial}{\partial x} + \Delta y\frac{\partial}{\partial y}\right) f\Big|_{(a,b)} + \frac{1}{2}\left(\Delta x\frac{\partial}{\partial x} + \Delta y\frac{\partial}{\partial y}\right)^2 f\Big|_{(a,b)} \\ &\quad + \frac{1}{3!}\left(\Delta x\frac{\partial}{\partial x} + \Delta y\frac{\partial}{\partial y}\right)^3 f\Big|_{(a,b)} + \cdots + \frac{1}{n!}\left(\Delta x\frac{\partial}{\partial x} + \Delta y\frac{\partial}{\partial y}\right)^n f\Big|_{(a,b)} \\ &\quad + \frac{1}{(n+1)!}\left(\Delta x\frac{\partial}{\partial x} + \Delta y\frac{\partial}{\partial y}\right)^{n+1} f\Big|_{(c_1,c_2)} \end{aligned}$$

where here the remainder term is evaluated at some (unknown) point (c_1,c_2) on the line connecting (a,b) and (x,y).



Taylor's Theorem is powerful for a couple of reasons. The first is that it allows us a methodical way of coming up with polynomial approximations to $f(x,y)$ near a point (a,b) for any degree polynomial that we like. The second is that it allows us to use the remainder term to get an upper bound on the error incurred when using the approximation.

Example 57

Use Taylor's Formula to find a cubic approximation to $f(x, y) = xe^y$ at the point $(0, 0)$.

If we want to do the cubic approximation then we need to evaluate the cubic term in the series. We have

$$\left(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y}\right)^3 f\bigg|_{(a,b)} = \Delta x^3 f_{xxx} + 3\Delta x^2 \Delta y f_{xxy} + 3\Delta x \Delta y^2 f_{xyy} + \Delta y^3 f_{yyy}$$

It turns out that you can easily get the coefficients of the expansion from Pascal's Triangle

$$\begin{array}{c} 1 \\ 1 \;\; 1 \\ 1 \;\; 2 \;\; 1 \\ 1 \;\; 3 \;\; 3 \;\; 1 \\ 1 \;\; 4 \;\; 6 \;\; 4 \;\; 1 \end{array}$$

To get the cubic terms in the example we need to evaluate some third-order partial derivatives

$$\begin{array}{ll} f_{xxx} = 0 & f_{xxx}(0,0) = 0 \\ f_{xxy} = 0 & f_{xxy}(0,0) = 0 \\ f_{xyy} = e^y & f_{xyy}(0,0) = 1 \\ f_{yyy} = xe^y & f_{yyy}(0,0) = 0 \end{array}$$

Then

$$\begin{aligned} f(x,y) &\approx x + xy + \frac{1}{3!}\left(0 \cdot (x-0)^3 + 3 \cdot 0 \cdot (x-0)^2 (y-0) + 3 \cdot 1 \cdot (x-0)(y-0)^2 + 0 \cdot (y-0)^3\right) \\ &= x + xy + \frac{xy^2}{2} \end{aligned}$$

If we use the cubic approximation to approximate the function at the point $(0.05, 0.05)$ we find that the exact error incurred is around 1×10^{-6}, which is again better than the linear and quadratic approximations.

Error in the Taylor Approximation

The remainder term in Taylor's Theorem gives us a way to find an upper bound on the error incurred by approximating a function $f(x, y)$ using a Taylor polynomial. The remainder term is always taken to be the next term in the series beyond those used in the approximation. The only difference is that the remainder term is evaluated at some unknown point (c_1, c_2) instead of (a, b). For instance, if we want to bound the error in the linear approximation, the remainder term is the quadratic term in the polynomial.

Recall (one more time) that the linear approximation of $f(x, y)$ at (a, b) is given by

$$L(x, y) = f(a, b) + f_x(a, b)\,(x - a) + f_y(a, b)\,(y - b)$$

Then from the theorem we see that

$$E(x, y) = f(x, y) - L(x, y) = \frac{1}{2}\left[f_{xx}\,(c_1, c_2)\,(x - a)^2 + 2f_{xy}\,(c_1, c_2)\,(x - a)\,(y - b) + f_{yy}\,(c_1, c_2)\,(y - b)^2\right]$$

Then to get a bound on the **worst-case scenario** error we put absolute values around everything in the remainder term. This guarantees that we don't get any helpful cancellation in the remainder from some terms being positive and some being negative.

$$|E(x, y)| \leq \frac{1}{2}\left[|f_{xx}|\,|x - a|^2 + 2\,|f_{xy}|\,|x - a|\,|y - b| + |f_{yy}|\,|y - b|^2\right]$$

If M is an upper bound on each of the second partial derivatives in the region of interest such that $|f_{xx}|, |f_{xy}|, |f_{yy}| \leq M$ then we have

$$|E(x, y)| \leq \frac{M}{2}\left[|x - a|^2 + 2\,|x - a|\,|y - b| + |y - b|^2\right] = \frac{M}{2}\left(|x - a| + |y - b|\right)^2$$

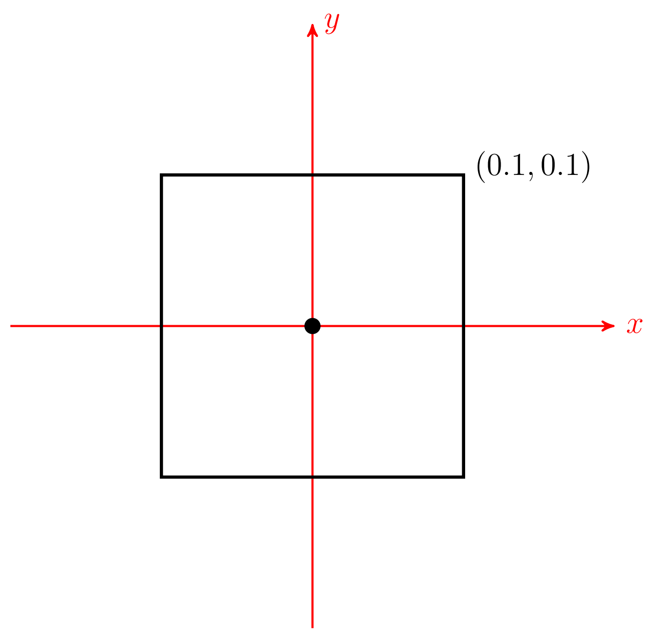
Example 58

Consider again the function $f(x, y) = xe^y$ near the point $(0, 0)$. Find a bound on the error if we use the linearization to approximate f for any x and y satisfying $|x| \leq 0.1$ and $|y| \leq 0.1$.

Note here that we want to find an upper bound when using the approximation to approximate f at **any** point in the region of interest. From the previous example we know that $L(x, y) = x$. To use the error formula we derived previously we need to find an upper bound on the second partial derivatives in the region $|x| \leq 0.1$ and $|y| \leq 0.1$. The second partials were

$$f_{xx} = 0 \quad f_{xy} = e^y \quad f_{yy} = xe^y$$

We want to find the worst-case scenario for the error when $|x| \leq 0.1$ and $|y| \leq 0.1$. So we need to choose points that make the second derivatives as large as possible in the given region shown below



To find M we need to figure out the largest values that any of the second partials can take on in the desired region. We have

$$\begin{aligned} |f_{xx}| &= 0 \\ |f_{xy}| &= |e^y| \leq e^{0.1} \\ |f_{yy}| &= |xe^y| \leq 0.1e^{0.1} \end{aligned}$$

The biggest value that the three partials take on in the given region is $M = e^{0.1}$, so we have

$$|E(x, y)| \leq \frac{e^{0.1}}{2} (|x| + |y|)^2 \leq \frac{e^{0.1}}{2} (0.1 + 0.1)^2 \approx 2.2 \times 10^{-2}$$

Note that this error bound is larger than the actual error incurred when we approximated f at the point $(0.05, 0.05)$. This makes sense because this error bound is valid for **any** point in the region of interest.

Example 59

Example. Use Taylor's Theorem to find the linear approximation to $f(x, y) = y \cos x$ at the point $(\pi, 0)$ and use it to approximate f at the point $(3.1, 0.15)$. Find a bound on the error if the linear approximation is used to approximate f for x in $[\pi - 0.1, \pi + 0.1]$ and y in $[-0.2, 0.2]$.

For the linearization we need to evaluate f and it's first partial derivatives at $(\pi, 0)$.

$$\begin{array}{ll} f = y\cos x & f(\pi,0) = 0 \\ f_x = -y \sin x & f_x(\pi, 0) = 0 \\ f_y = \cos x & f_y(\pi, 0) = -1 \end{array}$$

Then the linearization of f at $(\pi, 0)$ is given by

$$L(x, y) = -y$$

Evaluating both f and L at $(3.1, 0.15)$ we find

$$f(3.1, 0.15) = -0.14987... \quad L(3.1, 0.15) = -0.15 \quad \text{and} \quad |E(x, y)| = 1.30 \times 10^{-4}$$

To find an upper bound on the error we need to bound the second-partial derivatives of f in the desired region. We have

$$\begin{aligned} f_{xx} &= -y\cos x \\ f_{xy} &= -\sin x \\ f_{yy} &= 0 \end{aligned}$$

In the region that we care about, we have the following bounds on the second-partial derivatives:

$$\begin{aligned} &|f_{xx}| = |-y\cos x| \leq 0.2 \\ &|f_{xy}| = |-\sin x| \leq |-\sin(\pi + 0.1)| \approx 0.1 \\ &|f_{xx}| = 0.0 \end{aligned}$$

So we pick $M = 0.2$. The error bound for general (x, y) is then

$$|E(x,y)| \leq \frac{0.2}{2} \left(|x - \pi| + |y|\right)^2$$

Then, plugging in $x = \pi + 0.1$ and $y = 0.2$ we have the following bound on the error the linear approximation

$$|E(x,y)| \leq \frac{0.2}{2} (0.1 + 0.2)^2 = 9 \times 10^{-3}$$

which is greater than the exact error for the approximation at the point $(3.1, 0.15)$.

Example 60

Consider again the function $f(x,y) = xe^y$ near the point $(0,0)$. Find a bound on the error if we use the quadratic approximation of f for any x and y satisfying $|x| \leq 0.1$ and $|y| \leq 0.1$.

To find a bound on the quadratic approximation to f we use the cubic remainder term in the Taylor Polynomial. For a general f we have

$$E(x,y) = f(x,y) - Q(x,y) = \frac{1}{3!}\left[f_{xxx}(c_1,c_2)(x-a)^3 + 3f_{xxy}(c_1,c_2)(x-a)^2(y-b) + \right.$$

$$\left. +3f_{xyy}(c_1,c_2)(x-a)(y-b)^2 + f_{yyy}(c_1,c_2)(y-b)^3\right]$$

Then putting absolute values around everything in the remainder term we get the following upper bound.

$$|E(x,y)| \leq \frac{1}{3!}\left[|f_{xxx}||x-a|^3 + 3|f_{xxy}||x-a|^2|y-b| + |f_{xyy}||x-a||y-b|^2 + |f_{yyy}||y-b|^3\right]$$

If M is an upper bound on each of the second partial derivatives in the region of interest such that $|f_{xxx}|, |f_{xxy}|, |f_{xyy}|, |f_{yyy}| \leq M$ then we have

$$|E(x,y)| \leq \frac{M}{3!}\left[|x-a|^3 + 3|x-a|^2|y-b| + 3|x-a||y-b|^2 + |y-b|^3\right] \quad \Rightarrow$$

$$|E(x,y)| \leq \frac{M}{3!}\left(|x-a| + |y-b|\right)^3$$

To determine the upper bound on the error in the example we need to bound each of the third partial derivatives in the region of interest. We have

$$\begin{aligned}
&|f_{xxx}| = 0 \\
&|f_{xxy}| = 0 \\
&|f_{xyy}| = |e^y| \leq e^{0.1} \\
&|f_{yyy}| = |xe^y| \leq 0.1e^{0.1}
\end{aligned}$$

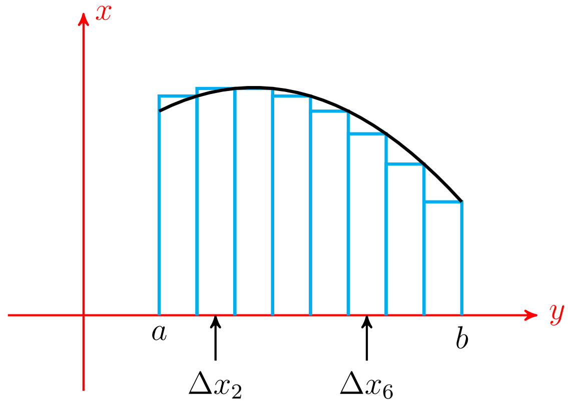
Again we see that the largest value that the third partial derivatives take on on the region is $M = e^{0.1}$. Then, plugging this into the error formula we have

$$|E(x,y)| \leq \frac{M}{3!}\left(|x-0| + |y-0|\right)^3 \leq \frac{e^{0.1}}{3!}(0.1+0.1)^3 = 1.47 \times 10^{-3}$$

This is the worst-case scenario error that can be incurred by using the quadratic approximation to approximate f in the region of interest.

12.1 Double Integrals over Rectangles

Recall that for functions of a single variable we can use a definite integral to compute the area under a curve



Then approximating the integral over $f(x)\ dx$ is done by approximating the area of the little chunks with area $f(x_k)\ \Delta x_k$ and summing them up.

$$S_N = \sum_{k=1}^{N} f(x_k)\ \Delta x_k$$

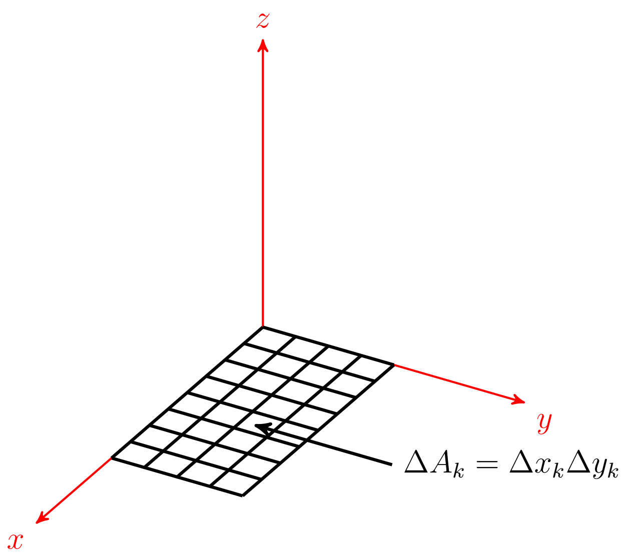
The definite integral is then defined as the limit as the number of subintervals goes to infinity.

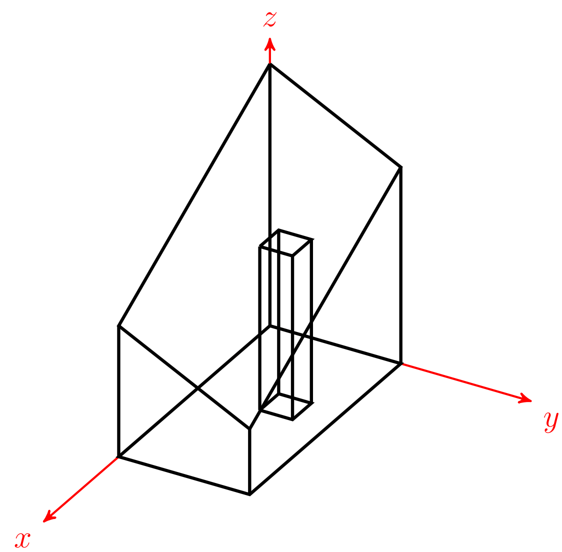
$$\lim_{N\to\infty} S_N = \int_a^b f(x)\ dx$$

For functions of two variables, we can use an integral to compute the volume under a surface defined by $z = f(x, y)$. Consider the volume under the surface $f(x, y) = 4 - x - y$ in region R where $R = \{(x, y) \mid\ 0 \le x \le 2, 0 \le y \le 1\}$.

Chris Ketelsen

APPM 2350
Chapter 12
November 19, 2015





Then approximating the volume under the surface over R is done by approximating the volume of the little chunks with volume $f(x_k, y_k)\ \Delta x_k \Delta y_k$ and summing them up. We have

$$S_N = \sum_{k=1}^{N} f(x_k, y_k)\ \Delta A_k = \sum_{k=1}^{N} f(x_k, y_k)\ \Delta x_k \Delta y_k$$

The exact value is obtained by taking the limit as the number of little volume chunks goes to infinity.

$$\lim_{N\to\infty} S_n = \iint_R f(x,y)\ dA = \iint_R f(x,y)\ dxdy$$

Properties of the Double Integral

1. $\displaystyle\iint_R k\left(f(x,y) + g(x,y)\right)\ dA = k\iint_R f(x,y)\ dA + k\iint_R g(x,y)\ dA$
2. If $f(x,y) \geq 0$ on R then $\displaystyle\iint_R f(x,y)\ dA \geq 0$
3. If $f(x,y) \geq g(x,y)$ on R then $\displaystyle\iint_R f(x,y)\ dA \geq \iint_R g(x,y)\ dA \geq 0$

Example: Find the volume under the function $f(x,y) = 4 - x - y$ over the region $R = [0,2] \times (0,1)$.

$$\iint_R 4 - x - y\, dA = \iint_R 4 - x - y\, dydx = \int_0^2 \left(\int_0^1 4 - x - y\, dy\right)\ dx =$$

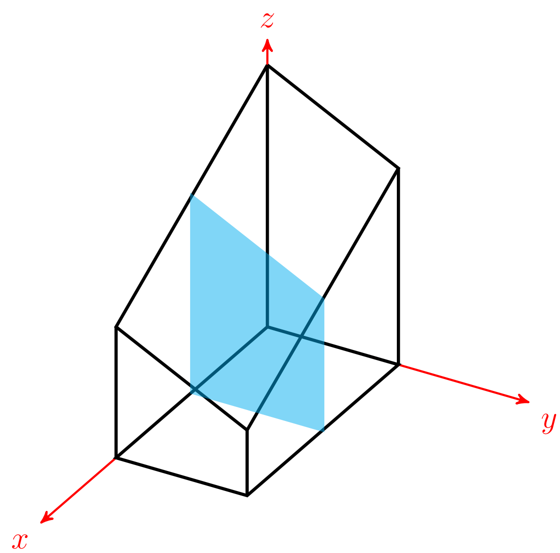
$$\int_0^2 \left(4y - xy - \frac{y^2}{2}\bigg|_0^1\right)\ dx = \int_0^2 \left(\frac{7}{2} - x\right)\ dx = \frac{7}{2}x - \frac{x^2}{2}\bigg|_0^2 = 7 - 2 = 5$$

Geometric Interpretation

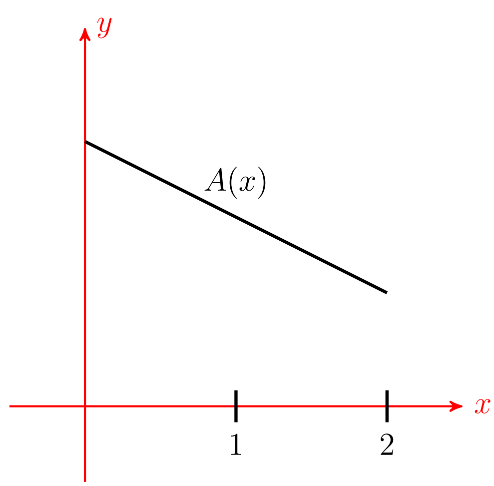
Notice that

$$\iint_R f(x,y)\ dA = \int_{x=0}^{2} A(x)\ dx \quad \text{with} \quad A(x) = \frac{7}{2} - x \quad \text{since} \quad \int_{y=0}^{1} 4 - x - y\, dy = \frac{7}{2} - x$$

$A\,(x)$ represents the area of a slice of the volume under the surface at the point x. Then the volume of a thin slice is $A(x)\,dx$. We get the entire volume by summing up all of the slices using an integral.



We've now reduced the problem to an integral of a function of a single variable. $\int_{x=0}^{2} A(x)\ dx$

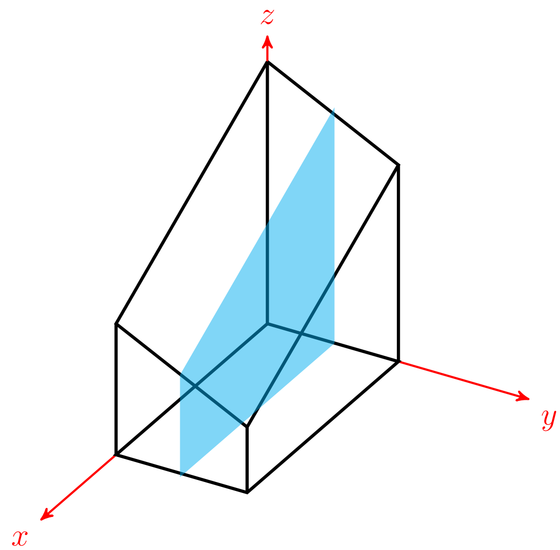


What would happen if we integrated over x first and y second?

$$\int_0^1 \int_{x=0}^{2} (4 - x - y)\ dxdy = \int_0^1 \left(4x - \frac{x^2}{2} - xy \Big|_0^2 \right) dy = \int_0^1 6 - 2y\, dy = 6y - y^2 \Big|_0^1 = 5$$

Geometric Interpretation

$$\int_0^1 \int_{x=0}^2 (4 - x - y)\ dxdy = \int_0^1 \left(4x - \frac{x^2}{2} - xy\Big|_0^2 \right) dy = \int_0^1 6 - 2y\, dy = \int_0^1 A(y)\ dy$$



Fubini's Theorem: If f is continuous on a rectangle $R = [a, b] \times [c, d]$, then

$$\iint_R f(x, y)\ dA = \int_a^b \int_c^d f(x, y)\ dydx = \int_c^d \int_a^b f(x, y)\ dxdy$$

Fubini's Theorem is more generally true as long as f is bounded on R and f is discontinuous only on a finite number of smooth curves in R.

Example: Find the volume under $f(x, y) = (1 - x) \sin(\pi y)$ on $R = [0, 2] \times [0, 1]$.

$$\int_0^2 \int_0^1 (1 - x) \sin(\pi y)\ dydx = \int_0^2 (1 - x) \left(\frac{-1}{\pi} \right) \cos(\pi y)\Big|_0^1 dx = \frac{2}{\pi} \int_0^2 1 - x\, dx$$

$$= \frac{2}{\pi} \left(x - \frac{x^2}{2}\Big|_0^2 \right) = \frac{2}{\pi}(2 - 2) = 0$$

Why would we get zero? Break up R into R_1 and R_2 with $R_1 = [0, 1] \times (0, 1)$ and $R_2 = [1, 2] \times [0, 1]$.

$$\iint_{R_1} f(x, y)\ dydx = \frac{1}{\pi} \qquad \iint_{R_2} f(x, y)\ dydx = -\frac{1}{\pi}$$

Volume over R_2 is below the $z = 0$ plane. The small volume elements have volume $f(x_k, y_k)\ \Delta A_k$ with $f(x_k, y_k) < 0$ which gives a negative volume!

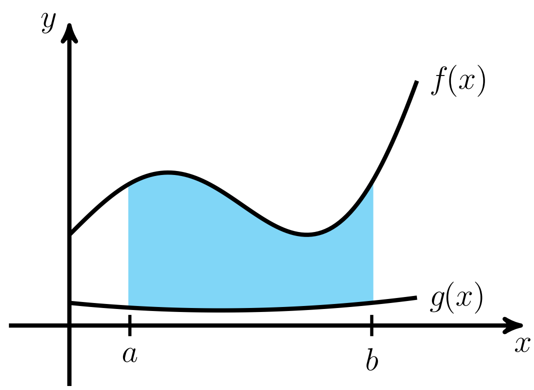
12.2 Double Integrals over General Regions

Recall that $\iint_R f(x,y)\ dA = \lim_{N\to\infty}\sum_{k=1}^{N} f(x_k,y_k)\ \Delta A_k$ gave the volume under the surface f over the region R.

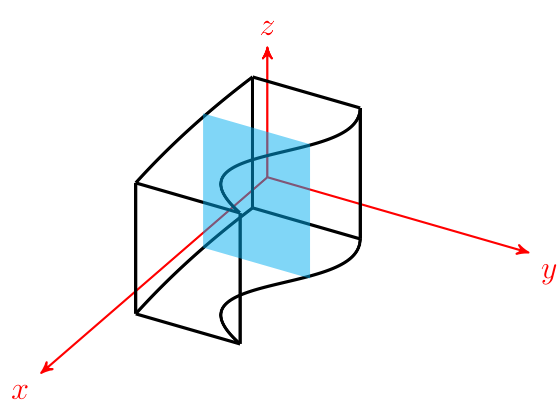
Suppose that f is the constant function $f=1$ Then $S_N = \sum_{k=1}^{N}\Delta A_k$ and

$$\lim_{N\to\infty} S_N = \iint_R dA = \text{Area of Region } R$$

OK, this is kinda boring if R is a rectangle. What if R is more complicated:



Think about integrating $f=1$ with order $dydx$ and remember the area slices we used to compute the volume under a surface.



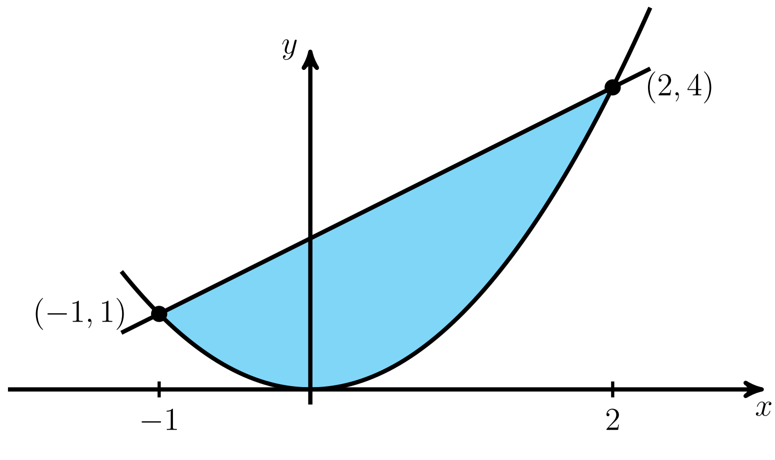
The area of this slice is then $A(x) = 1\cdot(g(x)-f(x))$.

So

$$\text{Area of } R = \iint_R dA = \int_a^b A(x)\ dx = \int_a^b (g(x)-f(x))\ dx = \int_a^b\int_{f(x)}^{g(x)} dydx$$

Example: Find the area of the region bounded by the curves $y = x + 2$ and $y = x^2$.

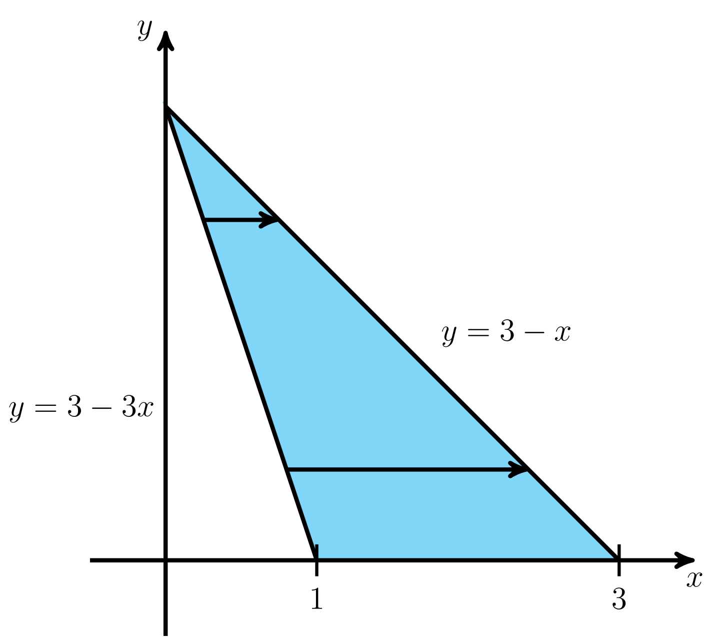
It's always a good idea to draw a picture for these problems.



$$A = \int_{-1}^{2}\int_{x^2}^{2+x} dydx = \int_{-1}^{2} y\,\Big|_{x^2}^{x+2} dx = \int_{-1}^{2} x + 2 - x^2\,dx = \frac{x^2}{2} + 2x - \frac{x^3}{3}\Big|_{-1}^{2}$$

$$= \left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right) = 8 - 3 - \frac{1}{2} = \frac{9}{2}$$

Example: Find the area of the region bounded by the x-axis, $y = 3 - x$, and $y = 3 - 3x$.

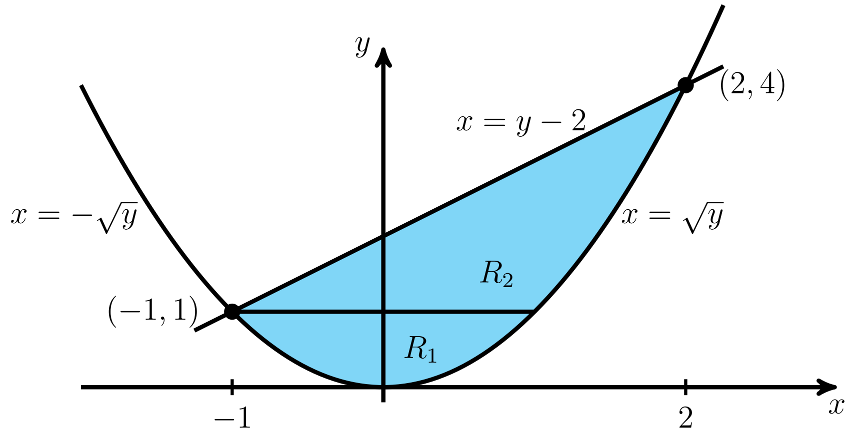


Solving for x in terms of y we have $x = 3 - y$ and $x = 1 - \dfrac{y}{3}$. Then

$$A = \int_0^3 \int_{1-y/3}^{3-y} dxdy = \int_0^3 2 - \frac{2y}{3}\,dy = 2y - \frac{y^2}{3}\Big|_0^3 = 6 - 3 = 3$$

Could we have done the first example in the order $dxdy$?

Example: If we integrate with respect first we have to break R into two regions.



$$\text{Area of } R_1 = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dxdy = \int_0^1 2\sqrt{y}\, dy = 2\left(\frac{2}{3}\right) y^{3/2}\Big|_0^1 = \frac{4}{3}$$

$$\text{Area of } R_2 = \int_1^4 \int_{y-2}^{\sqrt{y}} dxdy = \int_1^4 \sqrt{y} + 2 - y\, dy = \frac{2}{3}y^{3/2} + 2y - \frac{y^2}{2}\Big|_1^4 = \frac{19}{6}$$

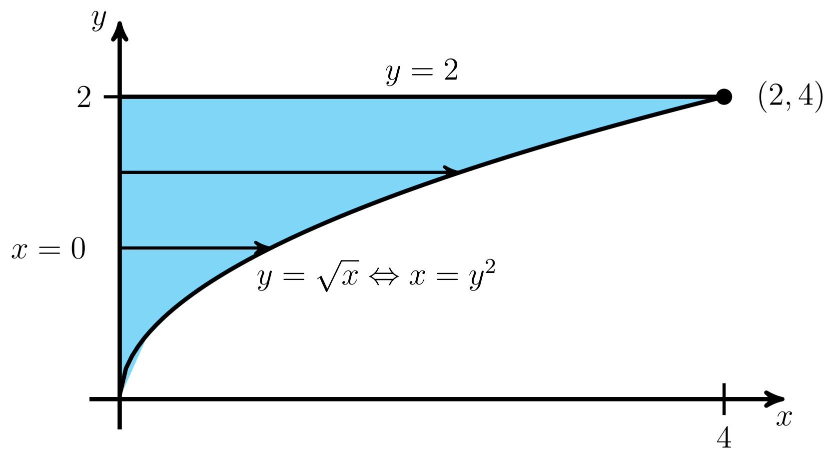
$$\text{Area of } R = \text{Area of } R_1 + \text{ Area of } R_2 = \frac{4}{3} + \frac{19}{6} = \frac{9}{2}$$

In general you should choose the order of integration that is easiest (usually the one that requires only one integral). Sometimes though, switching the order of integration makes the integral easier (or possible) to do.

Switching the Order of Integration

Example: Evaluate $\displaystyle\int_0^4 \int_{\sqrt{x}}^2 \frac{1}{y^3+1}\, dydx$

It is not obvious how to evaluate the inner integral with respect to y, but maybe if we switch the order of integration something nice will happen. In order to switch the order of integration we need to figure out the correct bounds on the integral. First we need to figure out what region we're actually intgrating over. Since y goes from $y = \sqrt{x}$ to $y = 2$ we plot these two functions



Since we're going to integrate with respect to x first we draw horizontal lines in the region. Since each line starts on the function $x = 0$ and ends on the function $x = y^2$, these define the limits of integration for dx. Then, we have $0 \le y \le 2$ for the whole region. We then have

$$\int_0^2 \int_0^{y^2} \frac{1}{y^3+1}\, dxdy = \int_0^2 \left. \frac{x}{y^3+1} \right|_0^{y^2} dy = \int_0^2 \frac{y^2}{y^3+1}\, dy$$

Switching the order of integration led to an integral that we can actually compute using u-substitution. We have

$$\begin{array}{rcl} u &=& y^3+1 \\ du &=& 3y^2\, dy \end{array} \quad \Rightarrow \quad y^2 dy = \frac{1}{3} du \quad \Rightarrow \quad \begin{array}{l} y = 0 \ \Rightarrow \ u = 1 \\ y = 2 \ \Rightarrow \ u = 9 \end{array}$$

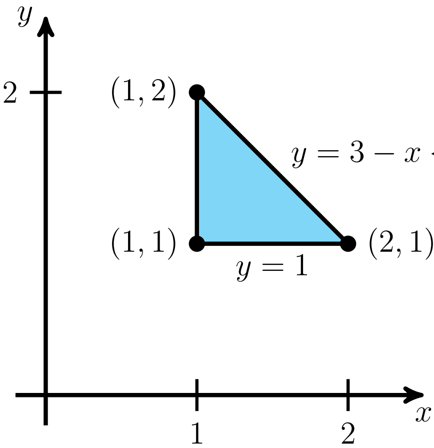
$$\Rightarrow \frac{1}{3} \int_1^9 \frac{1}{u}\, du = \frac{1}{3}(\ln 9 - \ln 1) = \frac{\ln 9}{3} = \ln 3$$

Volume Under and Between Surfaces

In the previous examples we integrated the constant function $f = 1$ over the region R to compute the area of R. Of course what we're really doing is computing the volume under the function $f = 1$, because in this case the volume under the function and the area of R are the same. If we want to compute the volume under some nonconstant function f over the region R we setup the problem in exactly the same way, but instead of having 1 as the integrand we have $f(x, y)$.

Example: Find the volume under $f(x, y) = xy$ over the region R defined by the triangle with vertices $(1, 1)$, $(2, 1)$, and $(1, 2)$.

Drawing the region R in the xy-plane we have



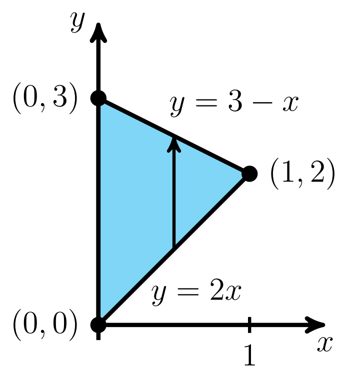
We then have

$$\int_1^2 \int_{x=1}^{x=3-y} xy\,dxdy = \int_1^2 \int_{y=1}^{y=3-x} xy\,dydx = \int_1^2 \left.\frac{xy^2}{2}\right|_1^{3-x} dx = \int_1^2 \frac{x}{2}\left[(3-x)^2 - 1\right]\,dx$$

$$\int_1^2 \frac{x^3}{2} - 3x^2 + 4x\,dx = \left.\frac{x^4}{8} - x^3 + 2x^2\right|_1^2 = (2-8+8) - \left(\frac{1}{8} - 1 + 2\right) = \frac{7}{8}$$

Example: Find the volume under $f(x,y) = 2xy$ over the triangular region with vertices $(0,0)\,, (1,2)\,,$ and $(0,3)$.

First we draw the picture.

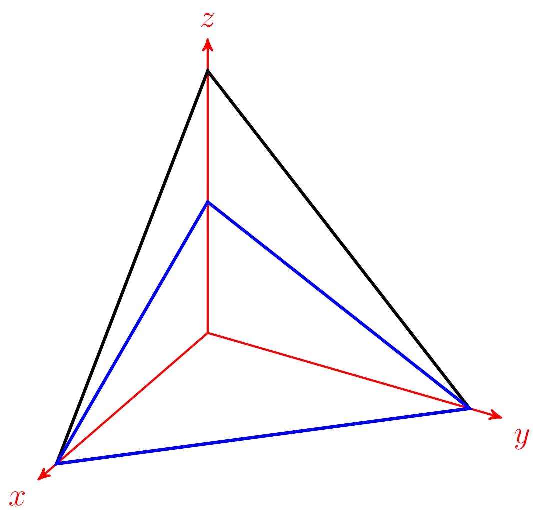


It is clear from the picture that we want to integrate w.r.t. y first. If we integrated w.r.t. x first we would have break R up into two different regions and compute two different integrals. We then have

$$\int_0^1 \int_{y=2x}^{y=3-x} 2xy\,dydx = \int_0^1 \left. xy^2 \right|_{2x}^{3-x} dx = \int_0^1 x\left[(3-x)^2 - (2x)^2\right]\,dx$$

$$\int_0^1 x\left[9 - 6x + x^2 - 4x^2\right]\,dx = \int_0^1 -3x^3 - 6x^2 + 9x\,dx = \left. -\frac{3}{4}x^4 - 2x^3 + \frac{9}{2}x^2 \right|_0^1 = -\frac{3}{4} - 2 + \frac{9}{2} = \frac{7}{4}$$

Example: Find the volume between the planes $z = 8 - 2x - 2y$ and $z = 4 - x - y$ over the previously defined triangular region.

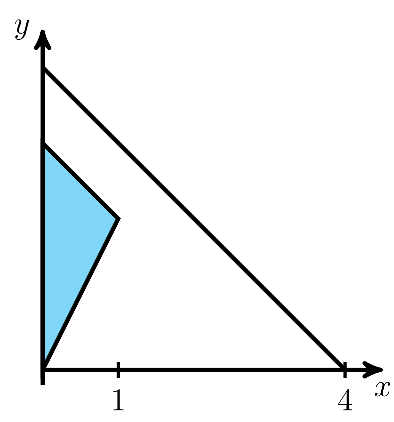


To find the volume between the two regions we want to integrate the difference between the top function and the bottom function. But, first we need to check that the top function is always the top function and the bottom function is always the bottom function over the region of integration. If we don't check this and the two surfaces cross, we will get areas with positive volume and areas with negative volume and they will cancel each other out.

The line of intersection of the two surfaces is given by

$$8 - 2x - 2y = 4 - x - y \qquad \Rightarrow \qquad y = 4 - x$$

Plotting this with the region of integration we have



We can see from the picture that the region R does not contain the line of intersection, so the functions don't switch top and bottom within R. We have

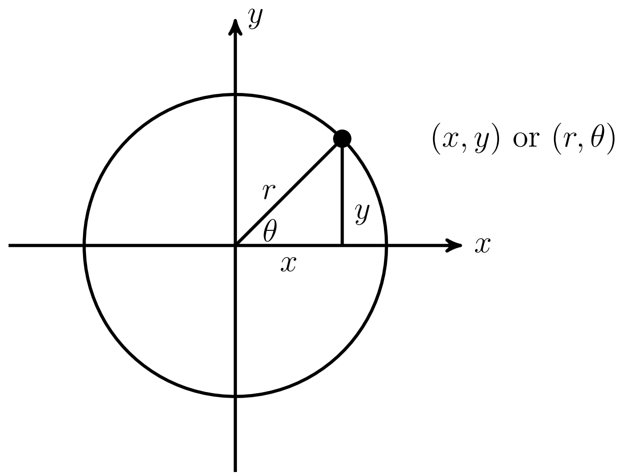
$$V = \int_0^1 \int_{2x}^{3-x} \left[(8-2x-2y)-(4-x-y)\right]\, dydx = \int_0^1 \int_{2x}^{3-x} 4{-}x{-}y\, dydx = \int_0^1 4y - xy - \frac{y^2}{2}\Big|_{2x}^{3-x}\, dx$$

$$= \int_0^1 \left[4(3-x) - x(3-x) - \frac{(3-x)^2}{2}\right] - \left[4(2x) - x(2x) - \frac{(2x)^2}{2}\right] dx =$$

$$= \frac{3}{2} \int_0^1 3x^2 - 8x + 5\, dx = \frac{3}{2}(1-4+5) = 2$$

12.3 Double Integral in Polar Coordinates

Polar Coordinates



The transformations between cartesian coordinates and polar coordinates are given by

$$x = r\cos\theta \qquad y = r\sin\theta \qquad x^2 + y^2 = r^2 \qquad \frac{y}{x} = \tan\theta$$

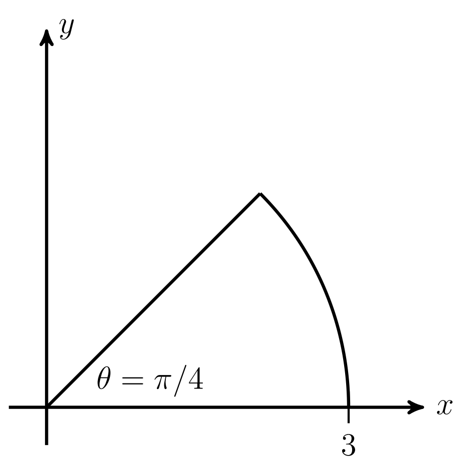
Example: Write down the equation for a circle of radius a centered at the origin in polar coordinates.

$$x^2 + y^2 = a^2 \qquad \Leftrightarrow \qquad r = a$$

Example: Write down the equation for the horizontal line $y = 5$ in polar coordinates

$$y = 5 \qquad \Leftrightarrow \qquad r\sin\theta = 5$$

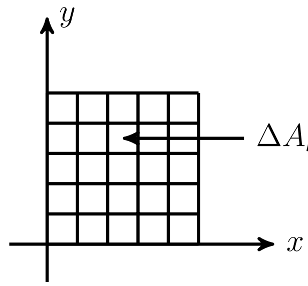
Sometimes regions of integration are more simply expressed in polar coordinates.



In rectangular coordinates the simplest shape is a rectangle. In polar coordinates the simplest shape is a **polar rectangle**:

$$R: \ 0 \le r \le 3, \qquad 0 \le \theta \le \pi/4$$

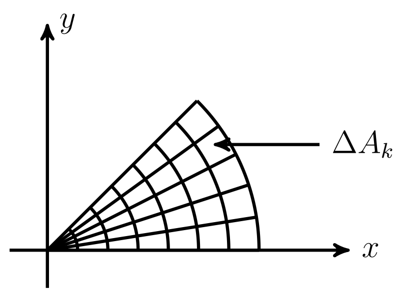
In rectangular coordinates we divided the region into little rectangular area elements.



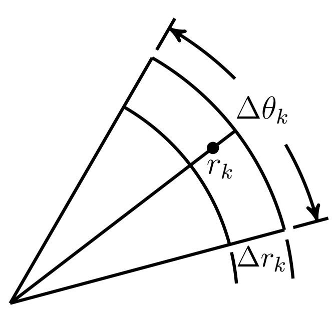
Then the integral representing the volume under a surface $f(x, y)$ is approximated by the sum of little pillars with height $f(x_k, y_k)$ and base area ΔA_k:

$$S_N = \sum_{k=1}^{N} f(x_k, y_k) \ \Delta A_k$$

In polar coordinates we divide the region into tiny polar rectangles.



So what does the small area element ΔA_k look like in polar coordinates?



We have

$$\Delta A_k = \text{Area of big circle sector} - \text{Area of small circle sector}$$

The area of a sector of a circle is given by $A = \pi r^2 \dfrac{\Delta\theta}{2\pi} = \dfrac{1}{2} r^2 \Delta\theta$. Then

$$\begin{aligned} \Delta A_k &= \text{Area of big circle sector} - \text{Area of small circle sector} \\ &= \frac{1}{2}\left(r_k + \frac{\Delta r}{2}\right)^2 \Delta\theta - \frac{1}{2}\left(r_k - \frac{\Delta r}{2}\right)^2 \Delta\theta \\ &= r_k \Delta r \Delta\theta \end{aligned}$$

Then the volume under a surface f and above the polar rectangle ΔA_k is approximated by

$$f(r_k, \theta_k)\, \Delta A_k = f(r_k, \theta_k)\, r_k \Delta r_k \Delta\theta_k$$

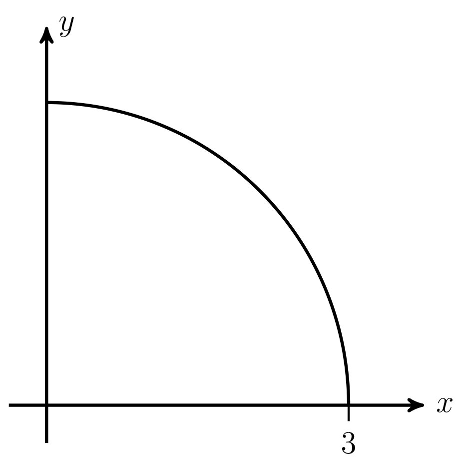
The total volume under the surface f over the region R is then approximated by

$$S_N = \sum_{k=1}^{N} f(r_k, \theta_k)\, r_k \Delta r_k \Delta\theta_k$$

The exact volume under the surface is obtained by taking the limit as the number of little polar area pieces goes to infinity:

$$\lim_{N\to\infty} S_N = \iint_R f(r\cos\theta, r\sin\theta)\ dA = \iint f(r\cos\theta, r\sin\theta)\ r dr d\theta$$

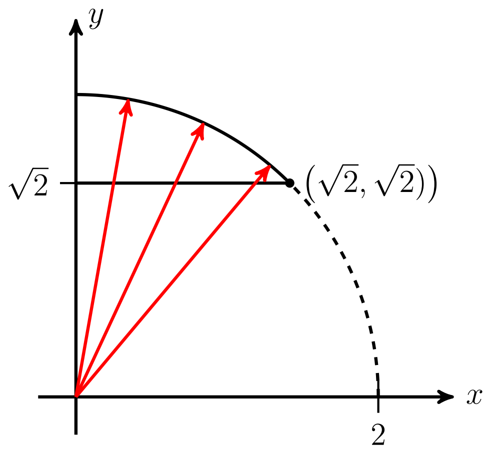
Example: Use a double integral to find the area of one fourth of a circle with radius 3.



$$A = \iint\ dA = \int_0^{\pi/2} \int_0^3 r dr d\theta = \int_0^{\pi/2} \frac{r^2}{2}\bigg|_0^3\ d\theta = \int_0^{\pi/2} \frac{9}{2}\, d\theta = \frac{9\pi}{4}$$

What if the region is more complicated?

Example: Setup an integral to integrate arbitrary function $f(r,\theta)$ over R where R is the region in the first quadrant bounded by the y-axis, a circle of radius 2, and the line $y=\sqrt{2}$.



If we decide to integrate in the order $dr\,d\theta$ then we draw **r-lines** obtained by holding θ fixed and drawing lines with varying radius. We then ask, on which function do the lines enter the region and on which function do they leave. From the picture we can clearly see that they enter at $y=\sqrt{2}$ and leave through the circle. This means that we will limits of integration on the r integral will be

$$\left(y=\sqrt{2}\right)\leq r\leq\left(x^2+y^2=4\right)\qquad\Leftrightarrow\qquad\sqrt{2}\csc\theta\leq r\leq 2$$

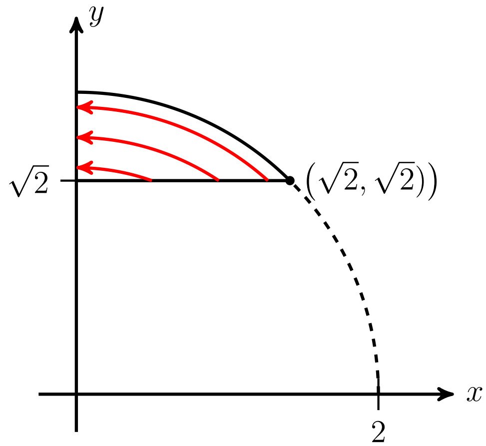
To find the limits of integration on θ we ask what the smallest and largest values of θ are in the region. Then we find that

$$\frac{\pi}{4}\leq\theta\leq\frac{\pi}{2}$$

The desired integral is then

$$\int_{\pi/4}^{\pi/2}\int_{\sqrt{2}\csc\theta}^{2} f(r,\theta)\ rdrd\theta$$

We could have course chosen to integrate in the order $d\theta dr$. In that case we do the same process but to figure out the limits of integration on θ we draw $\theta - lines$, which are curves obtained by fixing r and letting θ vary.



Now we ask what function do the θ-lines start and end on. We find

$$\left(y = \sqrt{2}\right) \leq \theta \leq \frac{\pi}{2} \qquad \Leftrightarrow \qquad \arcsin\left(\sqrt{2}/r\right) \leq \theta \leq \frac{\pi}{2}$$

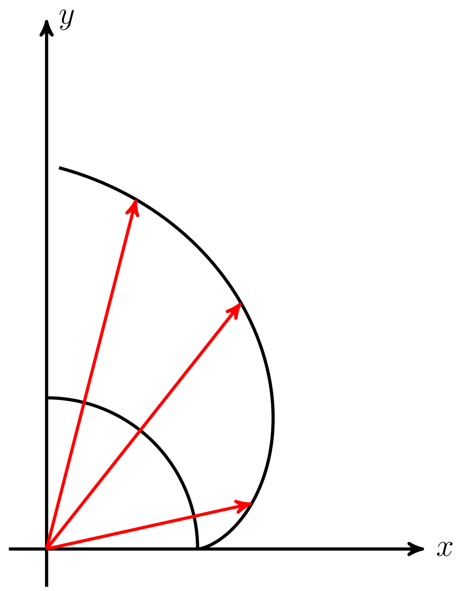
Then we find the smallest and largest value of r in the region of integration. We have

$$\sqrt{2} \leq r \leq 2$$

Then the integral in the $d\theta dr$ order is given by

$$\int_{\sqrt{2}}^{2} \int_{\arcsin\left(\sqrt{2}/r\right)}^{\pi/2} f(r, \theta)\ r d\theta dr$$

Example: Setup the integral to compute the area of the region in the first quadrant between the circle with radius 1 and the cardiod $r = 1 + \sin\theta$.



We notice that each r-line enters on the circle and exits on the cardiod. So we have

$$1 \leq r \leq 1 + \sin\theta \qquad \text{and} \qquad 0 \leq \theta \leq \frac{\pi}{2} \qquad \Rightarrow \qquad A = \int_0^{\pi/2} \int_1^{1+\sin\theta} r dr d\theta$$

Converting Cartesian Integrals to Polar Integrals

Suppose we have a double integral expressed in terms of Cartesian coordinates. If the region of integration is such that it would be easier to do in polar coordinates, it is straightforward to convert from x's and y's to r's and θ's.

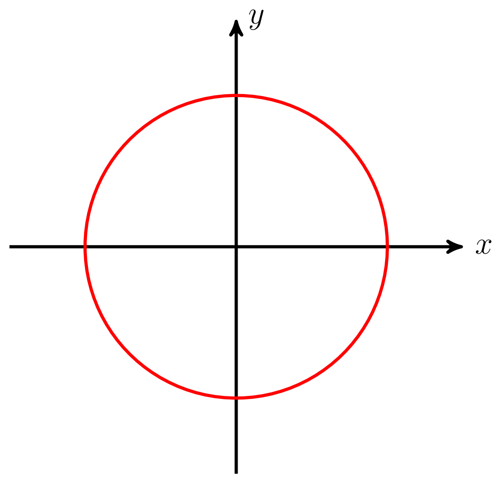
1. Use the bounds on the cartesian integral to draw a picture of the region of integration.
2. Determine limits of integration for the region in terms of polar coordinates.
3. Substitute $x = r\cos\theta$ and $y = r\sin\theta$ and $dxdy = rdrd\theta$.

Example: Compute $\displaystyle\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \left(x^2 + y^2\right) dxdy$

First we determine the region of integration from the limits. Since we're integrating in the order $dxdy$ we see that

$$-\sqrt{1-y^2} \leq x \leq \sqrt{1-y^2}$$

The bounds on x are the left and right halfs of the circle centered at the origin with radius 1. Since y is bounded by -1 and 1 we know that the region is the entire circle. So we have



Then the limits of integration are simply

$$0 \leq r \leq 1 \qquad \text{and} \qquad 0 \leq \theta \leq 2\pi$$

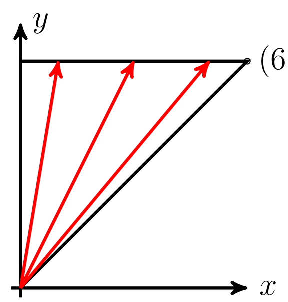
Converting the integrand we have $x^2 + y^2 = r^2$. So

$$\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \left(x^2 + y^2\right) dxdy = \int_0^{2\pi} \int_0^1 \left(r^2\right) r \, drd\theta$$

$$= \int_0^{2\pi} \int_0^1 \left(r^3\right) \, drd\theta = \int_0^{2\pi} \left.\frac{r^4}{4}\right]_0^1 d\theta = \int_0^{2\pi} \frac{1}{4} \, d\theta = \frac{\pi}{2}$$

Example: Compute $\int_0^6 \int_0^y x \, dxdy$ using polar coordinates.

The region of integration is the triangle bounded by the y-axis, the line $y = 6$, and the line $x = y$.



Drawing several r-lines we see that they all start at $r = 0$ and stop on the line $y = 6$. So we have

$$0 \le r \le (y = 6) \quad \Rightarrow \quad 0 \le r \le (r \sin\theta = 6) \quad \Rightarrow \quad 0 \le r \le 6 \csc\theta$$

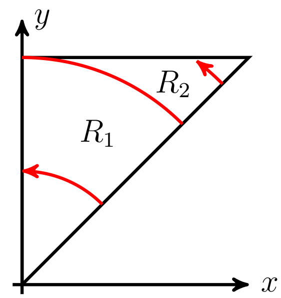
The smallest value of θ that occurs in the region is $\pi/4$ and the largest is $\pi/2$. Converting the integrand to polar coordinates we then have

$$\int_{\pi/4}^{\pi/2} \int_0^{6\csc\theta} (r\cos\theta)\, r \, drd\theta = \int_{\pi/4}^{\pi/2} \int_0^{6\csc\theta} \left(r^2 \cos\theta\right) \, drd\theta = \int_{\pi/4}^{\pi/2} \left.\cos\theta \, \frac{r^3}{3}\right]_0^{6\csc\theta} d\theta$$

$$= 72 \int_{\pi/4}^{\pi/2} \frac{\cos\theta}{\sin^3\theta} \, d\theta \quad \Rightarrow \quad (u = \sin\theta \Rightarrow du = \cos\theta d\theta) \quad \Rightarrow \quad 72 \int_{\sqrt{2}/2}^{1} \frac{1}{u^3} \, du$$

$$= -36 \left[\ \frac{1}{u^2} \ \right]_{\sqrt{2}/2}^{1} = -36\,(1 - 2) = 36$$

What if we want to integrate in the order $d\theta dr$?



We need to set up integrals for each of the regions R_1 and R_2 separately.

Region 1: We have

$$R_1: \quad \int_0^6 \int_{\pi/4}^{\pi/2} (r\cos\theta)\ r\, d\theta dr = \int_0^6 \int_{\pi/4}^{\pi/2} r^2 \cos\theta\, dr d\theta = \int_0^6 r^2 \sin\theta \Big]_{\pi/4}^{\pi/2} d\theta dr =$$

$$\int_0^6 r^2 \left(1 - \frac{\sqrt{2}}{2}\right) dr = \left(1 - \frac{\sqrt{2}}{2}\right) \frac{r^3}{3}\Big]_0^6 = 36\left(2 - \sqrt{2}\right)$$

Region 2: We have

$$R_2: \quad \int_6^{6\sqrt{2}} \int_{\pi/4}^{\arcsin(6/r)} r^2 \cos\theta\, d\theta dr = \int_6^{6\sqrt{2}} r^2 \sin\theta \Big]_{\pi/4}^{\arcsin(6/4)} dr =$$

$$\int_6^{6\sqrt{2}} r^2 \left(\frac{6}{r} - \frac{\sqrt{2}}{2}\right) dr = \int_6^{6\sqrt{2}} 6r - r^2\frac{\sqrt{2}}{2}\, dr = 3r^2 - r^3\frac{\sqrt{2}}{6}\Big]_6^{6\sqrt{2}} = 36\left(\sqrt{2} - 1\right)$$

Then

$$\text{Area } R = \text{Area } R_1 + \text{Area } R_2 = 36\left(2 - \sqrt{2}\right) + 36\left(\sqrt{2} - 1\right) = 36$$

12.4 Applications of Double Integrals

Average Value of a Function

Recall that for function $y = f(x)$, the average value of f over the interval (a, b) can be computed by

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\ dx$$

This concept is easily extended to functions $f(x, y)$ over a region R. We have

$$f_{avg} = \frac{1}{\text{Area of } R}\iint_R f(x,y)\ dA = \frac{\iint_R f(x,y)\ dA}{\iint_R dA}$$

Mass

Suppose that some thin material is described by the region R in the xy-plane and its density is described by $\rho(x, y)$. Then we can compute the mass of the object by integrating over the density:

$$M = \iint_R \rho(x,y)\ dA$$

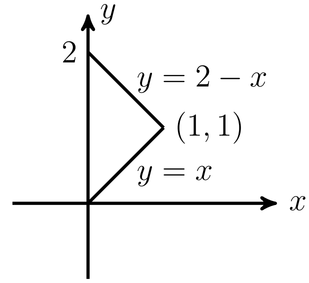
This should make intuitive sense because the integrand together with the small area unit is

$$\rho(x,y)\ dA \approx \rho(x_k, y_k)\,\Delta A_k.$$

So when taking the double integral we're summing up the masses of all the little area elements in R.

Example: Find the mass of the thin plate bounded by the y-axis and the lines $y = x$ and $y = 2 - x$ with density $\rho(x, y) = 6x + 3y + 3$.

Solution: First we draw a picture of the plate

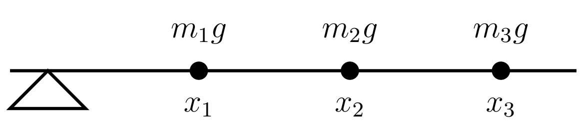


To calculate the mass M of the plate we integrate $\rho(x,y) = 6x + 3y + 3$ over the triangular region

$$
\begin{aligned}
M &= \int_0^1 \int_x^{2-x} 6x + 3y + 3 \; dydx \\
&= \int_0^1 6xy + \frac{3}{2}y^2 + 3y \Big]_x^{2-x} \; dx \\
&= \int_0^1 12 - 12x^2 \; dx \\
&= 12x - 4x^3 \Big]_0^1 \\
&= 12 - 4 \\
&= 8
\end{aligned}
$$

Moments

For functions and domains of a single variable (think about a metal rod) we compute moments in the following way



Summing up the torques from the point masses we have

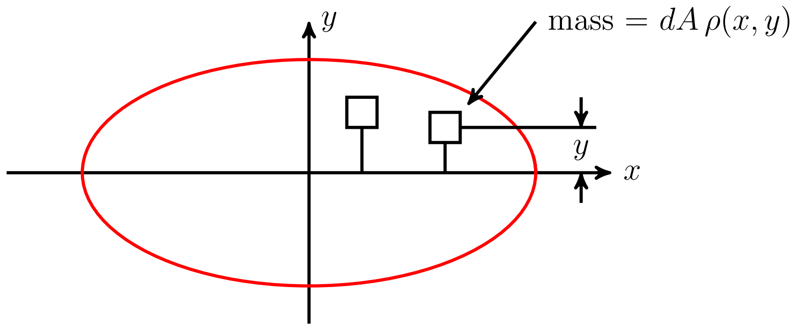
$$\text{Sum of Torques} = (m_1 x_1 g + m_2 x_2 g + m_3 x_3 g) = g\,(m_1 x_1 + m_2 x_2 + m_3 x_3)$$

The quantity in parentheses is the first moment. Now, if we suppose that the line is a thin metal rod that has nonconstant density, we can compute the first moment using an integral:

$$\text{First Moment} = \int_a^b x\rho(x) \; dx$$

Extending this argument to two variables is fairly straightforward. In this case we have two first moments to worry about. One that causes the object to want to rotate about the x-axis, and one that causes it to want to rotate about the y-axis.

Moment About x-Axis



Then $y\rho(x,y)\ dA$ is the first moment about the x-axis for just one of the little area units. To get the first moment of the object we have to sum up these infinitesimal moments using an integral.

$$\begin{aligned}\text{First Moment about } x\text{-axis} &= M_x = \iint_R y\rho(x,y)\ dA \\ \text{First Moment about } y\text{-axis} &= M_y = \iint_R x\rho(x,y)\ dA\end{aligned}$$

The center of mass of a thin metal plate is the point $(\bar{x},\bar{y})$ such that the moments in the x-direction and y-direction are zero. One way to think of this is that the x-component of the center of mass is the line $x=\bar{x}$ such that M_y about this point is zero. Then we have

$$0 = \iint_R (x-\bar{x})\,\rho(x,y)\ dA \quad \Rightarrow \quad \bar{x}\iint_R \rho(x,y)\ dA = \iint_R x\rho(x,y)\ dA \quad \Rightarrow$$

$$\bar{x}M = M_y \quad \Rightarrow \quad \bar{x} = \frac{M_y}{M}$$

Similarly, the y-component of the center of mass is the line $y=\bar{y}$ such that M_x about this point is zero. We again have

$$0 = \iint_R (y-\bar{y})\,\rho(x,y)\ dA \quad \Rightarrow \quad \bar{y}\iint_R \rho(x,y)\ dA = \iint_R y\rho(x,y)\ dA \quad \Rightarrow$$

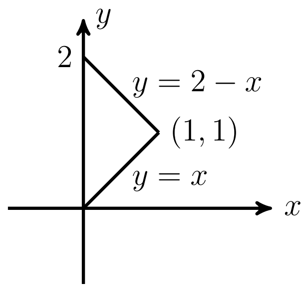
$$\bar{y}M = M_x \quad \Rightarrow \quad \bar{y} = \frac{M_x}{M}$$

So we have

$$(\bar{x},\bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$$

Example: Find the center of mass of the thin plate bounded by the y-axis and the lines $y = x$ and $y = 2 - x$ with density $\rho(x, y) = 6x + 3y + 3$.

Solution: First we draw a picture of the plate



In order to find the center of mass of the plate we first need to calculate the mass M, and the first moments M_x and M_y.

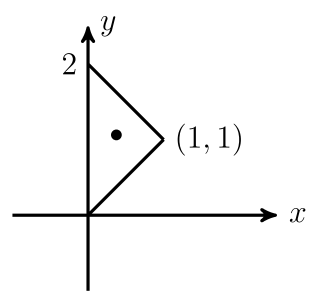
$$
\begin{aligned}
M &= \int_0^1 \int_x^{2-x} 6x + 3y + 3 \; dydx \\
&= \int_0^1 6xy + \frac{3}{2}y^2 + 3y \Big]_x^{2-x} \; dx \\
&= \int_0^1 12 - 12x^2 \; dx \\
&= 12x - 4x^3 \Big]_0^1 \\
&= 12 - 4 \\
&= 8
\end{aligned}
$$

$$
\begin{aligned}
M_x &= \int_0^1 \int_x^{2-x} y\,(6x + 3y + 3) \; dydx \\
&= \int_0^1 3xy^2 + y^3 + \frac{3}{2}y^2 \Big]_x^{2-x} \; dx \\
&= \int_0^1 14 - 6x - 6x^2 - 2x^3 \; dx \\
&= 14x - 3x^2 - 2x^3 - \frac{1}{2}x^4 \Big]_0^1 \\
&= 14 - 3 - 2 - \frac{1}{2} \\
&= \frac{17}{2}
\end{aligned}
$$

$$
\begin{aligned}
M_y &= \int_0^1 \int_x^{2-x} x\,(6x+3y+3)\ dydx \\
&= \int_0^1 6x^2y + \frac{3}{2}xy^2 + 3xy \Big]_x^{2-x}\ dx \\
&= \int_0^1 12x - 12x^3\ dx \\
&= 6x^2 - 3x^4 \Big]_0^1 \\
&= 6 - 3 \\
&= 3
\end{aligned}
$$

Then the center of mass is given by

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{3}{8}, \frac{17}{16}\right)$$



Moments of Inertia

Moments of inertia, sometimes called second moments, have to do with the amount of torque that would be necessary to acheive a desired angular acceleration about an axis. Since it takes more force to get heavy object rotating with a certain angular acceleration the moments of inertia are kind of mass terms. They are similar to first moments, but instead of using the distance from the axis of rotation we used the squared distance. We have

$$I_x = \iint_R y^2\rho(x,y)\ dA \quad \text{and} \quad I_y = \iint_R x^2\rho(x,y)\ dA$$

So I_x and I_y tell us the torque necessary to rotate an object with a particular angular acceleration about the x- and y-axis, respectively. We can also ask about the torque necessary to rotate an object about a point. This is called the polar moment of inertia and is given by

$$I_0 = \iint_R \left(x^2 + y^2\right)\rho(x,y)\ dA$$

Notice that $I_0 = I_x + I_y$.

Example: Find the moments of inertia about the x and y-axes and the origin for the metal plate in the previous example.

Solution:

$$\begin{aligned}
I_x &= \int\int_R y^2 \rho(x,y)\ dA \\
&= \int_0^1 \int_x^{2-x} y^2\left(6x+3y+3\right)\ dy\,dx \\
&= \int_0^1 2xy^3 + \frac{3}{4}y^4 + y^3\ \Big]_x^{2-x}\ dx \\
&= \int_0^1 20 - 20x + 4x^3 - 4x^4\ dx \\
&= 20x - 10x^2 + x^4 - \frac{4}{5}x^5\Big]_0^1\ dx \\
&= 20 - 10 + 1 - \frac{4}{5} \\
&= \frac{51}{5}
\end{aligned}$$

$$\begin{aligned}
I_y &= \int\int_R x^2 \rho(x,y)\ dA \\
&= \int_0^1 \int_x^{2-x} x^2\left(6x+3y+3\right)\ dy\,dx \\
&= \int_0^1 6x^3y + \frac{3}{2}x^2y^2 + 3x^2y\ \Big]_x^{2-x}\ dx \\
&= \int_0^1 12x^2 - 12x^4\ dx \\
&= 4x^3 - \frac{12}{5}x^5\Big]_0^1\ dx \\
&= 4 - \frac{12}{5} \\
&= \frac{8}{5}
\end{aligned}$$

$$\begin{aligned}
I_0 &= \int\int_R \left(x^2+y^2\right)\rho(x,y)\ dA \\
&= \int\int_R x^2\rho(x,y)\ dA + \int\int_R y^2\rho(x,y)\ dA \\
&= I_y + I_x \\
&= \frac{8}{5} + \frac{51}{5} \\
&= \frac{59}{5}
\end{aligned}$$

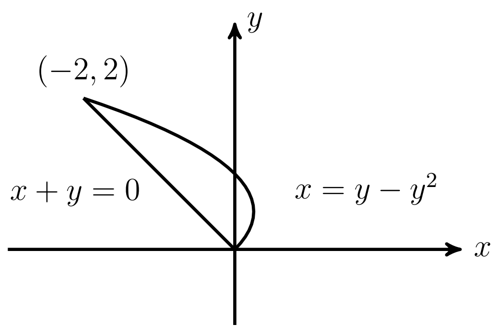
Radii of Gyration

The radius of gyration is kind of like the ceneter of mass but with second moments instead of first moments. It tells you that a particle of mass M at distance R away from a given axis has the same moment of inertia as the plate. The formulas for the radii of gyration are given by

$$
\begin{aligned}
R_x &= \sqrt{\frac{I_x}{M}} \\
R_y &= \sqrt{\frac{I_y}{M}} \\
R_0 &= \sqrt{\frac{I_0}{M}}
\end{aligned}
$$

Example: Find the moments of inertia and radii of gyration for the thin plate bounded by the curves $x = y - y^2$ and $x + y = 0$ if the density is given by $\rho(x, y) = x + y$.

Solution: The plate looks like the following



Notice that if we integrated in the order $dy\,dx$ we would have to use two different integrals to cover R. Instead we integrade inthe order $dx\,dy$. We have

$$M = \int_0^2 \int_{-y}^{y-y^2} x + y \ dx\,dy = \frac{8}{15}$$

$$I_x = \int_0^2 \int_{-y}^{y-y^2} y^2\,(x + y) \ dx\,dy = \frac{64}{105}$$

$$I_y = \int_0^2 \int_{-y}^{y-y^2} x^2\,(x + y) \ dx\,dy = \frac{64}{315}$$

$$I_0 = I_x + I_y = \frac{64}{105} + \frac{65}{315} = \frac{256}{315}$$

Then the radii of gyration are given by

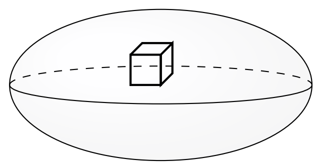
$$R_x = \sqrt{\frac{I_x}{M}} = \sqrt{\frac{64/105}{8/15}} = \sqrt{\frac{8}{7}}$$

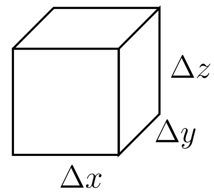
$$R_y = \sqrt{\frac{I_y}{M}} = \sqrt{\frac{64/315}{8/15}} = \sqrt{\frac{8}{21}}$$

$$R_0 = \sqrt{\frac{I_0}{M}} = \sqrt{\frac{256/315}{8/15}} = \sqrt{\frac{32}{21}}$$

12.5 Triple Integrals

Let E be some three-dimensional region in space





Let $\Delta x_k \Delta y_k \Delta z_k = \Delta V_k$ be the kth volume element and (x_k, y_k, z_k) be some point in ΔV_k. Then we have

$$\lim_{N\to\infty} \sum_{k=1}^{n} f\left(x_k, y_k, z_k\right)\, \Delta V_k = \iiint_E f(x,y,z)\; dV$$

The simplest region E is the box

$$E = [a,b] \times [c,d] \times [r,s] = \{\,(x,y,z) \mid a \le x \le b,\; c \le y \le d,\; r \le z \le s\}$$

Example: Evaluate the following integral

$$\begin{aligned}
\int_0^1 \int_0^2 \int_0^\pi xy \sin(z)\; dzdydx &= -\int_0^1 \int_0^2 \left(xy\cos(x)\,\big|_0^\pi\right)\, dydx \\
&= 2\int_0^1 \int_0^2 xy\, dydx = 2\int_0^1 \left(\left.\frac{xy^2}{2}\right|_0^2\right)\, dx \\
&= 2\int_0^1 2x\, dx = 2x^2\,\big|_0^1 = 2
\end{aligned}$$

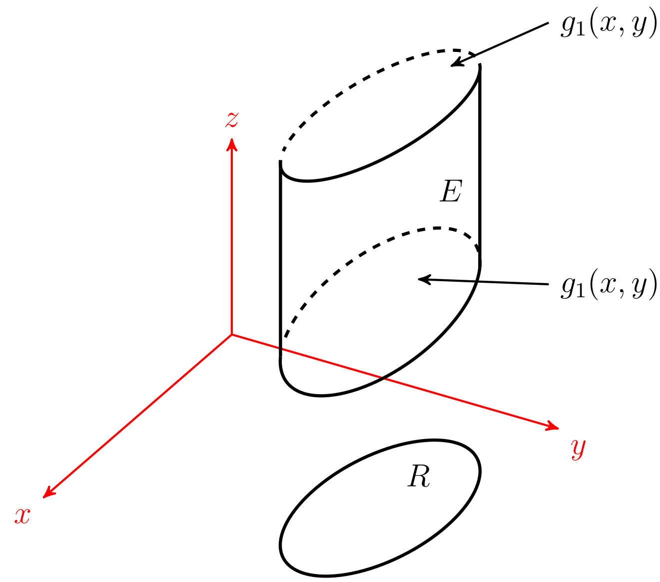
Note that the 3D version of Fubini's Theorem says that as long as f is continuous we're allowed to switch the order of integration. So we could have done

$$\int_0^\pi \int_0^1 \int_0^2 xy \sin(z)\; dydxdz \quad = \quad 2$$

What if the region E is more complicated than a box? First let's notice that

$$\text{Volume of } E = \iiint_E dV$$

so for simplicity let's take $f(x,y,z) = 1$ for now.



To setup the limit of integration with respect to dz first, we draw z-lines similar to the way we drew x- and y-lines to figure out double integrals. If all of the z-lines enter the region E at $z = g_1(x,y)$ and leave through $z = g_2(x,y)$ then we have

$$\text{Volume of } E = \iint_R \int_{g_1(x,y)}^{g_2(x,y)} dz\, dA$$

The region R is the projection of E into the xy-plane. We then determine the region of integration for R in the usual way. In this case we have

$$\text{Volume of } E = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} dzdydx$$

Geometric Interpretation

$$\text{Volume of } E = \iint_R \int_{g_1(x,y)}^{g_2(x,y)} dz\, dA = \iint_R \left(g_2(x,y) - g_1(x,y)\right) dA$$

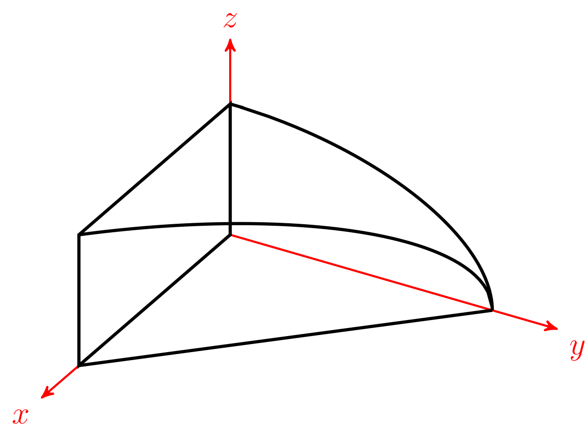
The integrand $[g_2(x,y) - g_1(x,y)]\, dA$ is the volume of the column with height $(g_2(x,y) - g_1(x,y))$ over the small area element dA. So

$$\iint_R \int_{g_1(x,y)}^{g_2(x,y)} dz\, dA = \iint_R \left(g_2(x,y) - g_1(x,y)\right) dA$$

says fill up E with columns and sum up over R.

Example: Find the volume of the region in the first quadrant bounded by $x + y = 4$ and $y^2 + 4z^2 = 16$.

First we draw a picture.



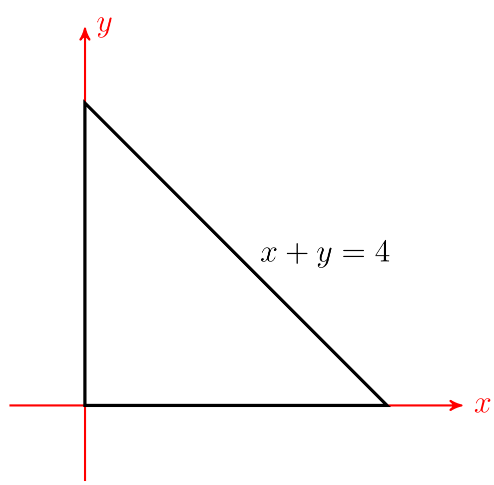
First we notice that if we draw lines in the z-direction that they will all enter the region on the lower surface $z = 0$ and leave the region on the top surface, which solving for z becomes

$$z = \sqrt{4 - \frac{y^2}{4}}$$

We can start setting up the integral with just the z limits in place. We have

$$\iint_R \int_0^{\sqrt{4-\frac{y^2}{4}}} dz\, dA$$

where R is the 2D region we obtain by projecting the surface onto the xy-plane. The region R looks like



To determine the x and y-limits in the triple integral based on the projected region R we can choose to integrate either in the order $dx\, dy$ or $dy\, dx$ just as easily. We'll do $dx\, dy$.

Integrating in x we see that all x-lines enter the region at $x = 0$ and leave at $x = 4 - y$. Finally, we see that the smallest value of y in the region is $y = 0$ and the largest is $y = 4$. Plugging this info into the triple integral we have

$$\int_0^4 \int_0^{4-y} \int_0^{\sqrt{4-\frac{y^2}{4}}} dz\, dx\, dy$$

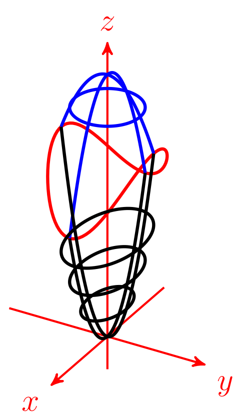
Evaluating this integral for the total volume in the region, we have

$$\begin{aligned}
\int_0^4 \int_0^{4-y} \int_0^{\sqrt{4-\frac{y^2}{4}}} dz\, dx\, dy &= \int_0^4 \int_0^{4-y} \sqrt{4 - \frac{y^2}{4}}\, dx\, dy \\
&= \int_0^4 (4-y) \sqrt{4 - \frac{y^2}{4}}\, dy \\
&= \int_0^4 4\sqrt{4 - \frac{y^2}{4}}\, dy - \int_0^4 y\sqrt{4 - \frac{y^2}{4}}\, dy \\
&= 8\pi - \frac{32}{3}
\end{aligned}$$

where here the first integral is evaluated using a trig substitution with $y = \sin\theta$ and the second with a u-sub with $u = 4 - \dfrac{y^2}{4}$.

Example: Find the volume of the region E bounded by $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.

The picture below displays the top surface in blue, the bottom surface in black, and the curve of intersection in red.



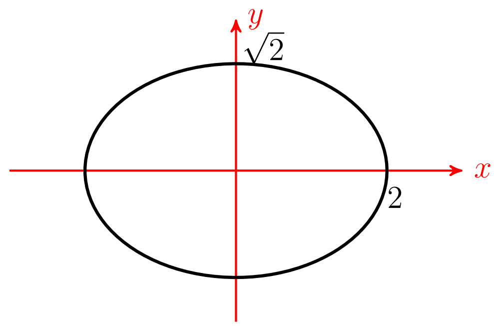
We notice that if we draw lines in the z-direction that they will all enter the region on the lower surface $z = x^2 + 3y^2$ and leave the region on the top surface $z = 8 - x^2 - y^2$. Plugging these into the integral we have

$$\iint_R \int_{x^2+3y^2}^{8-x^2-y^2} dz\, dA$$

where R is the 2D region we obtain by projecting the surface onto the xy-plane. To find this region we need to project the curve of intersection of the two surfaces into the xy-plane. Setting the two surfaces equal we have

$$x^2 + 3y^2 = 8 - x^2 - y^2 \quad \Rightarrow \quad x^2 + 2y^2 = 4$$

which is an ellipse in the xy-plane.



If we integrate in the order $dy\,dx$ then the y limits are defined by

$$-\sqrt{2 - \frac{x^2}{2}} \leq y \leq \sqrt{2 - \frac{x^2}{2}}$$

Finally, the smallest x value encountered in the region is -2 and the largest is 2. Plugging these into the integral we have

$$\int_{-2}^{2} \int_{-\sqrt{2-\frac{x^2}{2}}}^{\sqrt{2-\frac{x^2}{2}}} \int_{x^2+3y^2}^{8-x^2-y^2} dz\,dy\,dx$$

If we evaluate this integral we find that the volume of the region E is $8\pi\sqrt{2}$.

Example: Find the average value of $f(x, y, z) = xyz$ in the region E in the first octant and bounded by the planes $x = 2$, $y = 2$ and $z = 2$.

To find the average value of a function we need to compute

$$f_{avg} = \frac{\iiint_E xyz\,dV}{\iiint_E dV}$$

Since the region E is just a box it's pretty easy to figure out the limits of integration. We have

$$
\begin{aligned}
\iiint_E xyz\, dV &= \int_0^2 \int_0^2 \int_0^2 xyz\, dx\, dy\, dz \\
&= \int_0^2 \int_0^2 \left. \frac{x^2}{2} yz \right]_0^2 dy\, dz \\
&= \int_0^2 \int_0^2 2yz\, dy\, dz \\
&= \int_0^2 \left. y^2 z \right]_0^2 dz \\
&= \int_0^2 4z\, dz \\
&= \left. 2z^2 \right]_0^2 \\
&= 8
\end{aligned}
$$

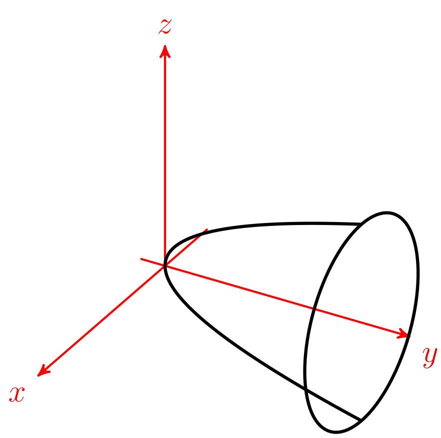
Since the integral in the denominator is just the volume of E which is a cube with side lengths 2 we know that

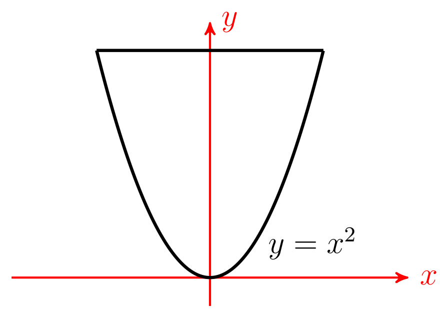
$$
\iiint_E dV = 8
$$

which gives $f_{avg} = \frac{8}{8} = 1$.

Example: Find the volume of the region E bounded by $y = x^2 + z^2$ and $y = 4$.

The first surface is a poraboloid that opens up along the positive y-axis. The second surface is the plane through $y = 4$ that is parallel to the xz-plane. If we decide to integrate in the order $dzdydx$ then regions E and R looks as follows:





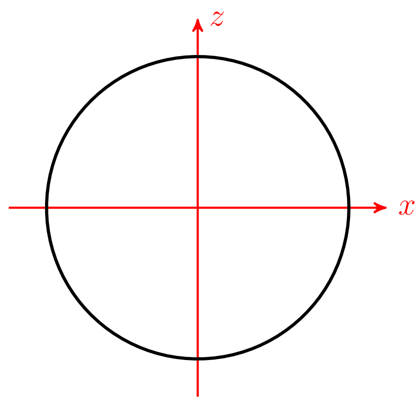
and we have

$$
\int_{-2}^{2} \int_{x^2}^{4} \int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} dzdydx
$$

If we integrate in the order $dydxdz$ then we have

$$x^2 + z^2 \leq y \leq 4$$

and the projection into the xz-plane looks like



and the integral becomes

$$\int_{-2}^{2} \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} \int_{x^2+z^2}^{4} \, dydxdz$$

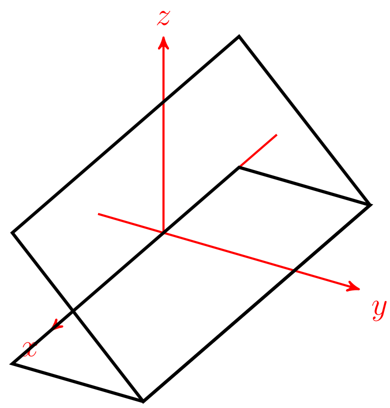
Even better, we could convert to polar coordinates for the integral in the xz-plane. Then we have

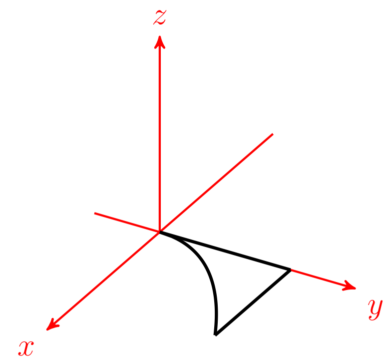
$$\int_{-2}^{2} \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} \int_{x^2+z^2}^{4} \, dydxdz = \int_{0}^{2\pi} \int_{0}^{2} \left[4 - r^2\right] \, rdrd\theta$$

Example: Switch the order of integration in the following integral to $dx\,dz\,dy$.

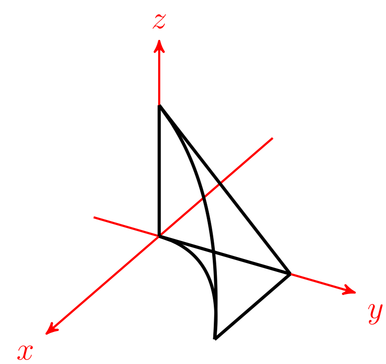
$$\int_{0}^{1} \int_{\sqrt{x}}^{1} \int_{0}^{1-y} f(x, y, z) \, dz\,dy\,dx$$

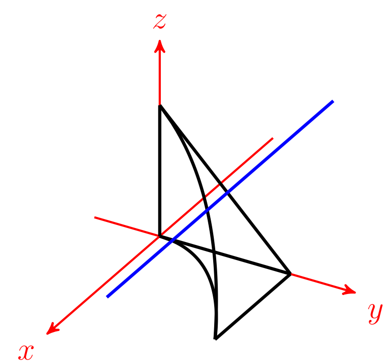
To switch the order of integration in a triple integral we need to be able to draw the region E described by the limits of integration. It's usually helpful to sketch the bounding surfaces in the inner-most integral as well as the region R described by the outer two integrals. The bounding surfaces and the region R look as follows





Combining these two pictures we get the following region E

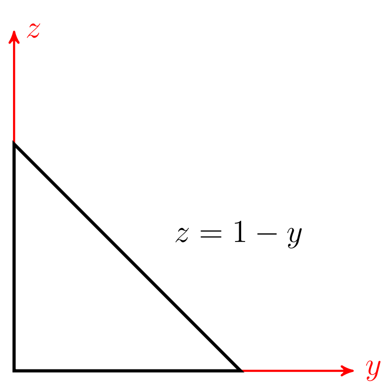




Since we're asked to integrate first with respect to x we draw x-lines as shown in blue in the picture above. From the picture we can see that the x-line enters the region E in the plane $x = 0$ and exits on the surface $y = \sqrt{x} \Leftrightarrow x = y^2$. Then the triple integral looks like

$$\iint_R \int_0^{y^2} f(x, y, z)\ dx\, dA$$

where R is the region obtained by projecting out the x-direction onto the yz-plane. This region looks like

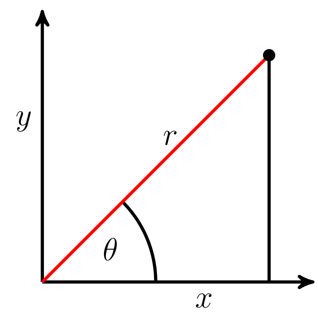


Then, from the picture of R we can fill in the limits on y and z. The final integral in the new order is

$$\int_0^1 \int_0^{1-y} \int_0^{y^2} f(x, y, z)\ dx\, dz\, dy$$

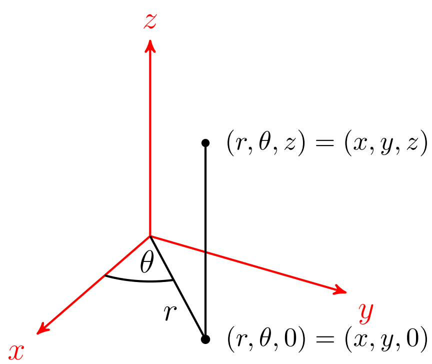
Triple Integration in Cylindrical Coordinates

Recall that in 2D we can represents points and curves in Polar Coordinates



$$
\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta \\
r^2 &= x^2 + y^2 \\
\tan\theta &= \frac{y}{x}
\end{aligned}
$$

Cylindrical coordinates give us a good way to describe surfaces and curves that are symmetric about the z-axis. They are essentially polar coordinates with height!

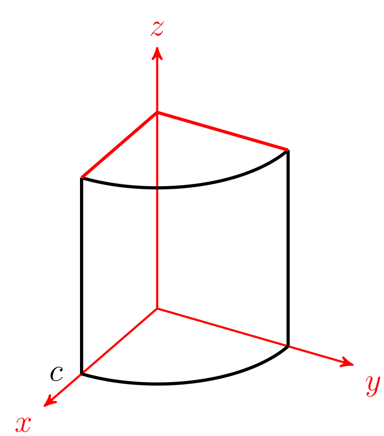


$$
\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta \\
z &= z
\end{aligned}
$$

Usually we define $r \geq 0$ by convention, but this is not strictly necessary.

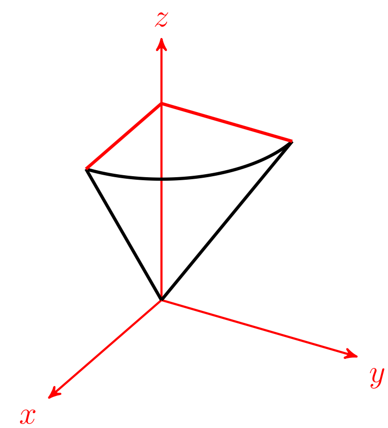
Example: Describe the surface $r = c$ in cylindrical coordinates where c is a constant.

$r = c$ in the xy-plane is a circle with radius c. Since z does not appear in the equation the circle in the xy-plane is extended in the z-direction and we get a cylinder. The portion of the first quadrant looks like the following



Example: Describe the surface $z = r$ in cylindrical coordinates.

Note that the level curves of the surface are $r = z_0$ where z_0 is a constant. These are circles in the $z = z_0$ plane with radius z_0 which traces out a cone. If we abide by the convention that $r \geq 0$ then this is a half-cone that lies above the xy-plane. The portion of the cone in the first quadrant looks like



Integration in Cylindrical Coordinates

A general integral of a function $f(x, y, z)$ over the 3D region E looks like

$$\iiint_E f(x, y, z)\ dV$$

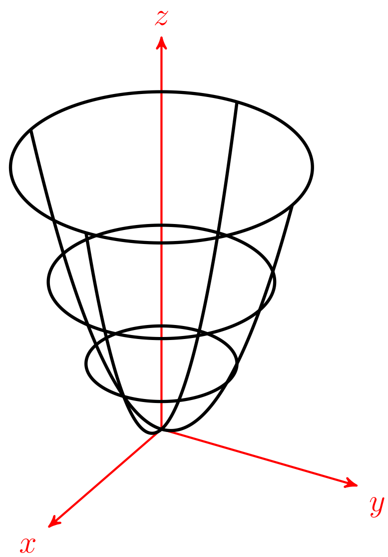
where in Cartesian coordinates $dV = dx\, dy\, dz$. We saw in class that the volume element in Cylindrical coordinates is

$$dV = r\, dz\, dr\, d\theta$$

Example: Evaulate the following integral using Cylindrical coordinates where the region E is bounded below by $z = x^2 + y^2$ and above by the $z = 4$ plane.

$$\iiint_E \sqrt{x^2 + y^2}\, dV$$

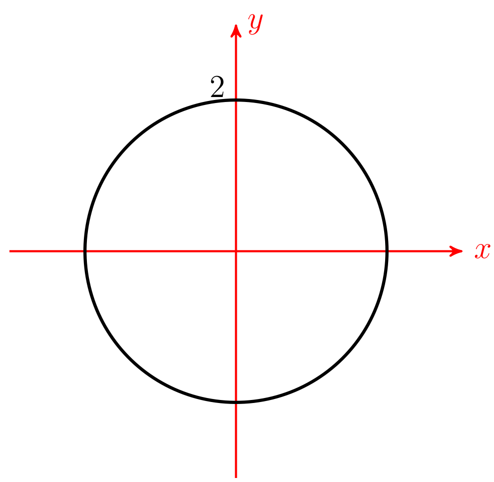
As always, the first step is to draw a picture. The lower bounding surface is a paraboloid that can be written in cylindrical coords as $z = x^2 + y^2 = r^2$. The top surface is just a plane parallel to the xy-plane at height 4 on the z-axis.



If we draw z-lines we note that they all enter the region E first through the paraboloid, and leave through the plane $z = 4$. Thus we can carry out the triple integral in the order $dz\,dr\,d\theta$, and the z-limits of the integral look like

$$\iint_R \int_{r^2}^{4} \sqrt{x^2 + y^2}\,dz\,dA$$

where the region R is the projection of the surface into the xy-plane. Since the curve of intersection of the two surfaces is a circle with radius 2, the region R looks like



We can now setup the outer limits of the triple integral. The radius ranges from 0 to 2 and the angle θ from 0 to 2π. We then have

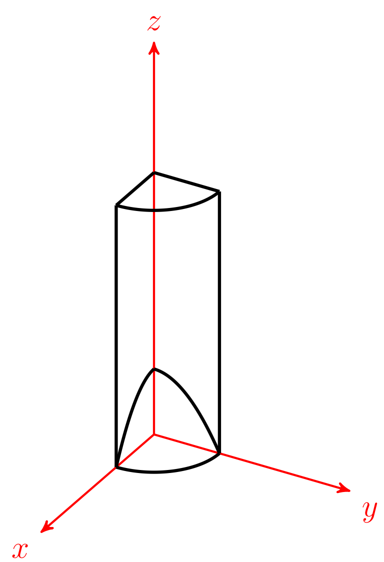
$$\int_0^{2\pi} \int_0^2 \int_{r^2}^{4} \sqrt{x^2 + y^2}\,r\,dz\,dr\,d\theta$$

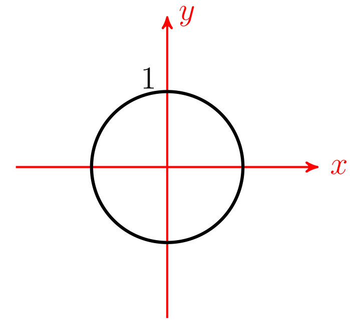
The last step is to write the integrand in polar coordinates. We have $\sqrt{x^2+y^2} = r$. So the final integral is

$$\int_0^{2\pi} \int_0^2 \int_{r^2}^4 r^2 \, dz \, dr \, d\theta = \frac{128\pi}{15}$$

Example: Find the volume of the solid E bounded on the outside by the cylinder $x^2+y^2 = 1$, above by $z = 4$ and below by $z = 1 - x^2 - y^2$.

The region is inside a cylinder of radius 1, below the plane $z = 4$, and above the upsidedown paraboloid $z = 1-x^2-y^2$. On the left I've drawn the region here in the first quadrant because it's easier to see, but keep in mind we're finding the volume of E in the whole domain. On the right I've drawn the projection of E onto the xy-plane.





If we draw z-lines for the region E we see that each line enters the region on the paraboloid on the bottom and leaves through the plane on the top. Note that in cylindrical coordinates, the paraboloid is given by

$$z = 1 - x^2 - y^2 = 1 - r^2$$

Setting up the r and θ limits based on the region R, which is a unit circle, we have

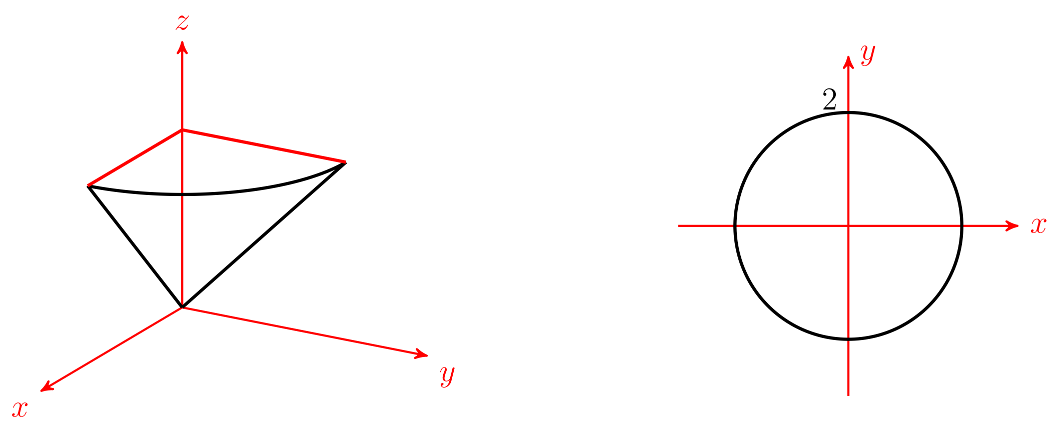
$$\int_0^{2\pi} \int_0^1 \int_{1-r^2}^4 r \, dz \, dr \, d\theta = \frac{7\pi}{2}$$

Example: Evaluate $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}\left(x^2+y^2\right)\,dz\,dy\,dx$

Clearly, there are too many square-roots in this integral to do it in Cartesian coordinates. But, there are enough things involved that look like circles that we should consider converting to cylindrical coordinates. Since we're integrating with respect to z first, the limits on the inner-most integral tell us that z starts on the surface $z = \sqrt{x^2+y^2}$ and stops on the plane $z = 2$. The bottom surface is the cone $z = \sqrt{x^2+y^2} = r$.

The projection of the region into the xy-plane is defined by the outer two integrals. The middle integral with respect to y indicates that y starts on the bottom of the circle or a radius 2, defined by $y = \sqrt{4-x^2}$, and stops on the top of the same circle. Since x ranges from -2 to 2 we know that it is the entire circle.

Thus, the region E (showing only the portion in the first quadrant) and the region R look as follows



We now have a good idea of the limits of integration. The last step is to convert the integrand into cylindrical coordinates . We have

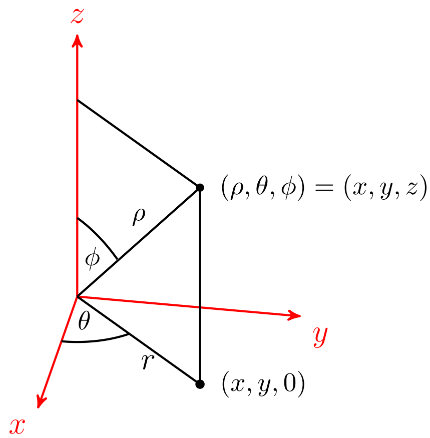
$$x^2 + y^2 = r^2$$

The integral in cylindrical coordinates is then

$$\int_0^{2\pi}\int_0^2\int_r^2 r^2\,r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2\int_r^2 r^3\,dz\,dr\,d\theta = \frac{16\pi}{5}$$

Triple Integration in Spherical Coordinates

Spherical Coordinates give us a good way to describe surfaces and curves that are symmetric about the origin.



$$
\begin{aligned}
x &= \rho \sin\phi \cos\theta \\
y &= \rho \sin\phi \sin\theta \\
z &= \rho \cos\phi \\
x^2 + y^2 + z^2 &= \rho^2
\end{aligned}
$$

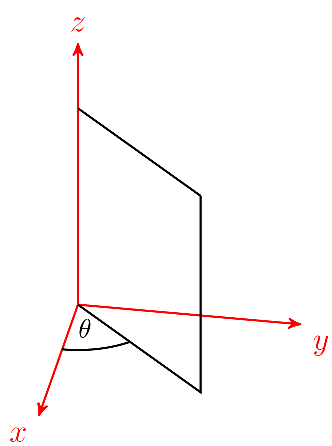
Usually we define $\rho \geq 0$ and $0 \leq \phi \leq \pi$ by convention, but this is not strictly necessary.

Example: Describe the surface $\rho = c$ where c is a constant.

The surface is defined by the fact that each point is exactly c units from the origin, which defines a sphere of radius c.

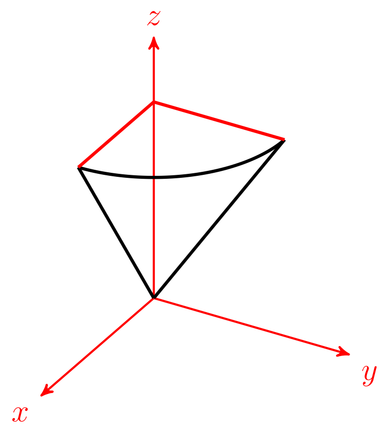
Example: Describe the surface $\theta = c$ where c is a constant.

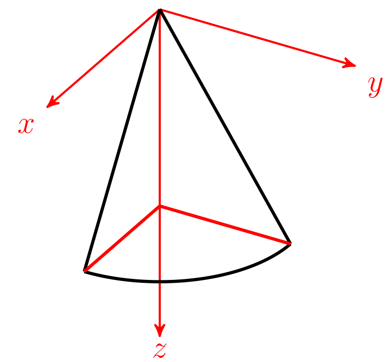
The surface $\theta = c$ contains all points where the points projection into the xy-plane makes an angle of c from the x-axis. This traces out a half plane which looks like



Example: Describe the surface $\phi = c$ where c is a constant.

The surface $\phi = c$ is a cone. Whether the cone opens upwards or downwards depends on if $0 \leq c \leq \frac{\pi}{2}$ or $\frac{\pi}{2} \leq c \leq \pi$. Two examples of the surface shown in the first quadrant are below





Volume Element: In spherical coordinates we have $dV = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$.

Example: Let E be the unit sphere $E = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$ and compute the integral

$$\iiint_E e^{\left(x^2+y^2+z^2\right)^{3/2}} \, dV$$

Since the region E is a sphere, it's pretty easy to set up the limits of integration. A sphere is the analogue of the box in Cartesian coordinates in the sense that the limits of integration will all be consant. For a sphere of radius 1 we have

$$0 \leq \rho \leq 1 \qquad 0 \leq \theta \leq 2\pi \qquad 0 \leq \phi \leq \pi$$

Next we need to convert the integrand into spherical coordinates. We have

$$e^{\left(x^2+y^2+z^2\right)^{3/2}} = e^{\left(\rho^2\right)^{3/2}} = e^{\rho^3}$$

Inserting the appropriate limits of integration, integrand, and volume element in spherical coordinates we have

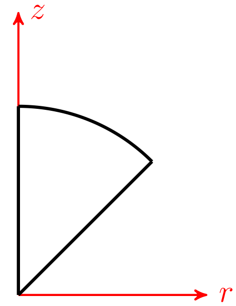
$$\int_0^{\pi} \int_0^{2\pi} \int_0^1 e^{\rho^3} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi = \frac{4\pi\,(e-1)}{3}$$

Example: Use spherical coordinates to find the volume of the ice cream cone bounded below by $z = \sqrt{x^2 + y^2}$ (the cone) and above by the sphere $x^2 + y^2 + z^2 = z$ (the ice cream).

As always, the first thing we want to do is draw a picture. But first we need to get a better handle on the sphere that makes the ice cream. Completing the square we have

$$x^2 + y^2 + z^2 = z \quad \Leftrightarrow \quad x^2 + y^2 + \left(z - \frac{1}{2}\right)^2 = \frac{1}{4}$$

so the ice cream is a sphere of radius $1/2$ centered at the point $(0, 0, 1/2)$. Now we can draw the picture. When we're certain that the region of integration is symmetric about the z-axis, it is often helpful to draw the picture in what we call the zr-plane, where the r axis is any line starting at the origin and laying completely in the xy-plane. The entire region can then be visualized by revolving the cross-section about the z-axis.

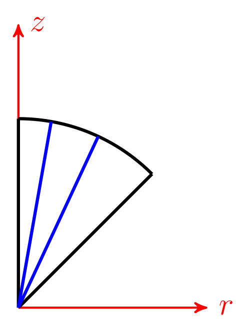


To get a better idea of the intersection points of the two surfaces, let's convert them to spherical coordinates. We have

$$\text{Cone}: \; z = \sqrt{x^2 + y^2} \quad \Leftrightarrow \quad z = r \quad \Leftrightarrow \quad \rho \sin\phi = \rho\cos\phi \quad \Leftrightarrow \quad \phi = \frac{\pi}{4}$$

$$\text{Sphere}: \; z = x^2 + y^2 + z^2 \quad \Leftrightarrow \quad \rho\cos\phi = \rho^2 \quad \Leftrightarrow \quad \cos\phi = \rho$$

Suppose we want to integrate over the region E in the order $d\rho\, d\phi\, d\theta$. To determine the limits for ρ we can draw ρ-lines, or lines on which θ and ϕ are held fixed and ρ is allowed to vary. Several such lines are shown in the rz-plane below.



We can see that each of the ρ-lines start at $\rho = 0$ and stop at the sphere defined by $\rho = \cos\phi$. To determine the limits on ϕ we notice that the smallest value ϕ takes on in the region is 0 and the largest is $\pi/4$. Since the ice cream cone goes all the way around the z-axis we must have $0 \le \theta \le 2\pi$. We can then compute the volume via the following integral

$$\int_0^{2\pi} \int_0^{\pi/4} \int_0^{\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{\pi}{8}$$

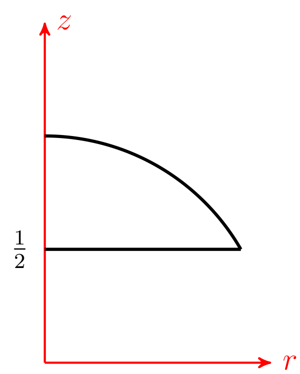
Example: Convert the following integral to spherical coordinates: $\displaystyle\int_0^{2\pi} \int_0^{\sqrt{3/4}} \int_{1/2}^{\sqrt{1-r^2}} r \, dz \, dr \, d\theta$

We begin by figuring out exactly what the region of integration is. Let's look at the limits of integraton in order.

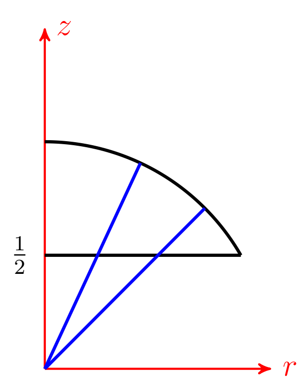
z: Starts at $z = \dfrac{1}{2}$ plane, stops at $z = \sqrt{1-r^2} \quad \Rightarrow \quad z^2 + r^2 = 1 \quad \Rightarrow \rho = 1$

r: Starts at $r = 0$, stops at $r = \sqrt{\frac{3}{4}}$. Note $z = \frac{1}{2} \Rightarrow \frac{1}{4} + r^2 = 1 \Rightarrow \quad r^2 = \frac{3}{4}$

θ: 1 full revolution



Now let's set up the integral in sphereical coordinates using the order $d\rho \, d\phi \, d\theta$. If we draw ρ-lines we get the following picture



From the picture you can see that ρ starts on the plane $z = \frac{1}{2}$ and stops on the sphere $\rho = 1$. Converting the plane to spherical coordinates we have

$$z = \frac{1}{2} \quad \Leftrightarrow \quad \rho\cos\phi = \frac{1}{2} \quad \Leftrightarrow \quad \rho = \frac{\sec\phi}{2}$$

So the bounds on ρ are: $\dfrac{\sec\phi}{2} \leq \rho \leq 1$.

The smallest value of ϕ is $\phi = 0$ and the largest is $\phi = \tan^{-1}\left(\dfrac{\sqrt{3/4}}{1/2}\right) = \tan^{-1}(\sqrt{3})$.

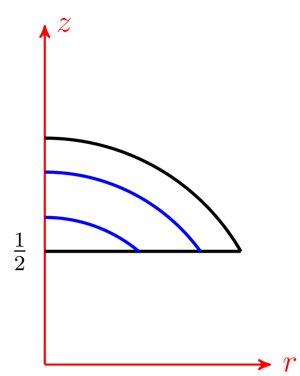
Since the region is one full revolution around the z-axis, we have $0 \leq \theta \leq 2\pi$.

Then the integral that gives us the volume of the region is

$$\int_0^{2\pi}\int_0^{\tan^{-1}(\sqrt{3})}\int_{\frac{\sec\phi}{2}}^{1} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

Example: Setup the previous integral in the order $d\phi\,d\rho\,d\theta$.

To determine the limits of integration with respsect to ϕ we need to draw ϕ-lines on our figure. ϕ-lines are curves that hold ρ and θ fixed and allowed ϕ to vary.



From the picture we can see that ϕ always starts at $\phi = 0$ and ends on the plane $z = \dfrac{1}{2}$. To set up the limits of integration we need to write the plane in terms of ϕ

$$z = \frac{1}{2} \quad \Leftrightarrow \quad \rho\cos\phi = \frac{1}{2} \quad \Leftrightarrow \quad \phi = \cos^{-1}\left(\frac{1}{2\rho}\right)$$

The smallest value of ρ in the region is $\rho = \dfrac{1}{2}$ and the largest is $\rho = 1$.

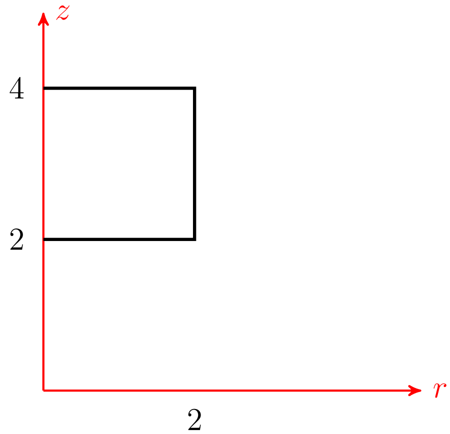
And again θ goes from 0 to 2π. We then have

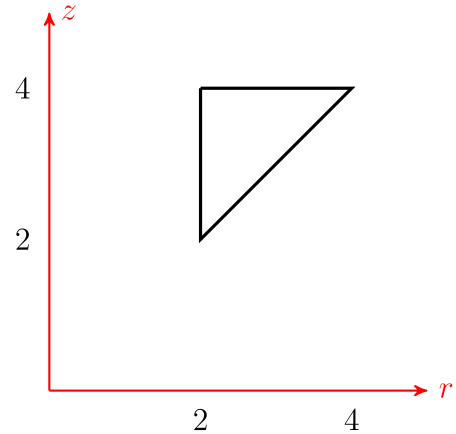
$$\int_0^{2\pi}\int_{1/2}^{1}\int_0^{\cos^{-1}\left(\frac{1}{2\rho}\right)} \rho^2\sin\phi\,d\phi\,d\rho\,d\theta$$

Example: Convert the following integral from cylindrical coordinates to spherical coordinates

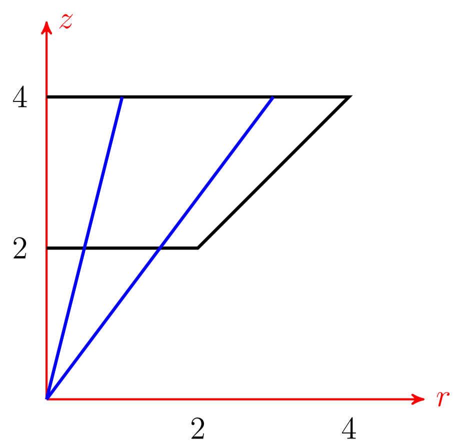
$$\int_0^{2\pi}\int_0^2\int_2^4 r\,dz\,dr\,d\theta + \int_0^{2\pi}\int_2^4\int_r^4 r\,dz\,dr\,d\theta$$

Let's draw seperate pictures for each integral and then combine them. Since $0 \leq \theta \leq 2\pi$ we know that the region makes one full revolution about the z-axis. As such, it's enough to draw the pictures in the rz-plane. These regions look as follows:





Combining these two regions we get



$$(z=2) \leq \rho \leq (z=4) \quad \Leftrightarrow \quad 2\sec\phi \leq \rho \leq 4\sec\phi$$

$$0 \leq \phi \leq \frac{\pi}{4} \qquad 0 \leq \theta \leq 2\pi$$

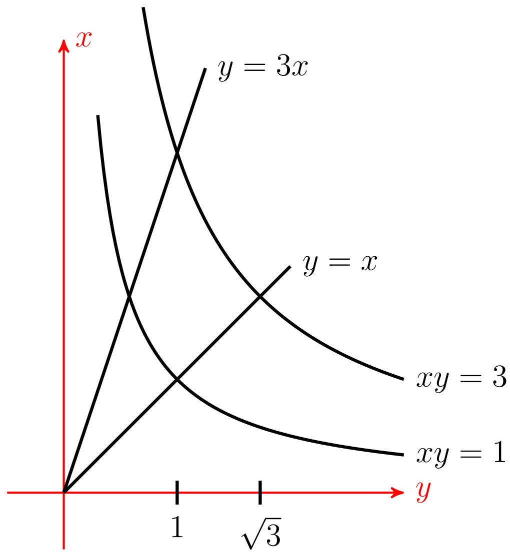
So we have

$$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_{2\sec\phi}^{4\sec\phi} \rho^2 \sin\phi\,d\phi\,d\rho\,d\theta = \frac{56\pi}{3}$$

12.8 Change of Variables in Multiple Integrals – Examples

Example: Evaluate $\iint_R xy\,dA$ where R is the region in the first quadrant bounded by the curves $y = x$, $y = 3x$, $xy = 1$, and $xy = 3$.

First we draw a picture of the region R.



At the very least we would need to break the region into two to compute the integral. Let's see what happens under a change of variables.

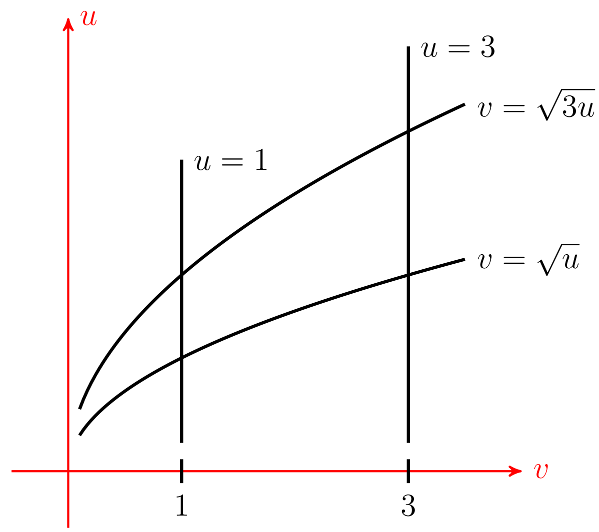
Notice that if we let $u = xy$ then the hyperbolas will turn into straight lines in the new coordinate system. Then, if we let $v = y$ then the straight lines will turn into porabolas. So we have

$$\begin{aligned} u &= xy \\ v &= y \end{aligned} \qquad \Leftrightarrow \qquad \begin{aligned} x &= \frac{u}{v} \\ y &= v \end{aligned}$$

To figure out the region of integration S in the new coordinate system we convert each of the boundaries of R into curves in terms of u and v. We have

R	S
$xy = 1$	$u = 1$
$xy = 3$	$u = 3$
$y = x$	$u = v^2$
$y = 3x$	$u = v^2/3$

The region S looks like



The Jacobian of the transformation is given by

$$\frac{\partial (x,y)}{\partial (u,v)} = \begin{vmatrix} 1/v & -u/v^2 \\ 0 & 1 \end{vmatrix} = \frac{1}{v}$$

Then

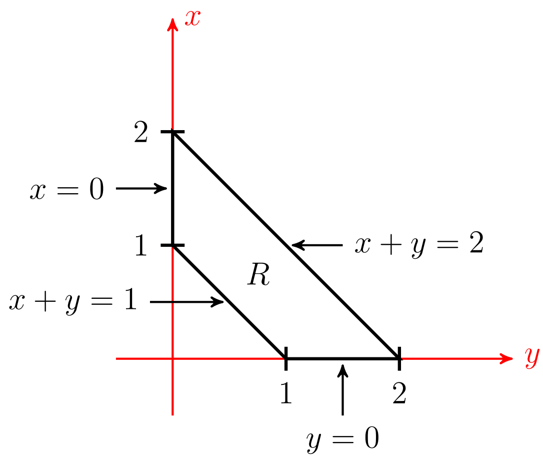
$$\iint_R xy\,dA = \iint_S u\left|\frac{1}{v}\right|\,dvdu = \iint_S \frac{u}{v}\,dvdu = \int_1^3\int_{\sqrt{u}}^{\sqrt{3u}} \frac{u}{v}\,dvdu$$

$$= \int_1^3 u\ln\left(\sqrt{3u}\right) - u\ln\left(\sqrt{u}\right)\,du = \int_1^3 \frac{u}{2}\ln\left(\frac{3u}{u}\right)\,du = \frac{\ln(3)}{2}\int_1^3 u\,du$$

$$= \frac{\ln(3)}{2}\left.\frac{u^2}{2}\right]_1^3 = \frac{\ln(3)}{4}\,(9-1) = 2\ln(3)$$

Example: Evaluate $\displaystyle\iint_R \cos\left(\frac{y-x}{y+x}\right)\, dA$ where R is the trapezoid in the first quadrant with vertices $(1,0)$, $(2,0)$, $(0,1)$ and $(0,2)$.

First we draw a picture of the region R.



We have to look at a couple of options for the change of variables. It's tempting to try to choose a transformation that would map the trapezoidal region R into a rectangular region in the uv-plane ($u = x + y$ and $v = y$ would do this). But, the bigger problem is that we don't know the anti-derivative of the integrand. To this end, we choose a transfomation that makes the anti-derivative easier to find. In this case that transformation is

$$u = y + x \quad \text{and} \quad v = y - x.$$

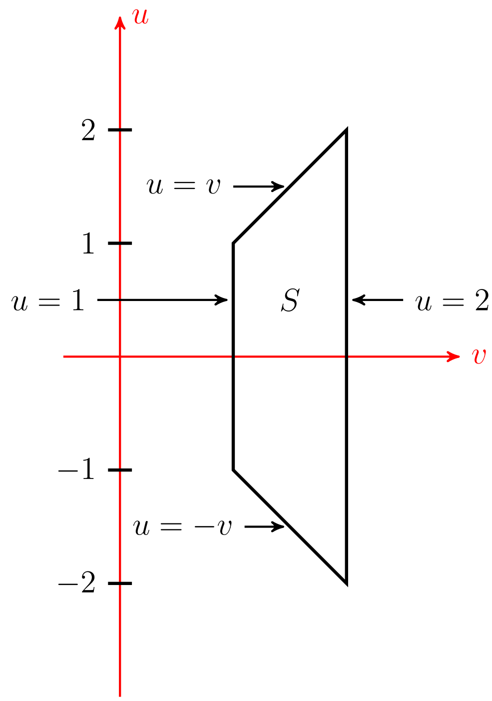
We need to solve for the reverse transformation which gives x and y in terms of u and v. To do this, we add the two transformations together and solve for y, and then subtract the two equations and solve for x:

$$u + v = 2y \quad \Rightarrow \quad y = \frac{u+v}{2} \qquad u - v = 2x \quad \Rightarrow \quad x = \frac{u-v}{2}$$

To get the image region S in the uv-plane we transform each of the boundaries of R:

R Boundaries	S Boundaries
$x + y = 2$	$u = 2$
$x + y = 1$	$u = 1$
$x = 0$	$u = v$
$y = 0$	$u = -v$

The the regions S looks like



OK, so the new region is not quite a rectangle, but at least it will take only one integral to evaluate.

The Jacobian of the transformation is given by

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{vmatrix} = \frac{1}{2}$$

Then

$$\iint_R \cos\left(\frac{y-x}{y+x}\right)\, dA = \iint_S \cos\left(\frac{v}{u}\right)\left|\frac{1}{2}\right|\, dvdu = \frac{1}{2}\int_1^2 \int_{-u}^{u} \cos\left(\frac{v}{u}\right)\, dvdu$$

$$\frac{1}{2}\int_1^2 u\sin\left(\frac{v}{u}\right)\Big]_{-u}^{u}\, du = \frac{1}{2}\int_1^2 u\left(\sin(1)-\sin(-1)\right)\, du = \sin(1)\int_1^2 u\, du = \frac{3}{2}\sin(1)$$

Example: Compute the volume of the general ellipsoid described by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

OK, this would kinda suck even in cylindrical or spherical coordinates. But with a change of variables it's a cinch! Consider the change of variables

$$\begin{aligned} x &= au \\ y &= bv \\ z &= cw \end{aligned} \quad \Rightarrow \quad u^2 + v^2 + w^2 = 1$$

Then

$$\text{Volume} = \iiint_E dV = \iiint_S \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| dudvdw$$

Where

$$\left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$

Then

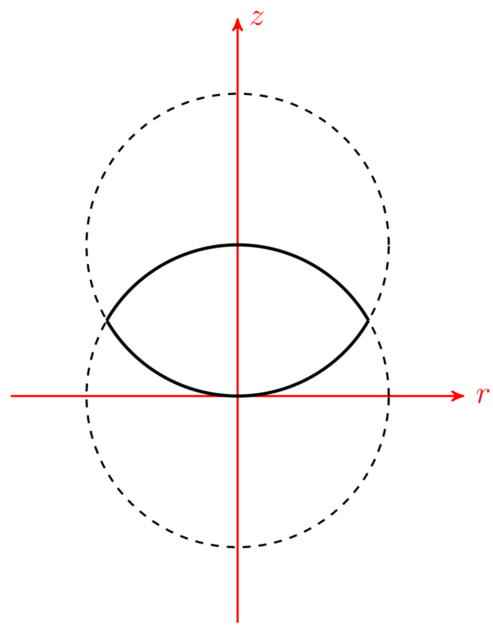
$$\text{Volume} = \iiint_E dV = abc \iiint_S dudvdw = abc\,(\text{Volume of Sphere with Radius 1}) = \frac{4}{3}\pi abc$$

Example: Compute the volume of the intersection of the surfaces $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 2z$.

The first surface is clearly a sphere of radius 1 centered at the origin. The second surface is also a sphere. To see this we put all the terms on one side and complete the square:

$$x^2 + y^2 + z^2 = 2z \quad \Rightarrow \quad x^2 + y^2 + z^2 - 2z = 0 \quad \Rightarrow \quad x^2 + y^2 + (z-1)^2 = 1$$

So the second surface is a sphere of radius one centered at the point $(0, 0, 1)$. Since both surfaces are symmetric about the z-axis, we sketch their intersection in the rz-plane:



Since the resulting solid is symmetric about the z-axis it actually makes more sense to integrate using cylindrical coordinates. The two surfaces in cylindrical coords are given by

$$x^2+y^2+z^2=1 \quad \Leftrightarrow \quad r^2+z^2=1 \quad \text{and} \quad x^2+y^2+(z-1)^2=1 \quad \Leftrightarrow \quad r^2+(z-1)^2=1$$

Integrating first with respect to z we have

$$V=\iiint_E dV=\iint_R \int_{1-\sqrt{1-r^2}}^{\sqrt{1-r^2}} dz\, dA$$

where the region R is the projection of E into the xy-plane. To find R we need to find the projection of the curve of intersection in the xy-plane. Since we've argued that the solid is symmetric about the z-axis, we can instead find the points of intersection of the two curves in the rz-plane and revolve them about the z-axis. We have

$$r^2=1-z^2 \quad \Rightarrow \quad 1-z^2+(z-1)^2=1 \quad \Rightarrow \quad -z^2+z^2-2z+1=0 \quad z=\frac{1}{2}$$

Substituting this into the first equation we have

$$r^2=1-\frac{1}{4} \quad \Rightarrow \quad r=\pm\frac{\sqrt{3}}{2}$$

So the projection of E into the xy-plane is a cricle of radius $\sqrt{3}/2$. Setting up the limits of integration for the circle we have

$$V=\iiint_E dV=\int_0^{2\pi}\int_0^{\sqrt{3}/2}\int_{1-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r dz dr d\theta$$

$$= \int_0^{2\pi} \int_0^{\sqrt{3}/2} \left[2r\sqrt{1-r^2} - r \right] \, dr d\theta = \int_0^{2\pi} \int_{1/4}^{1} \sqrt{u} \, du d\theta - \int_0^{2\pi} \int_0^{\sqrt{3}/2} r \, dr d\theta$$

$$= \int_0^{2\pi} \left. \frac{2}{3} u^{3/2} \right]_{1/4}^{1} d\theta - \int_0^{2\pi} \left. \frac{r^2}{2} \right]_0^{\sqrt{3}/2} d\theta = \frac{2}{3} \int_0^{2\pi} \left(1 - \frac{1}{8} \right) d\theta - \frac{3}{8} \int_0^{2\pi} d\theta = \frac{2}{3} \frac{1}{8} 2\pi$$

$$\left(\frac{2}{3} \right) \left(\frac{7}{8} \right) 2\pi - \left(\frac{3}{8} \right) 2\pi = \frac{5\pi}{12}$$

Standard Problem 1: Consider the surface described by

$$\left(x^2 + y^2 + z^2 + A^2 - a^2\right)^2 = 4A^2\left(x^2 + y^2\right) \quad \text{where} \quad A > a$$

(a) Using cylindrical coordinates, find a **simple** equation for the cross-section of the surface in the rz-plane (that is, an arbitrary plane of constant θ) and sketch the cross-section.

(b) Using the result of (a), find an equation in cylindrical coordinates for the cross-section (or cross-sections) of the surface in the xy-plane and sketch the cross-section.

(c) Using the result of (a) and (b) identify the surface.

(d) Using a triple integral in cylindrical coordinates, find the volume inside the surface.

(a) First we need to get a better handle on just what this thing is, which is a little easier in cylindrical coordinates. We have

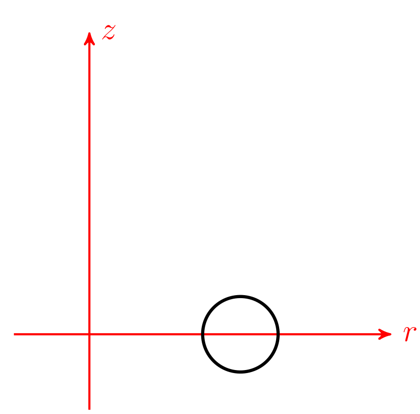
$$\left(r^2 + z^2 + A^2 - a^2\right)^2 = 4A^2r^2$$

Taking square roots of both sides we have

$$\left(r^2 + z^2 + A^2 - a^2\right)^2 = 4A^2r^2 \quad \Rightarrow \quad r^2 + z^2 + A^2 - a^2 = 2Ar$$

$$r^2 + z^2 + A^2 - a^2 = 2Ar \quad \Rightarrow \quad r^2 - 2Ar + A^2 + z^2 = a^2 \quad \Rightarrow \quad (r - A)^2 + z^2 = a^2$$

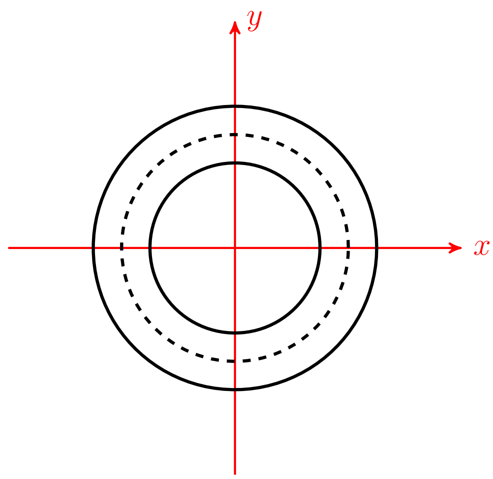
This is a circle of radius a centered at the point $(A, 0)$ in the rz-plane.



(b) Let's look at cross-sections of the surface in the xy-plane by setting $z = 0$:

$$(r - A)^2 = a^2 \quad \Rightarrow \quad r = A \pm a$$

So the cross-sections in the xy-plane are circles of radii $A + a$ and $A - a$:



(c) It should be clear now that the surface described by the given equation is a torus where A is the radius of the big circle and a is the radius of the little cross-sectional circles.

(d) We can compute the volume of the torus directly in cylindrical coordinates. Integrating with respect to z first we see that the cross-section in the rz-plane is bounded by the top and bottom halfs of the little circle, described by

$$z = \pm\sqrt{a^2 - (r - A)^2}$$

Then r goes from $A - a$ to $A + a$. We then have

$$V = \int_0^{2\pi}\int_{A-a}^{A+a}\int_{-\sqrt{a^2-(r-A)^2}}^{\sqrt{a^2-(r-A)^2}} r\,dzdrd\theta = 2\int_0^{2\pi}\int_{A-a}^{A+a} r\sqrt{a^2 - (r - A)^2}\,drd\theta$$

Using the substitution $(r - A) = a\cos t$ we have $dr = -a\sin t\,dt$ which gives

$$\begin{aligned}
V &= 2\int_0^{2\pi}\int_\pi^0 (A + a\cos t)\sqrt{a^2(1 - \cos^2 t)}\,(-a\sin t)\,dtd\theta \\
&= 2\int_0^{2\pi}\int_0^\pi (A + a\cos t)\sqrt{a^2(1 - \cos^2 t)}\,(a\sin t)\,dtd\theta \\
&= 2\int_0^{2\pi}\int_0^\pi \left(Aa^2\sin^2 t + a^3\cos t\sin^2 t\right)dtd\theta \\
&= 2\int_0^{2\pi}\int_0^\pi Aa^2\sin^2 t\,dtd\theta + 2\int_0^{2\pi}\int_0^\pi a^3\cos t\sin^2 t\,dtd\theta \\
&= 4\pi Aa^2\ \frac{t}{2} - \frac{\sin(2t)}{4}\bigg|_0^\pi = 2\pi^2 Aa^2
\end{aligned}$$

Note that the second integral is zero because $\cos t\sin^2 t$ is odd about $t = \frac{\pi}{2}$ on $[0, \pi]$.

Standard Problem 2

(a) Consider an arbitrary region R with area A in the rz-plane. The location of the r-component of the center of mass is given by

$$\bar{r} = \frac{\iint_R r\,dr\,dz}{\iint_R dr\,dz} = \frac{1}{A}\iint_R r\,dr\,dz$$

Then notice that we can write

$$A\bar{r} = \iint_R r\,dr\,dz$$

which will be useful later.

(b) The volume of revolution in cylindrical coordinates is given by

$$V = \iint_R \int_0^{2\pi} r\,d\theta\,dr\,dz$$

(c) Since the integrand does not depend on θ, we have

$$V = 2\pi \iint_R r\,dr\,dz$$

Now using the result from part (a) we have

$$V = 2\pi A\bar{r}$$

which is Pappus' Theorem for volumes of revolution.

(d) We now want to use this result to determine the volume of a cone with height H and radius R. We have

$$\begin{aligned} V &= 2\pi \int_0^R \int_0^{H-(H/R)r} r dz dr = 2\pi \int_0^R \left(Hr - \frac{H}{R}r^2\right) dr \\ &= 2\pi \left(H\frac{r^2}{2} - \frac{H}{R}\frac{r^3}{3}\Big|_0^R \right) = \frac{4}{3}\pi R^2 \end{aligned}$$

which is what we expect for the volume of a cone.

(e) Now we verify the result of the theorem using a sphere of radius R. For simplicity, we will only consider the sphere for $z > 0$ and multiply by 2 to get the total volume. We have

$$
\begin{aligned}
V &= 2\,(2\pi)\,A\bar{r} \;=\; 4\pi \int_0^R \int_0^{\sqrt{R^2-z^2}} r\,dr\,dz \;=\; 2\pi \int_0^R R^2 - z^2\,dz \\
&= 2\pi \left(R^2 z - \frac{z^3}{3}\bigg|_0^R \right) \;=\; 2\pi \frac{2}{3} R^2 = \frac{4}{3}\pi R^3
\end{aligned}
$$

(f) Newton lived from 1642 to 1727 and Leibniz lived from 1646 to 1716.

(g) Pappus lived from 290 to 350.

13.1/13.2 Vector Fields, Work, Flow, and Line Integrals

In many physical applications we want to show how a body acts in the presence of a force field. This could be a **gravitational force field, electromagnetic force field** or the force a particle feels while moving through a **fluid velocity field**.

When the direction and magnitude of a force depend on the point in space we represent this with a **vector field**.

Definition: A vector field is a function $\mathbf{F}$ which assigns a vector to each point in space.

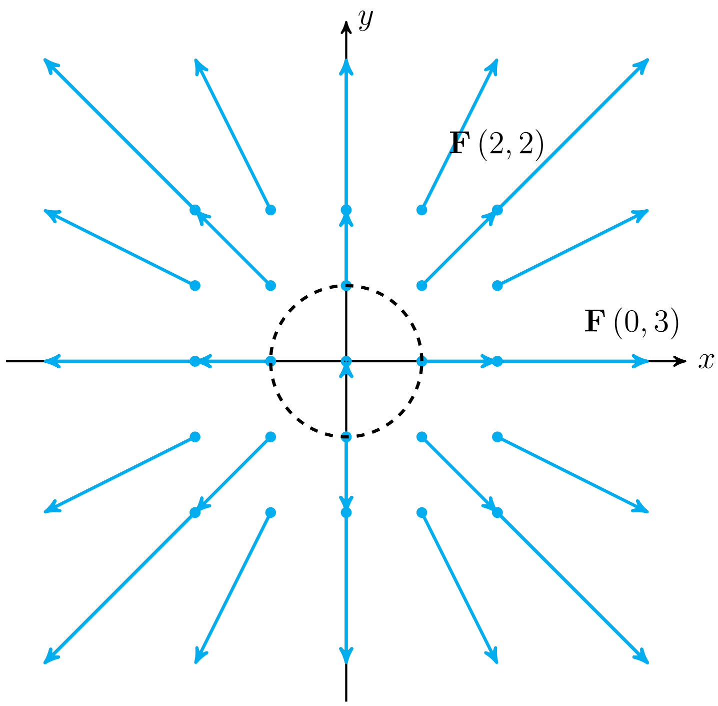
In 2D: $\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j} = \langle P(x, y)\,, Q(x, y)\rangle$

In 3D: $\mathbf{F}(x, y, z) = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k} = \langle P(x, y, z)\,, Q(x, y, z)\,, R(x, y, z)\rangle$

If the component functions P, Q, and R are continuous (differentiable) then we say $\mathbf{F}$ is continuous (differentiable).

It's often helpful to visualize a vector field $\mathbf{F}$ by plotting it

Example: $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$



Chris Ketelsen
APPM 2350
Chapter 13
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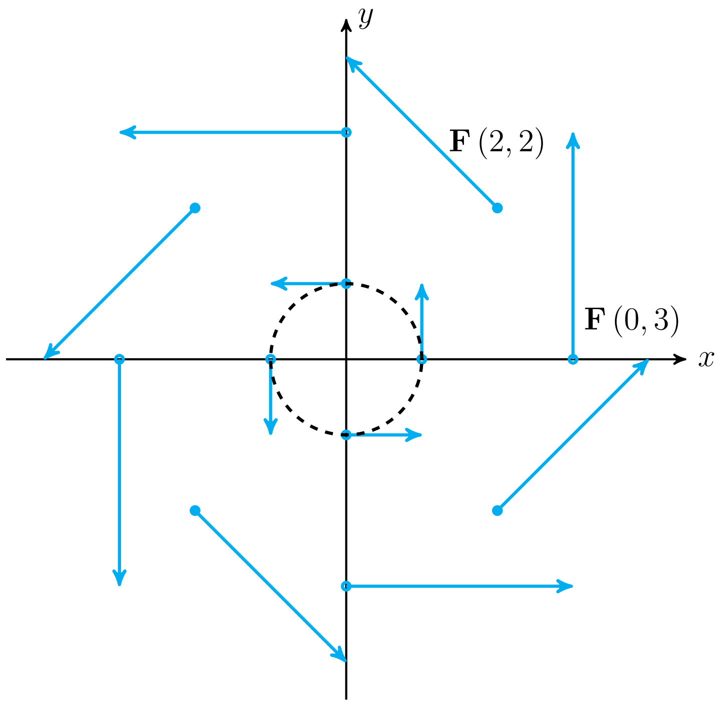
Example: $\mathbf{F}(x,y) = -y\mathbf{i} + x\mathbf{j}$



Note that it looks like for this vector field, each vector points in a direction tangent to the circle centered at the origin on which the point lies. We can check this analytically by noting that for any point (x,y) the vector $\mathbf{G} = -x\mathbf{i} - y\mathbf{j}$ points directly from the point to the origin. Then, dotting these two fields together, we have

$$\mathbf{F} \cdot \mathbf{G} = -xy + xy = 0$$

Gradient Fields

We've already seen vector fields in this class. Note that for any scalar function $f(x,y)$, the gradient $\mathbf{F} = \nabla f$ is a vector field. Specifically, it's the vector field that points in the direction of greatest increase of the scalar function f.

Definition: If $\mathbf{F} = \nabla f$ then we call f the **potential function** of $\mathbf{F}$.

Example: Compute the vector field $\mathbf{F}$ associated with the potential function

$$f(x,y,z) = \frac{mMG}{\sqrt{x^2+y^2+z^2}}$$

$$\mathbf{F} = \nabla f = -\frac{mGMx}{(x^2+y^2+z^2)^{3/2}}\mathbf{i} - \frac{mGMy}{(x^2+y^2+z^2)^{3/2}}\mathbf{j} - \frac{mGMz}{(x^2+y^2+z^2)^{3/2}}\mathbf{k}$$

This is the gravitational field acting on a body with mass m towards a body with mass M centered at the origin. Notice that for any (x,y,z) the field $\mathbf{F}$ points towards the origin and has magnitude inversely proportional to the distance from the origin.

Every scalar field f gives rise to a vector field $\mathbf{F}$ through the gradient, but does every field $\mathbf{F}$ have a potential function?

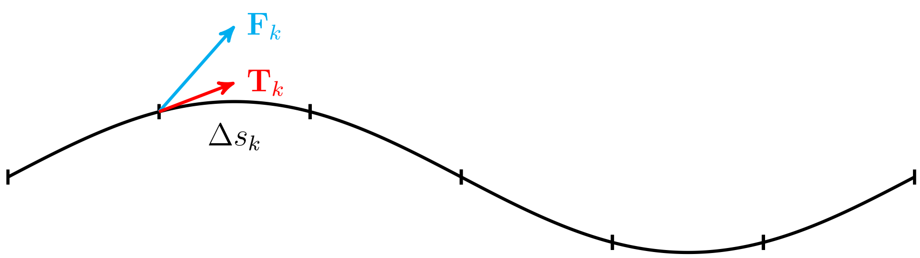
The answer is **NO**. Fields that do have a potential function are special and they're called **conservative vector fields**.

Work Done by Force Along a Curve

How much work is done by a gravitational field on a space ship as it travels along a specified curve?

Given a force field $\mathbf{F} = \mathbf{P}\mathbf{i} + \mathbf{Q}\mathbf{j} + \mathbf{R}\mathbf{k}$ and smooth curve $c : \mathbf{r}(t) = g(t)\,\mathbf{i} + h(t)\,\mathbf{j} + k(t)\,\mathbf{k}$ for $a \le t \le b$.

Recall that $\quad Work = Force \times Distance$



$$\begin{aligned} [\text{Work}]_k &= [\text{Component of } \mathbf{F}_k \text{ in direction of } \mathbf{T}_k]\,(\Delta s_k) \\ &= (\mathbf{F}_k \cdot \mathbf{T}_k)\,\Delta s_k \end{aligned}$$

Summing up all the work chunks along the curve we have

$$W_N = \sum_{k=1}^{N} (\mathbf{F}_k \cdot \mathbf{T}_k)\,\Delta s_k$$

$$\lim_{N\to\infty} W_N = W = \int_{t=a}^{t=b} \mathbf{F} \cdot \mathbf{T} ds$$

There are many different ways to write this integral. First notice that

$$\mathbf{T} ds = \frac{\mathbf{v}}{|\mathbf{v}|} ds = \frac{\mathbf{dr}}{dt}\frac{dt}{ds} = \mathbf{dr}$$

So we could instead write

$$W = \int_{t=a}^{t=b} \mathbf{F} \cdot \mathbf{dr} \quad \text{or} \quad W = \int_{t=a}^{t=b} \mathbf{F} \cdot \frac{\mathbf{dr}}{dt} dt$$

Note that the last expression is the first one that is useful for computation.

Example: Find the work done by the field $\mathbf{F} = xy\mathbf{i}+y\mathbf{j}-yz\mathbf{k}$ over the curve $\mathbf{r}(t) = t\mathbf{i}+t^2\mathbf{j}+t\mathbf{k}$ for $0 \le t \le 1$.

First we need to compute $d\mathbf{r}/dt$. We have

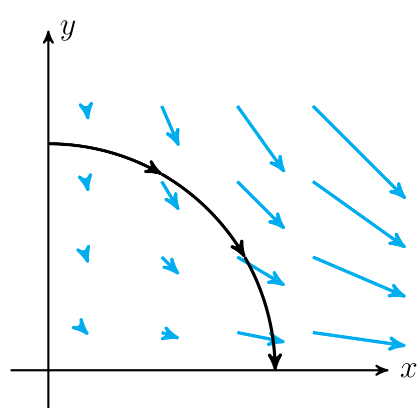
$$\mathbf{F} = t^3\mathbf{i} + t^2\mathbf{j} - t^3\mathbf{k} \quad \Rightarrow \quad \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + \mathbf{k}$$

Then

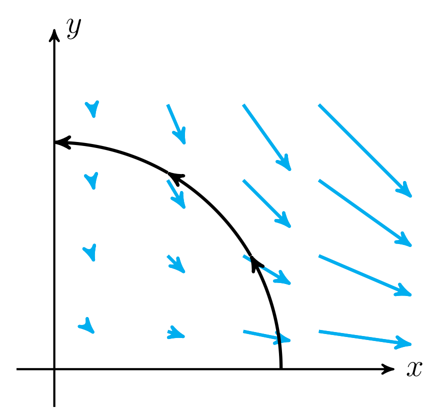
$$W = \int_0^1 \left(t^3\mathbf{i} + t^2\mathbf{j} - t^3\mathbf{k}\right) \cdot \left(\mathbf{i} + 2t\mathbf{j} + \mathbf{k}\right)\, dt = \int_0^1 \left(t^3 + 2t^3 - t^3\right)\, dt = \int_0^1 2t^3\, dt = \frac{1}{2}$$

Estimating Work Done by a Vector Field

Suppose that a particle is traveling in a gravitational field along some smooth curve C. If the particle is moving with the vector field, then the field is doing work on the particle. If the particle is traveling against the vector field, then the particle is the one doing the work, or you can say that the vector field is doing **negative** work on the particle.



Positive Work



Negative Work

Orientation **matters** when integrating over vector fields!

Given a parameterization of a curve C, the curve $-C$ indicates that traversal of the curve backwards. Then

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r}$$

This makes sense since we compute the integral over C as a regular integral over the interval $a \le t \le b$:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt = -\int_b^a \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt = -\int_C \mathbf{F} \cdot d\mathbf{r}$$

Fluid Flow

Suppose $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a velocity field of a fluid. Let C be a smooth curve described by the parameterization $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ for $a \le t \le b$. The analogue of work done by a force field in the fluids setting is called the **flow** of the fluid. It is a measure of how much the fluid field moves along the curve C.

$$\text{Flow} = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_a^b \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\,dt = \int_a^b Pdx + Qdy + Rdz$$

The last formulation of the flow/work integral is new, but it follows directly from the usual definition. We have

$$\begin{aligned}
\mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\,dt &= [P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}]\cdot\left[\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}\right]\,dt \\
&= [P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}]\cdot[dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}] \\
&= Pdx + Qdy + Rdz
\end{aligned}$$

If the curve C is closed then the flow is called **circulation**.

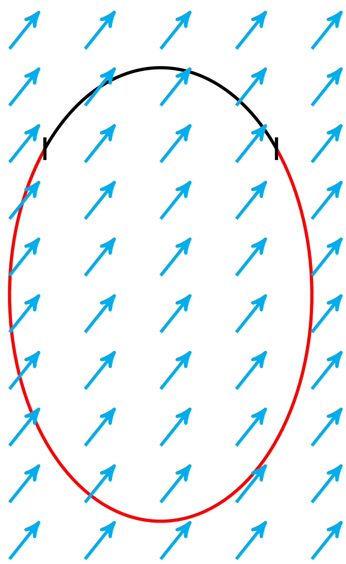
Example: Find the flow of the fluid velocity field $\mathbf{F} = (x - z)\,\mathbf{i} + x\mathbf{k}$ along the curve C parameterized by $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{k}$ for $0 \le t \le \pi$.

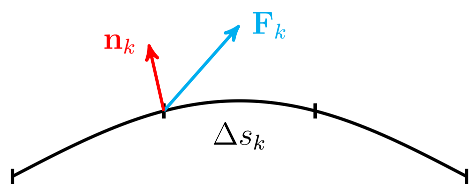
$$\begin{aligned}
\mathbf{F} &= (\cos t - \sin t)\,\mathbf{i} + \cos t\mathbf{k} \\
\frac{d\mathbf{r}}{dt} &= -\sin t\mathbf{i} + \cos t\mathbf{k}
\end{aligned}$$

$$\begin{aligned}
\text{Flow} &= \int_0^\pi \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\,dt \\
&= \int_0^\pi \left[-\cos t\sin t + \sin^2 t + \cos^2 t\right]\,dt \\
&= \int_0^\pi [1 - \cos t\sin t]\,dt \\
&= \int_0^\pi \left[1 - \frac{\sin 2t}{2}\right]\,dt \\
&= t + \frac{\cos 2t}{4}\Big]_0^\pi \\
&= \left(\pi + \frac{1}{4}\right) - \left(0 + \frac{1}{4}\right) \\
&= \pi
\end{aligned}$$

Flux Across a Plane Curve

Consider a planar flow $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ and a smooth planar curve C parameterized by $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$. The fluid **flux** is the rate at which fluid crosses a curve. If the curve is a closed loop then the flux is the rate at which fluid is entering or leaving the loop.





$$
\begin{aligned}
[\text{Flux}]_k &= [\text{Component of } \mathbf{F}_k \text{ in direction of } \mathbf{n}_k]\,(\Delta s_k) \\
&= (\mathbf{F}_k \cdot \mathbf{n}_k)\,\Delta s_k
\end{aligned}
$$

Summing all of the contributions to the flux around the curve and taking the limit as the number of subintervals goes to infinity we have

$$
\text{Flux} = \sum_{k=1}^{N} [\mathbf{F}_k \cdot \mathbf{n}_k]\,\Delta s_k \quad \xrightarrow{N\to\infty} \quad \int_C \mathbf{F}\cdot\mathbf{n}\,ds
$$

The normal vector $\mathbf{n}$ is a vector orthogonal to both $\mathbf{T}$ and $\mathbf{k}$. Both $\mathbf{T}\times\mathbf{k}$ and $\mathbf{k}\times\mathbf{T}$ satisfy this, but by convention we assume $\mathbf{n} = \mathbf{T}\times\mathbf{k}$ and that the curve C is traversed in the counterclockwise direction. Then $\mathbf{n}$ is called the **outward pointing normal** and is given by

$$
\mathbf{n} = \mathbf{T}\times\mathbf{k} = \left[\frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j}\right]\times\mathbf{k} = \frac{dy}{ds}\mathbf{i} - \frac{dx}{ds}\mathbf{j} \quad \Rightarrow \quad \mathbf{F}\cdot\mathbf{n} = P\frac{dy}{ds} - Q\frac{dx}{ds}
$$

Then $$\text{Flux} = \oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \oint_C P\,dy - Q\,dx = \int_{t=a}^{t=b} P\,dy - Q\,dx$$

Example: Find the flux and circulation of $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$ around the curve C parameterized by $\mathbf{r}(t) = \cos t\mathbf{i} + 4\sin t\mathbf{j}$ for $0 \le t \le 2\pi$.

We have

$$\begin{aligned} P &= -4\sin t & Q &= \cos t \\ dx &= -\sin t\, dt & dy &= 4\cos t\, dt \end{aligned}$$

$$\begin{aligned} \text{Flux} &= \int_0^{2\pi} P dy - Q dx \\ &= \int_0^{2\pi} \left[-16\sin t\cos t + \sin t\cos t\right]\, dt \\ &= -15\int_0^{2\pi} \sin t\cos t\, dt \\ &= -\frac{15}{2}\int_0^{2\pi} \sin 2t\, dt \\ &= -\frac{15}{4}\cos 2t\Big]_0^{2\pi} \\ &= 0 \end{aligned}$$

$$\begin{aligned} \text{Circulation} &= \int_0^{2\pi} P dx + Q dy \\ &= \int_0^{2\pi} \left[4\sin^2 t + 4\cos^2 t\right]\, dt \\ &= \int_0^{2\pi} 4\, dt \\ &= 8\pi \end{aligned}$$

General Line Integrals

The integrals that we used to compute work, flow, flux, etc. in the previous section have been special applications of line integrals.

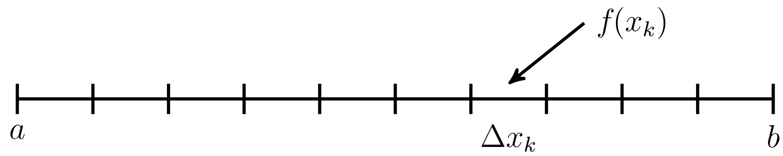
In 1D we integrate a function $f(x)$ over some interval $[a, b]$:

$$I = \int_a^b f(x)\, dx$$

If $f \equiv f(x, y)$ we can do something similar along a smooth curve in space.

Let C be a smooth curve parameterized by $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$

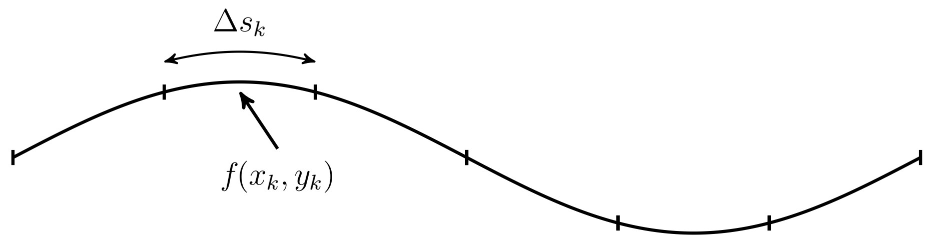
In 1D we found $I = \displaystyle\int_a^b f(x)\ dx$ by breaking up $[a, b]$ into tiny chunks



Then $$S_N = \sum_{k=1}^{N} f(x_k)\ \Delta x_k \quad \xrightarrow{N\to\infty} \quad \int_a^b f(x)\ dx$$

If $f \equiv f(x, y)$ we can do something similar along a smooth curve in space.

Let C be a smooth curve parameterized by $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$



Then $$S_N = \sum_{k=1}^{N} f(x_k, y_k)\ \Delta s_k \quad \xrightarrow{N\to\infty} \quad \int_C f(x, y)\ ds$$

Recall that $\dfrac{ds}{dt} = |\mathbf{v}(t)| = \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}\, dt$. Then equating $ds = |\mathbf{v}(t)|\ dt$ we have

$$\int_C f(x, y)\ ds = \int_a^b f(x(t)\,, y(t))\, |\mathbf{v}(t)|\ dt = \int_a^b f(x(t)\,, y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example: Evaluate $\int_C xy^4\,ds$ where C is the right half of the circle described by $x^2+y^2=16$.

First we need to parametrize the curve C. We have $\mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j}$ for $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$.

Then $\quad f(x(t)\,, y(t)) = 4^5\cos t\sin^4 t \quad$ and $\quad |\mathbf{v}(t)| = \sqrt{(-4\sin t)^2 + (4\cos t)^2} = 4$

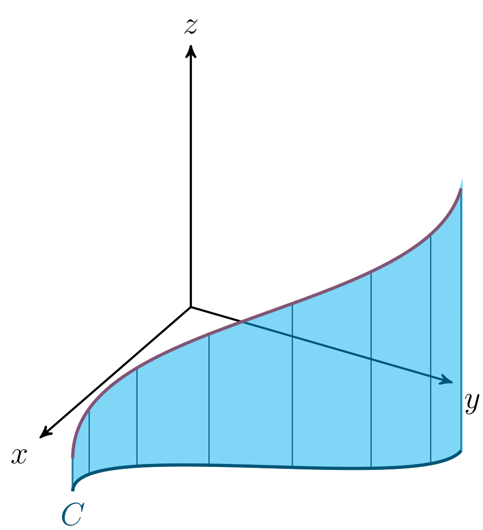
$$\int_C xy^4\,ds = \int_{-\pi/2}^{\pi/2} \left(4^5\cos t\sin^4 t\right) 4\,dt = 4^6 \int_{-\pi/2}^{\pi/2} \cos t\sin^4 t\,dt$$

$$\text{Let } u = \sin t \;\Rightarrow\; du = \cos t\,dt \;\Rightarrow\; -1 \leq u \leq 1$$

$$= 4^6 \int_{-1}^{1} u^4\,du = 4^6 \frac{u^5}{5}\Big]_{-1}^{1} = 4^6\frac{2}{5}$$

Geometric Interpretation

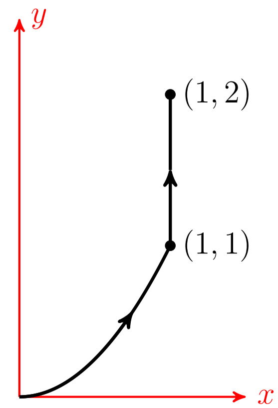
If $f(x,y) \geq 0$ on C then $\int_C f(x,y)\;ds$ is the area over C under the curve $f(x(t)\,, y(t))$. Think of this as the area of a curtain hanging from the rod described by the curve with x- and y-components described by $\mathbf{r}(t)$ and height $f(x(t)\,, y(t))$:



What if the curve C is not smooth? If C can be written as a union of curves that are smooth, say $C = C_1 \cup C_2$ then we have

$$\int_C f\;ds = \int_{C_1} f\;ds + \int_{C_2} f\;ds$$

Example: Evaluate $\int_C 2x\,ds$ where C consists of curves C_1 and C_2 where C_1 is the parabola $y = x^2$ from $(0,0)$ to $(1,1)$ and C_2 is the line segment connecting $(1,1)$ to $(1,2)$.



$$\begin{array}{llll} C_1: & x = t & y = t^2 & 0 \le t \le 1 \\ C_2: & x = 1 & y = t & 1 \le t \le 2 \end{array}$$

$$\int_{C_1} 2x\,ds = \int_0^1 2t\,|\mathbf{v}(t)|\,dt = \int_0^1 2t\sqrt{1+4t^2}\,dt = \frac{5\sqrt{5}-1}{6}$$

$$\int_{C_2} 2x\,ds = \int_1^2 2\,(1)\,|\mathbf{v}(t)|\,dt = \int_1^2 2\,(1)\,\sqrt{1^2}\,dt = \int_1^2 2\,dt = 2$$

$$\int_C 2x\,ds = \int_{C_1} 2x\,ds + \int_{C_2} 2x\,ds = \frac{5\sqrt{5}+11}{6}$$

Line Integrals in 3D

Consider the function $f(x,y,z)$ and the curve C parameterized by $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ for $a \le t \le b$. Then the integral of $f(x,y,z)$ over the curve C is given by

$$\begin{aligned} \int_C f(x,y,z)\,ds &= \int_a^b f(x(t)\,,y(t)\,,z(t))\,|\mathbf{v}(t)|\,dt \\ &= \int_a^b f(x(t)\,,y(t)\,,z(t))\,\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt \end{aligned}$$

Example: Find the mass of a wire in the shape of a helix described by the curve $C : x = t,\ y = \cos t,\ z = \sin t$ for $0 \le t \le 2\pi$ if the density at any point is equal to the square of the distance from the origin.

We have

$$\delta(x,y,z) = x^2 + y^2 + z^2 = t^2 + \cos^2 t + \sin^2 t = t^2 + 1 \qquad |\mathbf{v}(t)| = \sqrt{1 + (-\sin t)^2 + \cos^2 t} = \sqrt{2}$$

$$M = \int_C \delta(x,y,z)\,ds = \int_0^{2\pi} \sqrt{2}\,(1+t^2)\,dt = \sqrt{2}\left(t + \frac{t^3}{3}\right)\Bigg]_0^{2\pi} = \sqrt{2}\left(2\pi + \frac{8\pi^3}{3}\right)$$

13.3 The Fundamental Theorem for Line Integrals

Recall that we said that some vector fields could be written as the gradient of a scalar function, i.e. $\mathbf{F} = \nabla f$. When this happened we said that f is the **potential function** of $\mathbf{F}$.

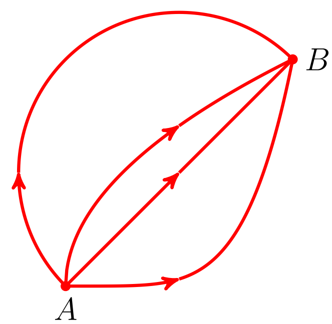
Definition: When a vector field $\mathbf{F}$ can be written as the gradient of a potential function, we say that $\mathbf{F}$ is a **conservative vector field**.

Lots of nice things happen when a vector field is conservative.

Recall that the work done by the vector field $\mathbf{F}$ on a particle moving along path C with initial point A and terminal point B is given by

$$\text{Work} = \int_C \mathbf{F} \cdot d\mathbf{r}$$

Consider several paths that all start at A and end at B:



If the work done by the vector field $\mathbf{F}$ is the same for **any** path that starts at A and ends at B then we say the work integral

$$W = \int_C \mathbf{F} \cdot d\mathbf{r}$$

is **path independent** and write

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_A^B \mathbf{F} \cdot d\mathbf{r}$$

It turns out that a work integral is path independent precisely when the vector field $\mathbf{F}$ is conservative.

OK, so if a planar field $\mathbf{F}$ is conservative, we know that work integrals over $\mathbf{F}$ are path independent. Suppose you wanted to compute

$$W = \int_C \mathbf{F} \cdot d\mathbf{r}$$

where the curve C starts and ends at the points A and B, respectively. How does the path independence of the integral make it easier to compute?

We could of course abandon the potentially complicated initial curve C and just compute W over a straight line connecting A and B. This would definitely be easier if C was something really nasty.

BUT IT'S TOTALLY BETTER THAN THIS!

The Fundamental Theorem for Line Integrals: Let $\mathbf{F}$ be a conservative vector field where $\mathbf{F} = \nabla f$ for some f. Let C be a smooth curve with initial point A and terminal point B. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_A^B \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A)$$

This means that if $\mathbf{F}$ is conservative, and we know its associated potential function, then we can evaluate the work integral simply by subtracting the value of the potential at the initial point from the value of the potential at the terminal point!

Notice that this expression looks kinda familiar. If $\mathbf{F} = \nabla f$ then ∇f is kinda like the derivative of f, or put another way, f is kinda like an anti-derivative of $\mathbf{F}$. The Calc I version of the fundamental theorem says

$$\int_a^b f'(x)\ dx = f(b) - f(a)$$

where f is the anti-derivative of $f'(x)$.

Example: The potential function for the field $\mathbf{F} = (y+2)\,\mathbf{i} + (x-2y)\,\mathbf{j}$ is $f(x,y) = 2x + xy - y^2$. Compute the work done by the vector field on a particle traveling on the path $\mathbf{r}(t) = \cos t\mathbf{i} + 2\sin t\mathbf{j}$ for $0 \leq t \leq \pi$.

Note that the start and end points of the path C are $A = (1,0)$ and $B = (-1,0)$. Then

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_A^B \mathbf{F} \cdot d\mathbf{r} = f\,(B) - f\,(A) = f\,(-1,0) - f\,(1,0) = -2 - 2 = -4$$

The Fundamental Theorem for Line Integrals works for space curves and vector fields in three dimensions just as easily.

Example: Consider the fluid velocity field $\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} = \nabla(xyz)$. Find the flow along any curve C that starts at $A = (-1, 3, 9)$ and ends at $B = (1, 6, -4)$.

We have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_A^B \mathbf{F} \cdot d\mathbf{r} = f(1, 6, -4) - f(-1, 3, 9) = (1)(6)(-4) - (-1)(3)(9) = -24 + 27 = 3$$

OK, so how the heck does this work?

Proof Sketch: Let $\mathbf{F} = \nabla f$ and let a smooth curve C be described by $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ for $a \leq t \leq b$, where the initial and terminal points are defined by $A = (x(a), y(a), z(a))$ and $B = (x(b), y(b), z(b))$, respectively.

Notice that on the curve C we have $f \equiv f(x(t), y(t), z(t))$. Then, using the Chain Rule to differentiate w.r.t. t we have

$$\begin{aligned} \frac{df}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \\ &= \left[\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right] \cdot \left[\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}\right] \\ &= \nabla f \cdot \frac{d\mathbf{r}}{dt} \end{aligned}$$

Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{t=a}^{t=b} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\,dt = \int_{t=a}^{t=b} \frac{df}{dt}\,dt = f(x(t), y(t), z(t))\big]_a^b = f(B) - f(A)$$

Cool Consequences of the Fundamental Theorem

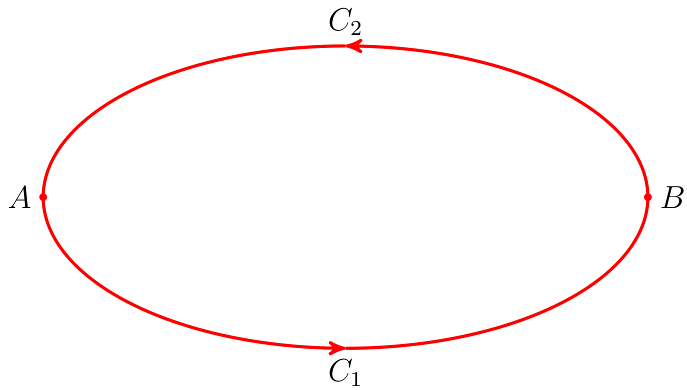
So far we have assumed that the curve C is smooth. In reality, the Fundamental Theorem works as long as C is piecewise smooth.

Theorem: Let $\mathbf{F} = \nabla f$ and suppose that $C = C_1 \cup C_2$ where C_1 starts at point A and ends at point D and C_2 starts at point D and ends at point B. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = (f(D) - f(A)) + (f(B) - f(D)) = f(B) - f(A)$$

Theorem: If $\mathbf{F}$ is conservative then $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed piecewise-smooth curve C.

Proof Sketch: Take an arbitrary closed curve C and break it up into C_1 and C_2 as follows.



Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = (f(B) - f(A)) + (f(A) - f(B)) = 0$$

OK, so two questions naturally arise:

1. How do we know if $\mathbf{F}$ is conservative?
2. If it is conservative, how do we find its potential function f?

Let's consider only planar vector fields for now. Let $\mathbf{F} = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$. If the component functions P and Q have continuous first partial derivatives, then if $\mathbf{F} = \nabla f$ for some f we have

$$P = \frac{\partial f}{\partial x} \quad \text{and} \quad Q = \frac{\partial f}{\partial y}$$

Since the partial derivatives of P and Q are continuous, we have by Clairaut's theorem

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial Q}{\partial x} \tag{1}$$

Thus a vector field $\mathbf{F}$ is conservative only if it satisfies (1). This also means that we can use (1) as a test to see if $\mathbf{F}$ is conservative.

Example: Determine if the field $\mathbf{F} = (y+2)\,\mathbf{i} + (x-2y)\,\mathbf{j}$ is conservative.

We have

$$\frac{\partial P}{\partial y} = 1 \;=\; 1 = \frac{\partial Q}{\partial x}$$

So $\mathbf{F}$ is conservative.

Once we know that $\mathbf{F}$ is conservative we can start looking for it's potential function. The process is straightforward. We illustrate it with an example.

Example: Consider the vector field from the previous example: $\mathbf{F} = (y+2)\,\mathbf{i} + (x-2y)\,\mathbf{j}$. We want to determine a scalar function f such that $\mathbf{F} = \nabla f$.

If $\mathbf{F} = \nabla f$ then equating the components of the two we have

$$\frac{\partial f}{\partial x} = P \;=\; (y+2) \tag{2}$$

$$\frac{\partial f}{\partial y} = Q \;=\; (x-2y) \tag{3}$$

Integrating (2) with respect to x we have

$$f(x,y) \;=\; \int P\,dx = \int (y+2)\;dx = xy + 2x + g(y) \tag{4}$$

The function $g(y)$ is there because it vanishes when taking the derivative with respect to x. Our goal is now to determine $g(y)$ which will fill out the expression for $f(x,y)$. Notice that we now have two ways to get to $\partial f/\partial y$. We know from (3) that it is equal to Q, but we also know that we can get it by taking the partial of (4) w.r.t. y. Setting these two expressions equal we have

$$x - 2y \;=\; x + g'(y) \tag{5}$$

Solving (5) for $g'(y)$ we have

$$g'(y) \;=\; -2y \quad\Rightarrow\quad g\,(y) = -y^2 + D \tag{6}$$

where D is some arbitrary constant. Substituting the expression for $g\,(y)$ into (4) we have

$$f(x,y) = xy + 2y - y^2 + D$$

So we've determine the potential function f up to an arbitrary constant. Of course, that constant doesn't matter for our purposes. We'll generally be using the potential function f in the Fundamental Theorem, for which the constant is irrelevant since

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A) = \left(\hat{f}(B) + D\right) - \left(\hat{f}(A) + D\right) = \hat{f}(B) - \hat{f}(A)$$

There is a similar process for checking to see if a three dimensional vector field is conservative,

Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ where P, Q, and R have continuous first partial derivatives. If $\mathbf{F}$ is conservative then there exists a potential function f such that $\mathbf{F} = \nabla f$. Then

$$\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

Proceeding as in the planar case, we have

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial Q}{\partial x}$$

So if $\mathbf{F}$ is conservative it must be the case that $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$. But that's not all...

Playing the same kind of games with the other two variable combinations we also find two more conditions that $\mathbf{F}$ must satisfy if it is conservative. Combining them here with the first condition we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

So, to check to see if a three dimensional vector field is conservative, we check the above conditions.

Example: Determine if the following field is conservative: $\mathbf{F} = 2xy\mathbf{i} + (x^2 - z^2)\,\mathbf{j} - 2yz\mathbf{k}$

We have

$$\begin{aligned} \frac{\partial P}{\partial y} &= 2x & \frac{\partial Q}{\partial x} &= 2x \quad \checkmark \\ \frac{\partial P}{\partial z} &= 0 & \frac{\partial R}{\partial x} &= 0 \quad \checkmark \\ \frac{\partial Q}{\partial z} &= -2z & \frac{\partial R}{\partial y} &= -2z \quad \checkmark \end{aligned}$$

So the field is conservative. Once we know that the field is conservative, we can find the potential function f using an integration method similar to the one done above for planar flows.

Example: Determine the potential function for the field $\mathbf{F} = 2xy\mathbf{i} + (x^2 - z^2)\,\mathbf{j} - 2yz\mathbf{k}$

Equating all components of the equation $\mathbf{F} = \nabla f$ we have

$$\begin{aligned} P &= \frac{\partial f}{\partial x} = 2xy && (7) \\ Q &= \frac{\partial f}{\partial y} = x^2 - z^2 && (8) \\ R &= \frac{\partial f}{\partial z} = -2yz && (9) \end{aligned}$$

Integrating (7) w.r.t. x we have

$$f(x, y) = \int 2xy\, dx = x^2y + g(y, z) \tag{10}$$

The function $g(y, z)$ is there because it vanishes when taking the partial derivative w.r.t. x. Then, since we know that $Q = \partial f/\partial y$ we take the partial of our current expression (10) w.r.t. y and sit it equal to Q. We have

$$x^2 - z^2 = \frac{\partial}{\partial y}\left(x^2y + g(y, z)\right) = x^2 + \frac{\partial g}{\partial y} \tag{11}$$

Solving for $\partial g/\partial y$ in (11) we have

$$\frac{\partial g}{\partial y} = -z^2 \tag{12}$$

Integrating (12) w.r.t. y will give us an expression for g. We have

$$g(y, z) = \int -z^2\, dy = -yz^2 + h(z) \tag{13}$$

where here we need the $h(z)$ function because it would vanish when taking the derivative w.r.t. y. So, updating our expression for $f(x, y)$ with our newfound $g(y, z)$ we have

$$f(x, y) = x^2y - yz^2 + h(z) \tag{14}$$

The only piece of information we haven't used yet is equation (9). Taking the derivative of f w.r.t. z and setting it equal to R we have

$$-2yz \quad = \quad \frac{\partial}{\partial z}\left(x^2y - yz^2 + h(y)\right) = -2yz + h'(z) \quad \Rightarrow \quad h'(z) = 0 \quad \Rightarrow \quad h(z) = C \quad (15)$$

Updating our expression for $f(x, y)$ we then have $f(x, y) = x^2y - yz^2 + C$ which, dropping the irrelevant constant, becomes

$$f(x, y) = x^2y - yz^2$$

13.4/13.5 Curl, Divergence, and Green's Theorem in the Plane

Last time we argued that the gradient operator acted like a kind of derivative. Specifically it's a derivative operator that takes a scalar function and returns a vector:

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \qquad \Rightarrow \qquad \nabla f(x, y, z) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

There are two other derivative-like operators that deal with vectors that are important in physics and engineering. The first one we'll talk about is called the **curl operator**:

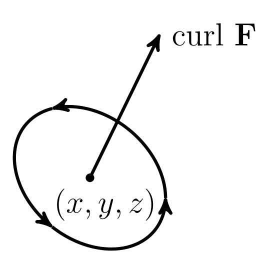
$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

Notice that, unlike the gradient operator, the curl operator takes a vector and returns a vector.

OK, so what does the curl of a vector field $\mathbf{F}$ tell us?

The curl tells us about rotations. It's easiest to picture when we interpret $\mathbf{F}$ as a fluid velocity field. curl $\mathbf{F}\,(x, y, z)$ tells us how the fluid particles are rotating about the point (x, y, z).

Notice that curl F is a vector, which means it has a direction and a magnitude. The direction tells us that the fluid at the point (x, y, z) is tending to rotate about the axis defined by curl F. The length of curl F tells us how fast the fluid is rotating.

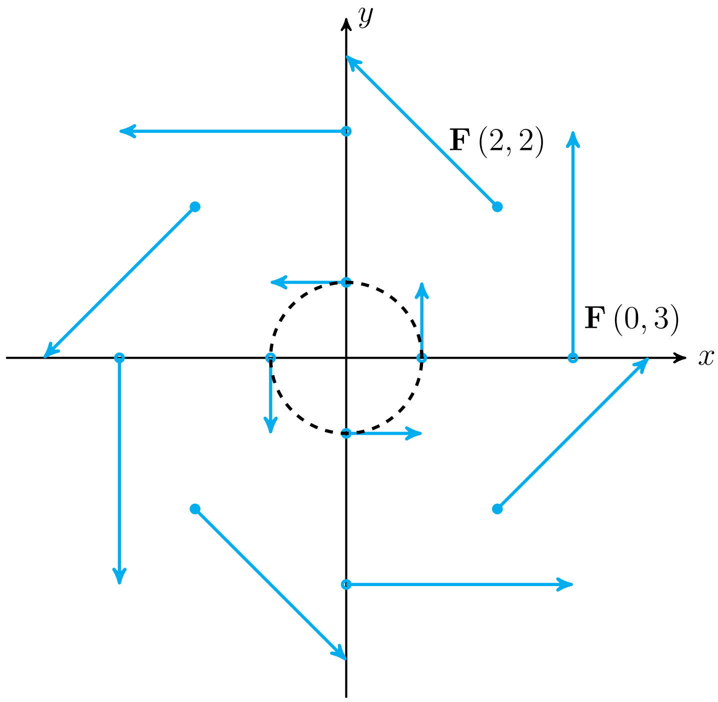


Example: Compute curl $\mathbf{F}$ for $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$

We have

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ -y & x & 0 \end{vmatrix} = (0-0)\,\mathbf{i} + (0-0)\,\mathbf{j} + (1+1)\,\mathbf{k} = 2\mathbf{k}$$

Recall that this vector field looks as follows



There are several things to notice in this example:

1. The field $\mathbf{F}$ is planar, so the $\mathbf{i}$ and $\mathbf{j}$ components of curl $\mathbf{F}$ will always be zero, meaning that any nonzero components of curl $\mathbf{F}$ will be in the $\mathbf{k}$-direction. This means that the main rotations that are happening are in the xy-plane, which makes sense for a planar flow.

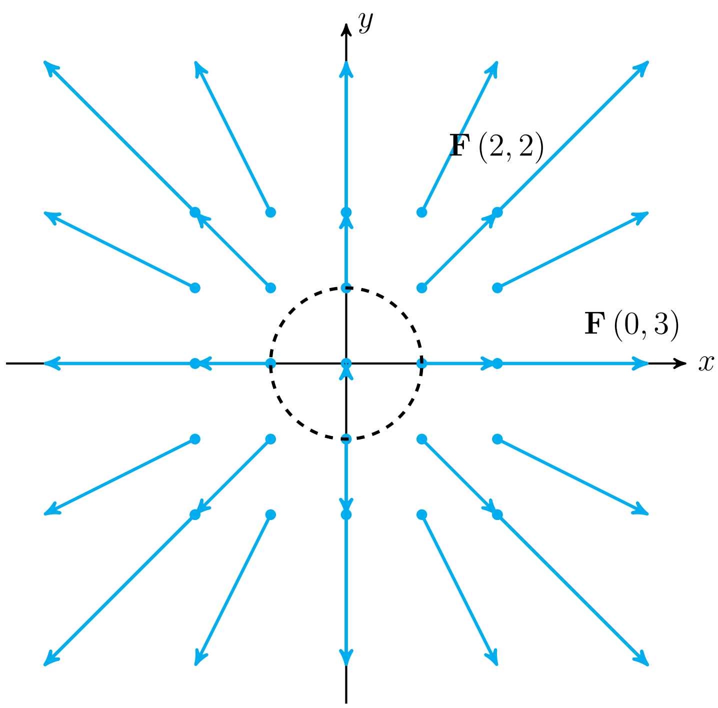
2. Since curl $\mathbf{F}$ is constant, all of the rotation in the fluid has the same magnitude. This means if we were to put a little paddle-wheel in the fluid, no matter where it went it would always rotate counter-clockwise at the same rate.

Example: Compute curl $\mathbf{F}$ for $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$

We have

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ x & y & 0 \end{vmatrix} = (0-0)\,\mathbf{i} + (0-0)\,\mathbf{j} + (1-1)\,\mathbf{k} = \mathbf{0}$$

Recall that this vector field looks as follows



Notice that for this planar flow the curl is zero everywhere. This means if we put a little paddle-wheel in the flow the boat would travel along the vector field arrows away from the origin, but it would not spin at all. When this happens we say that a flow is **irrotational**.

Fact: Let f be some scalar function with continuous second partial derivatives . Then

$$\text{curl } (\nabla f) = \nabla \times \nabla f = \mathbf{0}$$

Proof: We have

$$\text{curl } \nabla f = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ f_x & f_y & f_z \end{vmatrix} = (f_{zy} - f_{yz})\,\mathbf{i} + (f_{xz} - f_{zx})\,\mathbf{j} + (f_{yx} - f_{xy})\,\mathbf{k} = \mathbf{0}$$

This is a useful vector calculus identity, but it also has some cool implications for vector fields. Recall that we said that $\mathbf{F}$ is conservative if it can be written as $\mathbf{F} = \nabla f$. Since a conservative vector field is a gradient of a scalar function, we can check whether $\mathbf{F}$ is conservative by checking to see if it behaves like a gradient when you take it's curl.

Fact: If $\mathbf{F}$ has component functions with continuous partial derivatives, and curl $\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is conservative.

Example: Verify that the field $\mathbf{F} = 2xy\mathbf{i} + (x^2 - z^2)\,\mathbf{j} - 2yz\mathbf{k}$ is conservative.

We do this by taking computing curl $\mathbf{F}$ and see if it's zero.

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ 2xy & (x^2 - z^2) & -2yz \end{vmatrix} = (-2z + 2z)\,\mathbf{i} + (0-0)\,\mathbf{j} + (2x - 2x)\,\mathbf{k} = \mathbf{0} \quad \checkmark$$

so $\mathbf{F}$ is conservative.

OK, so now we have a handle of what the curl of a vector field does. We're now prepared to state a special case of one of the most important theorems in vector calculus.

Green's Theorem in the plane gives us a relationship between circulation and flux through a closed curve and an area integral over the region inclosed by the curve. There are two forms of Green's Theorem. One for circulation and one for flux. Today we'll only tackle the circulation form.

Let $\mathbf{F} = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$ be a planar flow and let C be a counter-clockwise oriented simple closed curve.

Recall that in the fluid setting, the circulation around a closed curve is given by

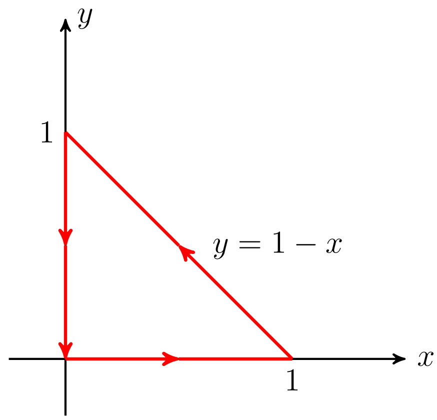
$$\text{Circulation} = \oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P\,dx + Q\,dy$$

Green's Theorem for circulation is as follows

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_D (\text{curl } \mathbf{F}) \cdot \mathbf{k}\,dA$$

This should make intuitive physical sense for fluid flows. It says that the total amount that the fluid is moving around a closed curve is equal to the sum of all the tiny rotations inside the curve. In this context, we can think of curl $\mathbf{F}$ as a **circulation density**.

Example: Use Green's Theorem to compute the circulation of the field $\mathbf{F} = 2y\mathbf{i} + x^2\mathbf{j}$ around the counter-clockwise oriented curve C defined by the lines $x = 0$, $y = 0$ and $y = 1 - x$ in the first quadrant.



Notice that if we wanted to compute the circulation with a line integral, we'd need three different integrals with three different parameterizations for the parts of the curve. But with Green's Theorem, we can turn it into one area integral over the interior of the triangular region.

First we need to compute curl $\mathbf{F}$. We have

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ 2y & x^2 & 0 \end{vmatrix} = (0-0)\,\mathbf{i} + (0-0)\,\mathbf{j} + (2x-2)\,\mathbf{k} = 2\,(x-1)\,\mathbf{k}$$

Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_D 2\,(x-1)\;dA = \int_0^1 \int_0^{1-x} 2\,(x-1)\;dydx = \int_0^1 -2\,(x-1)^2\;dx = -\frac{2}{3}$$

Geometric Extensions

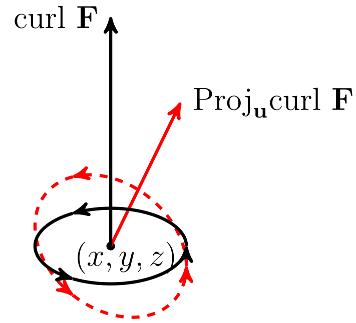
Consider a river whose fluid velocity is described by the planar flow $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$. Now, rivers have depth, so if the flow is truly planar this means that underneath in a give z-cross-section, the fluid is behaving the same as it is on the surface. If the fluid field has some non-trivial curl near a point (x, y, z) then we consider how much a little chunk of fluid is rotating in a plane other than the xy-plane.

Now, think about what would happen if we could see just the fluid flow in a plane that has unit normal vector $\mathbf{u}$ that is not parallel to $\mathbf{k}$. In that little slice, is the fluid rotating?

The answer is yes. But how much is it rotating? Think about the case when $\mathbf{u}$ is makes a 45 degree angle with $\mathbf{k}$. Then we still see some rotation, but it's not as great. Now think about the case when we look at the rotation in a plane with normal vector $\mathbf{i}$ or $\mathbf{j}$. Because the flow is planar, we don't observe any fluid rotation in this plane.

OK, so how do we quantify this? It turns out that the rotation about $\mathbf{u}$ will be described by the projection of curl $\mathbf{F}$ onto $\mathbf{u}$. If we just want to know how fast the fluid is rotating in that plane, we look for the component of curl $\mathbf{F}$ in the direction of $\mathbf{u}$, which we compute using a dot product.

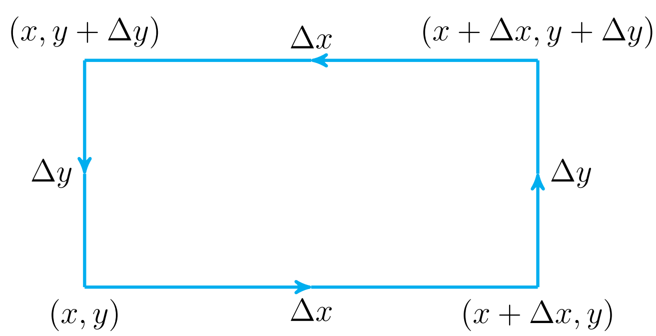
$$\text{Component of Rotation in Direction of Unit Vector } \mathbf{u} = \text{curl } \mathbf{F} \cdot \mathbf{u}$$



Recall that the gradient vector ∇f gave a vector that pointed in the direction of maximum increase of f. Under the above interpretation of the curl, we see that curl $\mathbf{F}$ is also a vector that tells us a maximum about a vector field. It tell us the plane in which the maximum rotation of the fluid occurs.

OK, let's take a second and see if we can figure out exactly where curl $\mathbf{F}$ comes from. For simplicity, we'll just consider a planar flow and attempt to derive the $\mathbf{k}$ component of the curl.

Let $\mathbf{F} = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$ and consider the counter-clockwise oriented curve traversing the boundary of the following box



Let's setup the circulation around the box. Since the circulation is just the sum of the flow around each side we have, we can approximate the flow around each side as follows

$$\begin{aligned}
\text{Top:} \quad & \mathbf{F}\,(x, y + \Delta y) \cdot (-\mathbf{i})\,\Delta x = -P\,(x, y + \Delta y)\,\Delta x \\
\text{Bottom:} \quad & \mathbf{F}\,(x, y) \cdot \mathbf{i}\Delta x = P\,(x, y)\,\Delta x \\
\text{Right:} \quad & \mathbf{F}\,(x + \Delta x, y) \cdot \mathbf{j}\Delta y = Q\,(x + \Delta x, y)\,\Delta y \\
\text{Left:} \quad & \mathbf{F}\,(x, y) \cdot (-\mathbf{j})\,\Delta y = -Q\,(x, y)\,\Delta y
\end{aligned}$$

Adding the contributations to the circulation from the top and the bottom we have

$$\text{Top + Bottom :} \qquad -\left(P(x,y+\Delta y)-P(x,y)\right)\Delta x \approx -\frac{\partial P}{\partial y}\Delta y \Delta x$$

Adding the contributations to the circulation from the left and the right we have

$$\text{Left + Right :} \qquad \left(Q(x+\Delta x,y)-Q(x,y)\right)\Delta y \approx \frac{\partial Q}{\partial x}\Delta x \Delta y$$

Adding the circulations together and dividing by the area of the box will give us a kind of **circulation density**:

$$\frac{\text{circulation around rectangle}}{\text{area of rectangle}} \approx \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = (\text{curl } \mathbf{F}) \cdot \mathbf{k}$$

Now, what happens if we do the same thing with **Flux**?

Writing the outward flux through each part of the curve we have

$$\begin{aligned}
\text{Top:} \quad & \mathbf{F}\left(x,y+\Delta y\right)\cdot \mathbf{j}\Delta x = Q\left(x,y+\Delta y\right)\Delta x \\
\text{Bottom:} \quad & \mathbf{F}\left(x,y\right)\cdot\left(-\mathbf{j}\right)\Delta x = -Q\left(x,y\right)\Delta x \\
\text{Right:} \quad & \mathbf{F}\left(x+\Delta x,y\right)\cdot \mathbf{i}\Delta y = P\left(x+\Delta x,y\right)\Delta y \\
\text{Left:} \quad & \mathbf{F}\left(x,y\right)\cdot\left(-\mathbf{i}\right)\Delta y = -P\left(x,y\right)\Delta y
\end{aligned}$$

Again, adding the contributions from the top and bottom gives

$$\text{Top + Bottom :} \qquad \left(Q(x,y+\Delta y)-Q(x,y)\right)\Delta x \approx \frac{\partial Q}{\partial y}\Delta y \Delta x$$

then adding the contributions from the left and right

$$\text{Left + Right :} \qquad \left(P(x+\Delta x,y)-P(x,y)\right)\Delta y \approx \frac{\partial P}{\partial x}\Delta x \Delta y$$

Adding the four contributions together and dividing by the area to get a density we have

$$\frac{\text{flux through rectangle}}{\text{area of rectangle}} \approx \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = \text{div } \mathbf{F}$$

The operator that gives us the **flux density** is the last of the vector related differential operators, called the **divergence operator**.

$$\text{div } \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial P}{\partial y} + \frac{\partial P}{\partial z} = \nabla \cdot \mathbf{F}$$

Example: Consider again the vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$. Compute div $\mathbf{F}$.

We have

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} = 1 + 1 = 2$$

This means that at any point (x, y) there is a net flux **out** of the region. If you have a vector field $\mathbf{F}$ such that div $\mathbf{F} < 0$ then that means there is a net inward flux into a point. Let's think about the physical scenarios where you could have a positive or negative flux into a point (or at least, a very small area around a point). One way that you could get an outward flux at a point is if there is a source at that point. Think about putting a garden hose into a stream and turning on the water. Fluid is rushing out the point at the end of the garden hose because new fluid is entering the system there. Alternatively, one way that you could have a negative flux at a point would be if the point is a sink. Think about the flow of water in a bathtube. Fluid is being removed from the system through the drain, so there is a net inward flux at that point.

Now consider the case where the flow has no sources or sinks. Can there still be a net flux into or out of a point (and by extension, a positive or negative divergence at that point)? The answer is of course yes. If div $\mathbf{F} > 0$ at a point this means that there is a net flux of fluid out of that point, which means that the fluid is expanding. Similarly, if div $\mathbf{F} < 0$ at a point then there is a net flux of fluid into the point. If the fluid is not disappearing (i.e. the point is not a sink) then the fluid must be compressing.

A fluid described by a field with no sources or sinks such that div $\mathbf{F} = 0$ is one where there is no compression or expansion anywhere. In this case we call the fluid **incompressible**.

Example: Consider again the vector field $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$. Compute div $\mathbf{F}$.

We have

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = -\frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(x) = 0$$

so the fluid is incompressible.

OK, we're now ready to formulate the flux version of Green's Theorem in the plane.

Let $\mathbf{F} = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$ be a planar flow and let C be a counter-clockwise oriented simple closed curve.

Recall that in the fluid setting, the flux through a closed curve is given by

$$\text{Flux} = \oint_C \mathbf{F}\cdot\mathbf{n}ds = \oint_C P\,dy - Q\,dx$$

Green's Theorem for Flux is as follows

$$\oint_C \mathbf{F}\cdot\mathbf{n}ds = \oint_C P\,dy - Q\,dx = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)\,dA = \iint_D \text{div }\mathbf{F}\,dA$$

This should make intuitive physical sense for fluid flows. It says that the net flux through a closed curve is equal to the sum of all the tiny fluxes inside the curve. In this context, we can think of div $\mathbf{F}$ as a **flux density**.

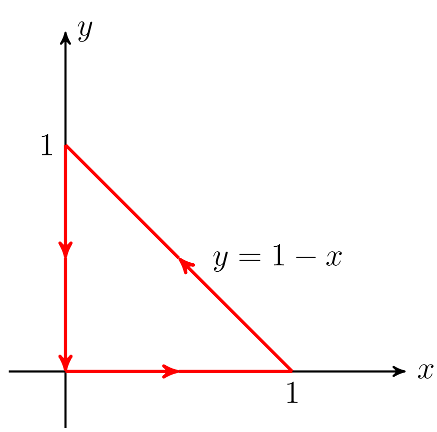
Example: Consider the fluid velocity field $\mathbf{F} = -x\mathbf{i} - y\mathbf{j}$ and let C be the counter-clockwise oriented circle of radius R centered at the origin. Find the outward flux through the curve.

Using Green's Theorem we have

$$\oint_C \mathbf{F}\cdot\mathbf{n}ds = \iint_D \text{div }\mathbf{F}dA = \iint_D (-1-1)\,dA = -2\int_0^{2\pi}\int_0^R r\,dr\,d\theta = -2\pi R^2$$

Notice that the flux is negative because fluid is rushing into the circle.

Example: Use Green's Theorem to compute the flux of the field $\mathbf{F} = 2y\mathbf{i} + x^2\mathbf{j}$ through the counter-clockwise oriented curve C defined by the lines $x = 0$, $y = 0$ and $y = 1 - x$ in the first quadrant.



Notice again that if we wanted to compute the flux with a line integral, we'd need three different integrals with three different parameterizations for the parts of the curve. But with Green's Theorem, we can turn it into one area integral over the interior of the triangular region.

But in this case it's even easier. Notice that

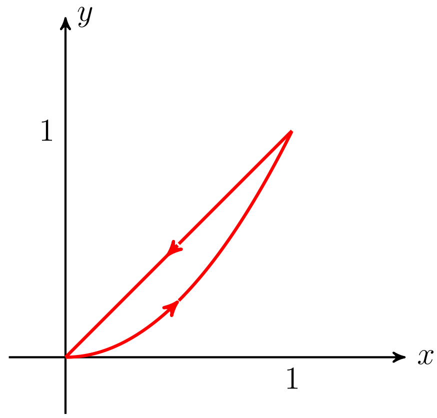
$$\text{div } \mathbf{F} = 0$$

indicating that the fluid is incompressible and there are no sources or sinks. Then we have

$$\oint_C \mathbf{F} \cdot \mathbf{n} ds = \iint_D \text{div } \mathbf{F} \, dA = \iint_D 0 \, dA = 0$$

Example: Use Green's Theorem to compute the counterclockwise circulation and outward flux of the field $\mathbf{F} = \langle xy, y^2 \rangle$ around and through the boundary of the region enclosed by the curves $y = x^2$ and $y = x$ in the first quadrant.

The curve and enclosed region looks as follows:



We need to compute the curl and the divergence of $\mathbf{F}$. We have

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ xy & y^2 & 0 \end{vmatrix} = (0 - x)\,\mathbf{k} = -x\mathbf{k} \quad \text{and} \quad \nabla \cdot \mathbf{F} = y + 2y = 3y$$

We then have

$$\text{Circ} = \iint_D -x \, dA = -\int_0^1 \int_{x^2}^x x \, dy \, dx = -\int_0^1 x^2 - x^3 \, dx = \frac{1}{4} - \frac{1}{3} = -\frac{1}{12}$$

$$\text{Flux} = \iint_D 3y \, dA = \int_0^1 \int_{x^2}^x 3y \, dy \, dx = \frac{3}{2} \int_0^1 x^2 - x^4 \, dx = \frac{1}{2} - \frac{3}{10} = \frac{1}{5}$$

Green Theorem's for Areas

We can also use Green's Theorem to compute the area of regions using line integrals. Note that this is a **trick**! There is **NO PHYSICS HERE**!

Recall that the two forms of Green's Theorem are

$$\oint_C P dy - Q dx = \iint_R \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA \quad \text{and} \quad \oint_C P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

and recall that if the integrand in the double integral is identically one then we get the area of the region R. So, our goal is to **MAKE UP** a vector field $\mathbf{F}$ such that its divergence is 1 or the k-component of its curl is 1, and then compute the line integral over that field instead. Notice that

$$\text{Area}(R) = \iint_R 1\, dA = \iint_R \left(\frac{1}{2} + \frac{1}{2} \right) dA$$

Then we could choose $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ such that

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y} = \frac{1}{2} \quad \text{for example} \quad P = \frac{x}{2} \quad \text{and} \quad Q = \frac{y}{2}$$

and then using the Flux form of Green's Theorem, we have

$$\iint_R \left(\frac{1}{2} + \frac{1}{2} \right) dA = \oint_C \frac{x}{2} dy - \frac{y}{2} dx \quad \Rightarrow \quad \text{Area}(R) = \frac{1}{2} \oint_C x dy - y dx$$

Similarly we could use the Circulation form of Green's Theorem with $Q = \dfrac{x}{2}$ and $P = -\dfrac{y}{2}$ and have

$$\iint_R \left(\frac{1}{2} + \frac{1}{2} \right) dA = \oint_C -\frac{y}{2} dx + \frac{x}{2} dy \quad \Rightarrow \quad \text{Area}(R) = \frac{1}{2} \oint_C x dy - y dx$$

We can come up with simpler formulae as well. Consider using the Flux form of Green's theorem with the (made up) vector field $\mathbf{F} = \langle x, 0 \rangle$. Then $P = x$ and we have

$$\iint_R (1 + 0)\ dA = \oint_C x\, dy \quad \Rightarrow \quad \text{Area}(R) = \oint_C x\, dy$$

Or we could use the Flux form of Green's theorem with the (made up) vector field $\mathbf{F} = \langle 0, y \rangle$. Then $Q = y$ and we have

$$\iint_R (0+1)\ dA = -\oint_C y\,dx \quad \Rightarrow \quad \text{Area}(R) = -\oint_C y\,dx$$

Any of these three formulae are equally valid, and can be used to compute the area of a the region enclosed by the curve C

$$\text{Area}(R) = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C xdy - ydx$$

Example: Use a line integral to find the area of a circle with radius a

We parameterize the curve as follows: $\mathbf{r}(t) = a\cos t\mathbf{i} + a\sin t\mathbf{j}$ for $0 \le t \le 2\pi$. Then

$$x = a\cos t \ \Rightarrow \ dx = -a\sin t\,dt \quad \text{and} \quad y = a\sin t \ \Rightarrow \ dy = a\cos t\,dt$$

Then

$$A = \frac{1}{2}\int_0^{2\pi} a^2\cos^2 t + a^2\sin^2 t\,dt = \frac{1}{2}\int_0^{2\pi} a^2\,dt = \frac{1}{2}a^2 2\pi = \pi a^2$$

Example: Use a line integral to find the area of an ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

We have $x(t) = a\cos t$ and $y = b\sin t$ for $0 \le t \le 2\pi$.

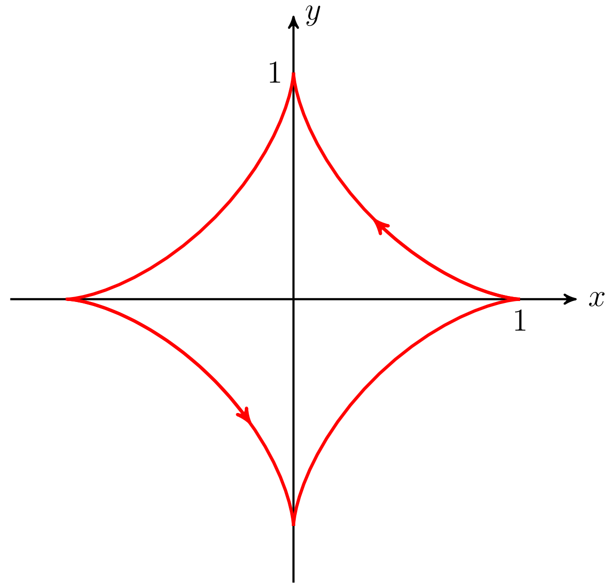
Using the formula $A = \displaystyle\oint_C x\,dy$ we have

$$x = a\cos t \quad \text{and} \quad dy = b\cos tdt$$

$$A = \int_0^{2\pi} ab\cos^2 t\,dt = \frac{ab}{2}\int_0^{2\pi} 1 + \cos(2t)\ dt = \frac{ab}{2}2\pi = ab\pi$$

Example: Use a line integral to find the area of an astroid parameterized by $\mathbf{r}(t) = \langle \cos^3 t, \sin^3 t \rangle$ for $0 \leq t \leq 2\pi$.

An astroid (not to be confused with an asteroid) looks like this



We'll use the area formula $A = -\oint_C y\,dx$ for this one. We have

$$x(t) = \cos^3 t \quad \Rightarrow \quad dx = -3\sin t \cos^2 t\,dt \quad \text{and} \quad y(t) = \sin^3 t$$

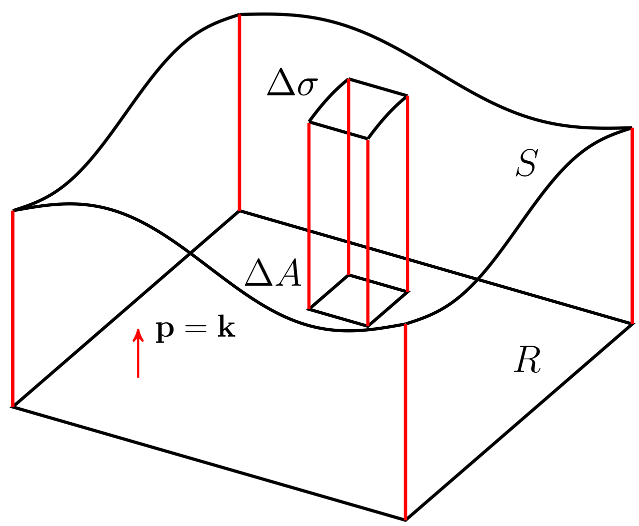
Then

$$A = 3\int_0^{2\pi} \sin^4 t \cos^2 t\,dt = \texttt{mathemagic} = \frac{3\pi}{8}$$

Note: You can also do similar tricks to turn double integrals for centers of mass and moments of inertia into line integrals. You'll explore this in your homework.

13.7 Surface Integration

Suppose we want to find the surface area of a surface S described by the function $g(x, y, z) = c$ where c is a constant. Or, similarly, suppose we know that the density of a thin metal shell described by surface S and we want to integrate over S to find the mass. Both of these things require us to be able to integrate over surfaces. Consider the following picture.



We do surface integration in a similar fashion to every other form of integration we've done so far. We come up with a formula for the surface area of the little surface area element $\Delta\sigma$, evaluate the function on the little surface area chunk, add up the contribution from all the little chunks, and then take the limit as the number of chunks goes to infinity. In the end we have something that looks like

$$\iint_S f(x, y, z)\, d\sigma$$

If we only want to compute the area of the surface then we perform the integral with $f(x, y, z) = 1$, or

$$SA = \iint_S d\sigma$$

We compute surface integrals similar to the way we compute line integrals. With line integrals we had to write the little arc length element ds in terms of a 1D element dt, i.e.

$$ds = |v| dt$$

where $|v|$ was the Jacobian of the transformation from the curved space to a line. Then we computed the line integral according to

$$\int_C f(x, y, z)\, ds = \int_a^b f(x(t)\,, y(t)\,, z(t))\, |v| dt$$

For surface integrals we will write the surface area element $d\sigma$ in S in terms of the area element dA in R obtained by projecting $d\sigma$ onto one of the coordinate planes. The transformation looks like

$$d\sigma = \frac{|\nabla g|}{|\nabla g \cdot \mathbf{p}|} dA$$

where g is the function defining the surface and $\mathbf{p}$ is the vector in the direction in which we project S onto the chosen coordinate plane. For example, in the picture above, imagine that the plane of projection is the xy-plane, then we project S along the vector $\mathbf{p} = \mathbf{k}$.

In general, we compute the surface integral according to

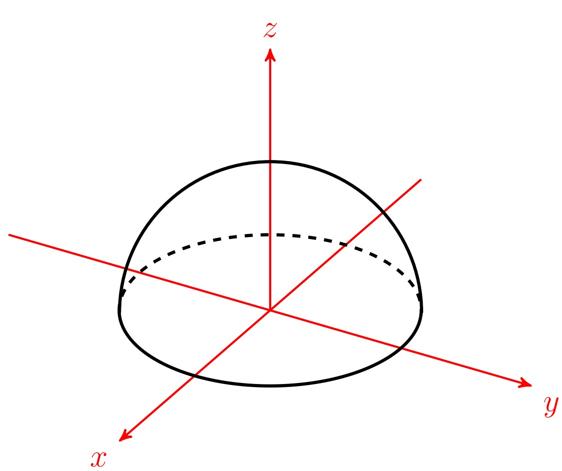
$$\iint_S f(x,y,z)\, d\sigma = \iint_R f(x,y,z)\, \frac{|\nabla g|}{|\nabla g \cdot \mathbf{p}|} dA$$

The steps needed to compute the surface integral are as follows

1. Choose coordinate plane to project surface onto and define unit vector $\mathbf{p}$
2. Determine region of projection R
3. Compute $|\nabla g|$ and $|\nabla g \cdot \mathbf{p}|$ and construct integrand (Jacobian + Function)
4. Eliminate extra variables using the surface equation $g(x,y,z) = c$
5. Evaluate regular area integral over R

Example 1: Compute the surface area of the hemisphere of radius 1 above the xy-plane.

The equation of a sphere of radius 1 is given by $g(x,y,z) = x^2 + y^2 + z^2 = 1$. If we draw the hemisphere we have



We choose to project the surface into the xy-plane. We make this choice because projecting into either of the other coordinate planes would cause the surface to fold over itself. This is not, strictly speaking, impossible, but it would require computing two surface integrals. For example, if we projected the surface into the yz-plane we would need to do the part of the surface defined for $x > 0$ separately from the part of the surface defined for $x < 0$.

If we project onto the xy-plane then we have $\mathbf{p} = \mathbf{k}$.

The projected region R is the circle described by $x^2 + y^2 = 1$.

To find the integrand, we need to compute $|\nabla g|$ and $|\nabla g \cdot \mathbf{p}|$

$$\nabla g = \langle 2x, 2y, 2z \rangle \quad \Rightarrow \quad |\nabla g| = \sqrt{4x^2 + 4y^2 + 4z^2} \quad \text{and} \quad |\nabla g \cdot \mathbf{p}| = |\nabla g \cdot \mathbf{k}| = |2z|$$

Note that since the surface lies above the xy-plane we do not need the absolute values around the $2z$ term. Since we're finding the surface area of the hemisphere, the only thing in the integrand is the Jacobian

$$\frac{|\nabla g|}{|\nabla g \cdot \mathbf{p}|} = \frac{\sqrt{4x^2 + 4y^2 + 4z^2}}{2z} = \frac{\sqrt{x^2 + y^2 + z^2}}{z}$$

Since we've projected the surface into the xy-plane we can't have z in the integrand. To eliminate the z variables we use the equation of the surface: $x^2 + y^2 + z^2 = 1$

$$\frac{\sqrt{x^2 + y^2 + z^2}}{z} = \frac{1}{\sqrt{1 - x^2 - y^2}}$$

We then have

$$\iint_S d\sigma = \iint_R \frac{1}{\sqrt{1 - x^2 - y^2}} dA$$

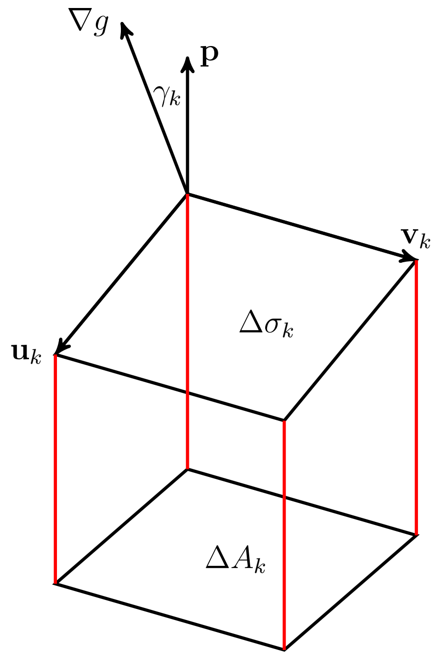
Since the region R is the unit circle in the xy-plane we convert to polar coordinates. The integral then becomes

$$\iint_S d\sigma = \iint_R \frac{1}{\sqrt{1 - x^2 - y^2}} dA = \int_0^{2\pi} \int_0^1 \frac{1}{\sqrt{1 - r^2}} r \, dr \, d\theta = 2\pi$$

This agrees with our geometric intuition because the surface area of a unit sphere is 4π and here we are just finding the area of the top half.

Deriving the Jacobian of the Transformation

So far we've stated the transformation between $d\sigma$ and dA as fact. Let's see where it comes from. First we make the assumption that we can approximate the surface area element $\Delta\sigma$ by it's tangent plane. This is reasonable since in the end we're going to take the limit as the size of $\Delta\sigma \to 0$. In this limit the surface area element appears very flat and the approximation is valid. Consider the following picture



We now try to find a relationship between the area elements $\Delta\sigma_k$ and ΔA_k. Since $\Delta\sigma_k$ is a rectangle, we can find it's area by taking the magnitude of the cross-product of vectors $\mathbf{u}_k$ and $\mathbf{v}_k$

$$\Delta\sigma_k = |\mathbf{u}_k \times \mathbf{v}_k|$$

Note that this area is not the same as ΔA_k unless ΔA_k and $\Delta\sigma_k$ are parallel. To get the area of the projection we use

$$\Delta A_k = |(\mathbf{u}_k \times \mathbf{v}_k) \cdot \mathbf{p}|$$

This should be believable because if $(\mathbf{u}_k \times \mathbf{v}_k)$ (which is parallel to ∇g) is parallel to $\mathbf{p}$ then the quantity ΔA_k as defined above will be large. If $(\mathbf{u}_k \times \mathbf{v}_k)$ is not parallel to $\mathbf{p}$ then ΔA_k gets smaller. Then, by the definition of the dot product

$$\begin{aligned}
\Delta A_k &= |(\mathbf{u}_k \times \mathbf{v}_k) \cdot \mathbf{p}| \\
&= |(\mathbf{u}_k \times \mathbf{v}_k)|\,|\mathbf{p}|\,|\cos(\gamma_k)| \\
&= \Delta\sigma_k\,|\cos(\gamma_k)|
\end{aligned}$$

Solving this expression for $\Delta\sigma_k$ we have

$$\Delta\sigma_k = \frac{\Delta A_k}{|\cos(\gamma_k)|}$$

Taking the limit as the size of the surface area elements goes to 0 we have

$$d\sigma = \frac{dA}{|\cos(\gamma)|}$$

Now we need to compute $|\cos(\gamma)|$ in terms of quantities that we know. We have

$$\begin{aligned} |\nabla g \cdot p| &= |\nabla g|\,|p|\,|\cos(\gamma)| \\ &= |\nabla g|\,|\cos(\gamma)| \end{aligned}$$

Rearranging this expression we have

$$\frac{1}{|\cos(\gamma)|} = \frac{|\nabla g|}{|\nabla g \cdot p|}$$

and finally

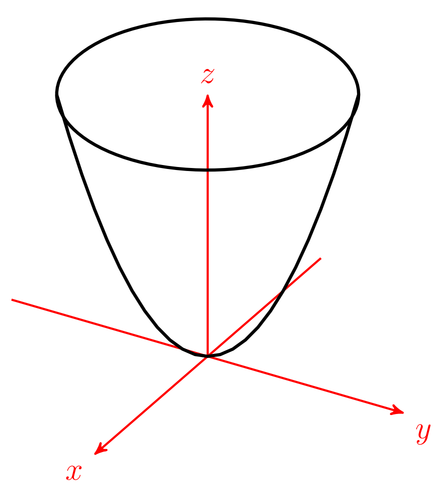
$$d\sigma = \frac{|\nabla g|}{|\nabla g \cdot p|} dA$$

Example 2: Find the area of the paraboloid $z = x^2 + y^2$ cut by the plane $z = 4$.

To determine the function $g(x, y, z) = c$ that describes the surface, we need to put all the variables on one side of the equals sign and all the constants on the other. Moving the z to the other side of the equation of the paraboloid we have

$$g(x, y, z) = x^2 + y^2 - z$$

To help choose the direction of projection it is usually a good idea to plot the surface.



We again want to project the surface into the xy-plane, so we choose $\mathbf{p} = \mathbf{k}$.

Since the paraboloid is cut by the plane $z = 4$, the projected region R is given by

$$z = x^2 + y^2 \quad \Rightarrow \quad 4 = x^2 + y^2$$

which is a circle of radius 2.

To find the integrand, we need to compute $|\nabla g|$ and $|\nabla g \cdot \mathbf{p}|$

$$\nabla g = \langle 2x, 2y, -1 \rangle \quad \Rightarrow \quad |\nabla g| = \sqrt{4x^2 + 4y^2 + 1} \quad \text{and} \quad |\nabla g \cdot \mathbf{p}| = |\nabla g \cdot k| = |-1| = 1$$

Then we have

$$\frac{|\nabla g|}{|\nabla g \cdot \mathbf{p}|} = \frac{\sqrt{4x^2 + 4y^2 + 1}}{1} = \sqrt{4x^2 + 4y^2 + 1}$$

Note that this time z does not appear in the integrand, so we don't need to eliminate any variables. We then have

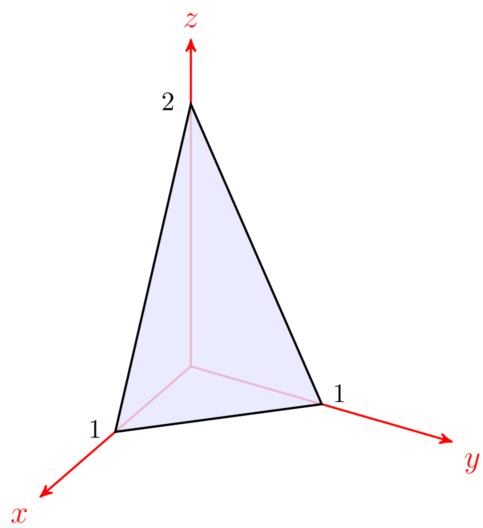
$$\iint_S d\sigma = \iint_R \sqrt{4x^2 + 4y^2 + 1}\, dA$$

Since R is a circle of radius 2 in the xy-plane we again want to use polar coordinates. So we have

$$\iint_S d\sigma = \iint_R \sqrt{4x^2 + 4y^2 + 1}\, dA = \int_0^{2\pi} \int_0^2 \sqrt{4r^2 + 1}\, r\, dr\, d\theta = \frac{\pi}{6}\left(17\sqrt{17} - 1\right)$$

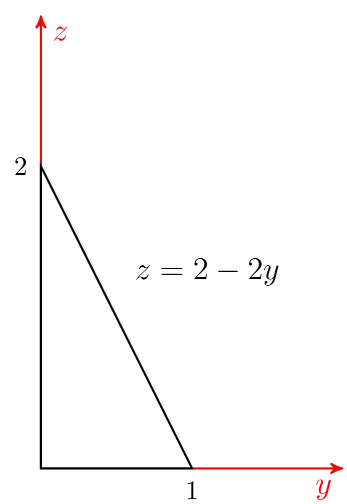
Example 3: Integrate the function $f(x, y, z) = x + y + z$ over the portion of the plane $2x + 2y + z = 2$ in the first octant.

The surface in the first octant looks like the following



The function describing the surface is $g(x, y, z) = 2x + 2y + z$

We have some freedom in choosing which coordinate plane to project the surface S onto. Just to mix things up, let's project S into the yz-plane. Then the projection vector is $\mathbf{p} = \mathbf{i}$. Then projection R in the yz-plane looks like



Computing the Jacobian we have

$$\nabla g = \langle 2, 2, 1 \rangle \quad \Rightarrow \quad |\nabla g| = 3 \quad \text{and} \quad |\nabla g \cdot \mathbf{p}| = |\nabla g \cdot \mathbf{i}| = 2$$

Substituting into the surface integral we have

$$\iint_S (x+y+z)\ d\sigma = \iint_R (x+y+z)\ \frac{3}{2}\,dA$$

Since we've projected S into the yz-plane we must eliminate x from the integrand. Solving the surface equation for x we have

$$2x+2y+z=2 \quad \Rightarrow \quad x = 1-y-\frac{z}{2}$$

Substituting this into the integrand and setting up the limits of integration for R we have

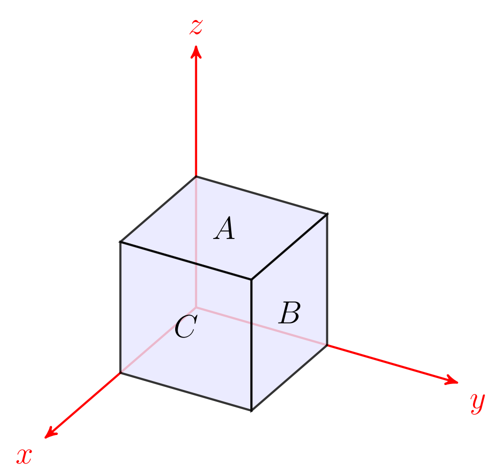
$$\iint_R (x+y+z)\ \frac{3}{2}\,dA = \frac{3}{2}\int_0^1 \int_0^{2-2y} \left(1+\frac{z}{2}\right)\ dz\,dy = 2$$

Sometimes we want to integrate over a surface S that is the union of multiple surfaces, say $S = S_1 \cup S_2 \cup S_3$. To integrate over S we break the integral up into multiple integrals over each of the constituent surfaces:

$$\iint_S f\ d\sigma = \iint_{S_1} f\ d\sigma + \iint_{S_2} f\ d\sigma + \iint_{S_3} f\ d\sigma$$

Example 4: Integrate the function $f(x,y,z) = xyz$ over the surface of the unit cube in the first octant.

The unit cube in the first octant looks like



The surface S of the cube is the union of its six faces. In theory we should compute six separate integrals, one for each face. However, we see that the function we're integrating, $f(x,y,z) = xyz$ is zero on the three sides of the cube lying in the coordinate planes, so we can skip them. We need to integrate over the top, side, and front facing sides which I've labeled A, B, and C, respectively. We handle each one separately.

Side A: The top surface of the cube lies in the plane $g(x, y, z) = z = 1$. We must project the surface into the xy-plane since projecting into either of the other coordinate planes with result in a region R that is just a straight line. Note that the projection in the y-plane is the unit square in the first quadrant.

Computing the Jacobian of the transformation, we have

$$\nabla g = \langle 0, 0, 1\rangle \quad \Rightarrow \quad |\nabla g| = 1 \quad \text{and} \quad |\nabla g \cdot \mathbf{p}| = |\nabla g \cdot \mathbf{k}| = 1$$

which gives $d\sigma = dA$. This should not be surprising since the surface is a plane parallel to the xy-plane. We then have

$$\iint_{S_A} (xyz)\ d\sigma = \iint_R (xyz)\ dA$$

We use the surface equation $z = 1$ to eliminate z from the integrand. Then

$$\iint_{S_A} (xyz)\ d\sigma = \iint_R (xyz)\ dA = \int_0^1 \int_0^1 xy\, dxdy = \frac{1}{4}$$

The other two surfaces of interest are similar. We have

Side B: The side surface of the cube lies in the plane $g(x, y, z) = y = 1$. Our only option is to project into the xz-plane choosing $\mathbf{p} = \mathbf{j}$. We then have

$$\nabla g = \langle 0, 1, 0\rangle \quad \Rightarrow \quad |\nabla g| = 1 \quad \text{and} \quad |\nabla g \cdot \mathbf{p}| = |\nabla g \cdot \mathbf{j}| = 1 \quad \Rightarrow \quad d\sigma = dA$$

Then

$$\iint_{S_B} (xyz)\ d\sigma = \iint_R (xyz)\ dA = \int_0^1 \int_0^1 xz\, dxdy = \frac{1}{4}$$

and for the front side we have

Side C: The side surface of the cube lies in the plane $g(x, y, z) = x = 1$. Projecting the surface into the yz-plane and choosing $\mathbf{p} = \mathbf{i}$ we have

$$\nabla g = \langle 1, 0, 0\rangle \quad \Rightarrow \quad |\nabla g| = 1 \quad \text{and} \quad |\nabla g \cdot \mathbf{p}| = |\nabla g \cdot \mathbf{i}| = 1 \quad \Rightarrow \quad d\sigma = dA$$

Then

$$\iint_{S_C} (xyz)\ d\sigma = \iint_R (xyz)\ dA = \int_0^1 \int_0^1 yz\, dxdy = \frac{1}{4}$$

Finally, we add the contributions from each nonzero face to obtain

$$\iint_S (xyz)\ d\sigma = \iint_{S_A} (xyz)\ d\sigma + \iint_{S_B} (xyz)\ d\sigma + \iint_{S_C} (xyz)\ d\sigma = \frac{3}{4}$$

Flux Through a Surface

We want to compute the flux of a vector field through some surface S. If you think of $\mathbf{F}$ as a fluid velocity field, and the surface S as a thin permeable membrane, then the flux through the surface S is the rate at which fluid is flowing through the membrane.

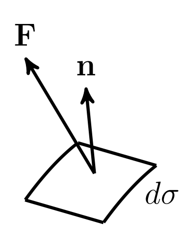
But before we can compute flux, we need to talk about what kinds of surfaces we can work on. The first requirement is that the surface be smooth, or at least a union of smooth surfaces. For example, the cube in the previous example is not smooth, but it is a union of its six faces which are themselves smooth. The second requirement on the surface is that it is *orientable*.

Definition: A smooth surface S is **orientable** or **2-sided** if it is possible to define an outward pointing unit normal vector $\mathbf{n}$ that varies continuously with position.

In other words, an orientable surface is one where if you move an outward pointing normal vector all the way around the surface it will be pointing in the same direction when it returns to where it started. Most surfaces that we can picture in real life are orientable, e.g. spheres, planes, etc. An example of a surface that is not orientable is the **Mobius Strip**.

We're now ready to talk about flux through a surface. We proceed in a similar fashion to the definition of flux through a curve for a planar flow. Suppose the fluid velocity field can be written as $\mathbf{F} = P(x,y,z)\,\mathbf{i} + Q(x,y,z)\,\mathbf{j} + R(x,y,z)\,\mathbf{k}$. We divide the surface into infinitesimally small surface area chunks $d\sigma$ and compute the flux through each one. We then add up the flux through the little chunks (via a surface integral) to obtain the total flux through the surface.

Consider the following surface area element shown with a **unit** outward pointing normal vector $\mathbf{n}$ and the fluid velocity field $\mathbf{F}$ evaluated at some point on $d\sigma$.



The fluid flowing through the surface area element does so in a direction normal to the surface. To compute this flux we take the dot product of $\mathbf{F}$ with the unit normal vector $\mathbf{n}$ and multiply by the area of the surface area element. In other words

$$\text{flux through } d\sigma = (\mathbf{F} \cdot \mathbf{n})\, d\sigma$$

To compute the total flux through the surface we add up all the small surface area elements to obtain

$$\text{Flux} = \iint_S (\mathbf{F} \cdot \mathbf{n}) \, d\sigma.$$

Note that the quantity $(\mathbf{F} \cdot \mathbf{n})$ is a scalar function and so this looks just like the surface integrals we described in the previous section.

Before we proceed, we should say what we mean by **outward** pointing unit normal vector $\mathbf{n}$. Usually, if the surface is curved in some way, we choose the normal vector to point towards the outside of the curve. For example, given the two possible normal vectors on the surface of a sphere, we choose the one that points **away** from the sphere's center.

We compute the unit normal vector to the surface as

$$\mathbf{n} = \pm \frac{\nabla g}{|\nabla g|}$$

where the sign is chosen so that the vector points outward from the surface. Then we have

$$\mathbf{F} \cdot \mathbf{n} = \mathbf{F} \cdot \pm \frac{\nabla g}{|\nabla g|}$$

Recalling the definition of the transformation between $d\sigma$ and dA, we have

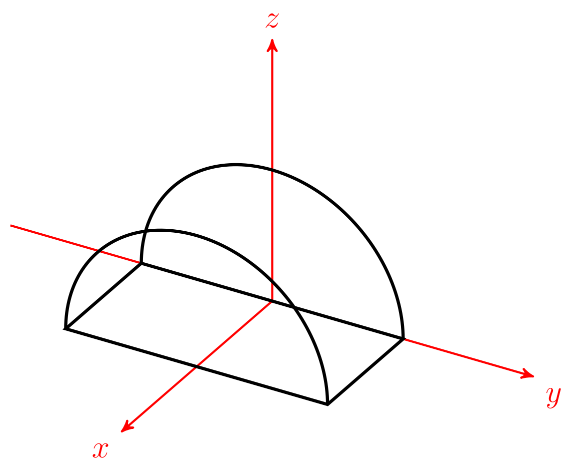
$$d\sigma = \frac{|\nabla g|}{|\nabla g \cdot \mathbf{p}|} dA$$

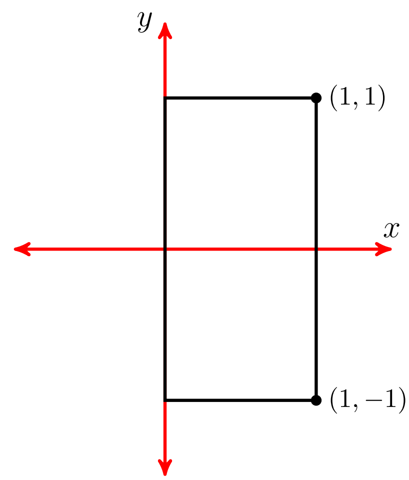
Combining these in the surface integral we have

$$\text{Flux} = \iint_S (\mathbf{F} \cdot \mathbf{n}) \, d\sigma = \iint_R \mathbf{F} \cdot \pm \frac{\nabla g}{|\nabla g|} \frac{|\nabla g|}{|\nabla g \cdot \mathbf{p}|} dA = \iint_R \mathbf{F} \cdot \frac{\pm \nabla g}{|\nabla g \cdot \mathbf{p}|} dA$$

Example 5: Find the flux of the fluid velocity field $\mathbf{F} = yz\mathbf{j} + z^2\mathbf{k}$ through the surface $y^2 + z^2 = 1$ cut by the planes $z = 0$, $x = 0$, and $x = 1$.

Let's draw the surface along with its projection into the xy-plane.





Let's compute the integrand of the Flux integral. First we need to choose the sign on the normal vector so that it's pointing outward from the surface. We eliminated $\mathbf{n}$ from the Flux integral, but ∇g points in the same direction, so we need to choose the sign on ∇g so that it points outward from the surface. We have

$$\nabla g = \langle 0, 2y, 2z \rangle$$

To make it easier to see the direction, we plug some point on the surface into ∇g. Let's pick the point $(0, 0, 1)$. Then

$$\nabla g(0, 0, 1) = \langle 0, 0, 2 \rangle$$

which points straight up and outward from the surface. So in the expression for the Flux integral we choose $+\nabla g$. We also have

$$|\nabla g \cdot \mathbf{p}| = |\nabla g \cdot \mathbf{k}| = |2z| = 2z$$

Note that we don't need the absolute value sign because z is greater than zero everywhere on the surface. Then

$$\mathbf{F} \cdot \frac{\nabla g}{|\nabla g \cdot \mathbf{p}|} = \left\langle 0, yz, z^2 \right\rangle \cdot \frac{\langle 0, 2y, 2z \rangle}{2z} = \frac{2y^2 z + 2z^3}{z} = y^2 + z^2$$

Setting up the integral we have

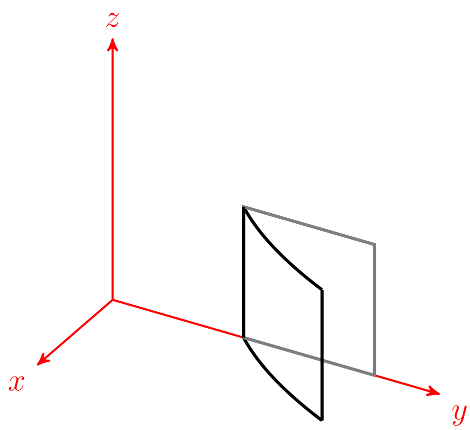
$$\text{Flux} = \iint_S (\mathbf{F} \cdot \mathbf{n})\, d\sigma = \int_{-1}^{1} \int_0^1 \left(y^2 + z^2\right) dx dy$$

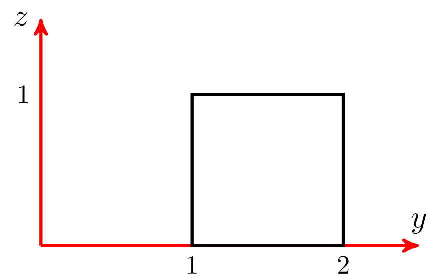
The integral is still not computable in it's current state because we're integrating with respect to x and y but have a z in the integrand. We need to use the equation of the surface to eliminate the z variable. Recalling that the equation of the surface is $y^2+z^2=1$ we have

$$\text{Flux} = \iint_S (\mathbf{F}\cdot\mathbf{n})\,d\sigma = \int_{-1}^{1}\int_0^1 dx dy = 2$$

Example 6: Find the flux of the fluid velocity field $\mathbf{F} = \langle -2, 2y, z\rangle$ through the cylinder $y = e^x$ in the first octant cut by the planes $y = 2$ and $z = 1$.

Note that the surface $y = e^x$ does not have a z in it, so the surface looks like the curve $y = e^x$ in the xy-plane and then propagated straight up in the z-direction. We have now plot the surface and its projection into the yz-plane.





Note that we chose $\mathbf{p} = \mathbf{i}$ and projected into the yz-plane. We could have just as easily chosen $\mathbf{p} = \mathbf{j}$ and projected into the xz-plane. Since we projected into the yz-plane our final integrand should not have any x's in it.

To get the function g that describes the surface, we need to move all the variables in the surface equation to one side of the equation. We then have

$$g(x, y, z) = y - e^x \quad\Rightarrow\quad \nabla g = \langle -e^x, 1, 0\rangle$$

We need to choose the sign on ∇g so that it points outward from the surface (or away from the yz-plane). We want the $\mathbf{i}$-component of ∇g to be pointing in the positive x-direction, which means we need to flip the sign on ∇g. So we choose

$$-\nabla g = \langle e^x, -1, 0\rangle \quad \text{and so} \quad |\nabla g\cdot p| = |\nabla g\cdot \mathbf{i}| = |e^x| = e^x$$

Then the integrand of the Flux integral is

$$\mathbf{F}\cdot\frac{-\nabla g}{|\nabla g\cdot p|} = \langle -2, 2y, z\rangle\cdot\frac{\langle e^x, -1, 0\rangle}{e^x} = \frac{-2e^x - 2y}{e^x} = -2 - 2\frac{y}{e^x}$$

But on the surface we have $y = e^x$ and so

$$\mathbf{F} \cdot \frac{-\nabla g}{|\nabla g \cdot p|} = -2 - 2\frac{y}{y} = -4$$

Plugging this into the Flux integral and setting up the limits of integration on R we have

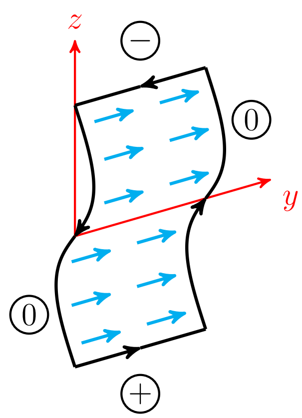
$$\text{Flux} = \iint_S (\mathbf{F} \cdot \mathbf{n})\, d\sigma = \int_0^1 \int_1^2 (-4)\, dydz = -4$$

13.8 Stokes's Theorem

Recall that Green's Theorem for Circulation in a planar flow $\mathbf{F} = P(x, y)\,\mathbf{i} + Q(x, y)$ relates the circulation around a closed curve to the integral of curl $\mathbf{F} \cdot \mathbf{k}$ over the region interior to the curve:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA$$

Now picture a surface in three dimensions and a fluid velocity field $\mathbf{F} = P\,(x, y, z)\,\mathbf{i} + Q\,(x, y, z)\,\mathbf{j} + R\,(x, y, z)\,\mathbf{k}$. Let C be the bounding curve of S oriented counterclockwise with respect to the surfaces outward pointing normal (in this case, pointing out of the page) and consider the circulation of $\mathbf{F}$ around C.

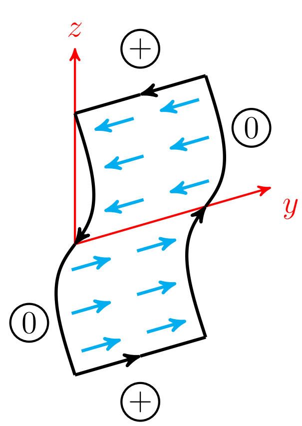


Notice that for the pictured vector field $\mathbf{F}$ the flow around the sides of the bounding curve is zero, and the flow along the top and bottom curves have equal and opposite flows. So we have

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$$

Further notice that on S we have curl $\mathbf{F} = \mathbf{0}$.

Now consider the same surface S and a slightly different vector field $\mathbf{F}$.

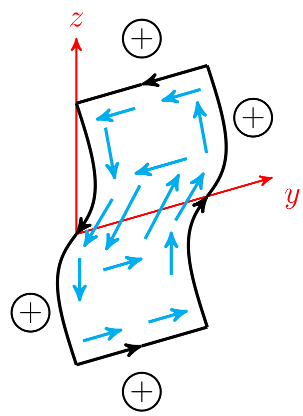


Notice that again the flow along the sides of the bounding curve C is zero, but this time the flow along the top and bottom are both positive. This means that we have

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \text{small and positive}$$

Notice also that curl $\mathbf{F} = \mathbf{0}$ on the majority of S, but curl $\mathbf{F} > 0$ in the part of S where the direction of the vector field changes.

Consider again the same surface S but with one more vector field $\mathbf{F}$.



This time we have a positive flow along each section of the bounding curve C, so

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \text{large and positive}$$

and this time we have a nontrivial curl $\mathbf{F}$ on all parts of the surface S.

So, it should at least be believable that curl $\mathbf{F}$ on the surface S is related to the circulation around the bounding curve C. In fact, this fact is expressed by one of the most important theorem's in Calculus.

Stokes's Theorem: Let $\mathbf{F}$ be a vector field with continuous first partial derivatives, and S a surface with counterclockise oriented bounding curve C. Then we have

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S [(\nabla \times \mathbf{F}) \cdot \mathbf{n}]\ d\sigma$$

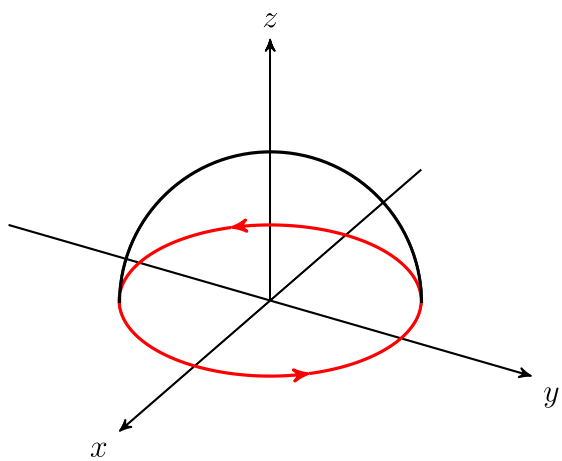
Note that as with the standard flux calculation, we have

$$\mathbf{n}\, d\sigma = \frac{\pm\nabla g}{|\nabla g \cdot \mathbf{p}|}\, dA \quad \Rightarrow \quad \iint_S [(\nabla \times \mathbf{F}) \cdot \mathbf{n}]\ d\sigma = \iint_R (\nabla \times \mathbf{F}) \cdot \frac{\pm\nabla g}{|\nabla g \cdot \mathbf{p}|}\, dA$$

where R is the projection of S along vector $\mathbf{p}$.

Example: Use Stokes's Theorem to compute the circulation of the field $\mathbf{F} = y^3\mathbf{i} - x^3\mathbf{j}$ around the counterclockwise oriented boundary curve C of the hemisphere $x^2+y^2+z^2 = 4$ for $z \geq 0$.

First, we draw a picture:



Note that C is described by $x^2 + y^2 = 4$ when $z = 0$. We then have

$$\nabla \times \mathbf{F} = \left(-3x^2 - 3y^2\right)\mathbf{k} \qquad \nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \qquad |\nabla g \cdot \mathbf{k}| = |2z| = 2z$$

$$\mathbf{n}\, d\sigma = \frac{\nabla g}{|\nabla g \cdot \mathbf{k}|}\, dA = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{z}\, dA$$

Then

$$\iint_R \left(-3x^2 - 3y^2\right)\mathbf{k} \cdot \left(\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{z}\right)\, dA = \iint_R \left(-3x^2 - 3y^2\right)\frac{z}{z}\, dA = -\iint_R \left(3x^2 + 3y^2\right)\, dA$$

Converting to polar coordinates we have

$$= -3\int_0^{2\pi}\int_0^2 r^2\, r\, dr\, d\theta = -3\int_0^{2\pi} \frac{r^4}{4}\bigg]_0^2 d\theta = -3\,(2\pi)\,(4) = -24\pi$$

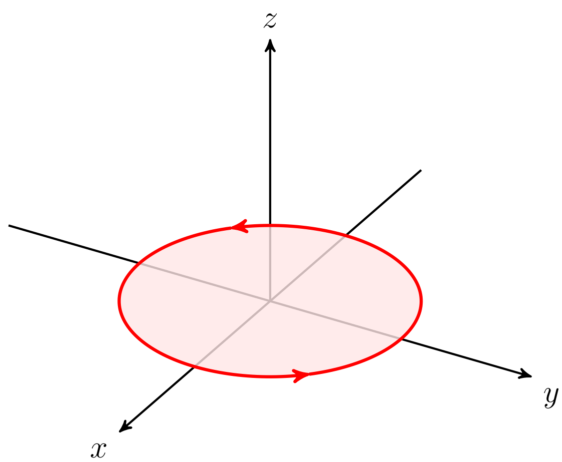
Let's look at some consequences of Stokes's Theorem a bit more...

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma$$

This says that circulation around the counterclockwise oriented curve C can be computed using **any** surface S that has C as it's boundary curve. Since we're free to pick any surface that has C as it's bounding curve, we can compute surface integrals of $(\nabla \times F) \cdot \mathbf{n}$ over complicated surfaces by choosing a simpler surface with the same bounding curve.

Example: Compute the flow of the vector field from the previous example around the bounding curve of the surface S described by $z = 0$ and bounded by $x^2 + y^2 = 4$.

Notice that S is the flat circular disc in the xy-plane with the same boundary curve as the hemisphere in the previous example. The picture looks as follows:



This time we have

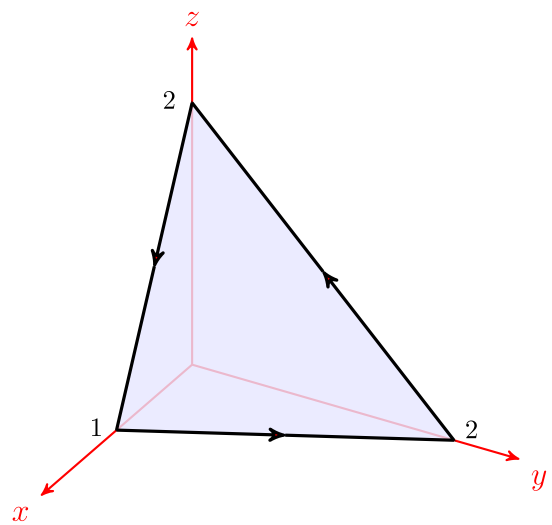
$$\nabla \times \mathbf{F} = \left(-3x^2 - 3y^2\right)\mathbf{k} \qquad \nabla g = \mathbf{k} \qquad |\nabla g \cdot \mathbf{k}| = |1| = 1$$

Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left(-3x^2 - 3y^2\right)\mathbf{k} \cdot \mathbf{k}\, dA = -3\iint_R \left(x^2 + y^2\right)\, dA = -24\pi$$

Remark: Notice that we got the same answer as before, and recovered Green's Theorem for planar flows.

Example: Use Stokes's Theorem to evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = (xz)\,\mathbf{i} + (xy)\,\mathbf{j} + (3xz)\,\mathbf{k}$ and C is the boundary of the plane $2x + y + z = 2$ in the first octant traversed counterclockwise as viewed from above.



We have

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ xz & xy & 3xz \end{vmatrix} = \langle 0, x - 3z, y \rangle \qquad \nabla g = \langle 2, 1, 1 \rangle \qquad |\nabla g \cdot \mathbf{k}| = 1$$

Then

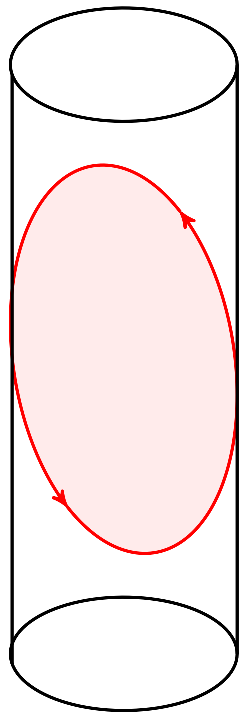
$$(\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \langle 0, x - 3z, y \rangle \cdot \langle 2, 1, 1 \rangle \ dA = (x + y - 3z) \ dA$$

Then by Stokes's Theorem we have

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \int_0^{2-2x} (x + y - 3z) \ dA = \int_0^1 \int_0^{2-2x} (7x + 4y - 6) \ dA = -1$$

Example: Use Stokes's Theorem to evaulate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = \langle x^2 z, xy^2, z^2 \rangle$ and C is the curve of intersection of the plane $x + y + z = 1$ and the cylinder $x^2 + y^2 = 9$ oriented counterclockwise when viewed from above.

Notice that C is the boundary curve of the elliptical disc that lies on the plane and inside the cylinder.



We have

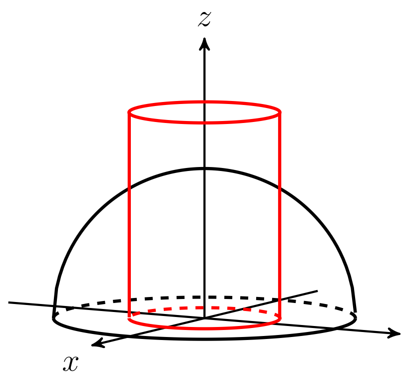
$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ x^2 z & xy^2 & z^2 \end{vmatrix} = \left\langle 0, x^2, y^2 \right\rangle \qquad \nabla g = \langle 1, 1, 1 \rangle \qquad |\nabla g \cdot \mathbf{k}| = 1$$

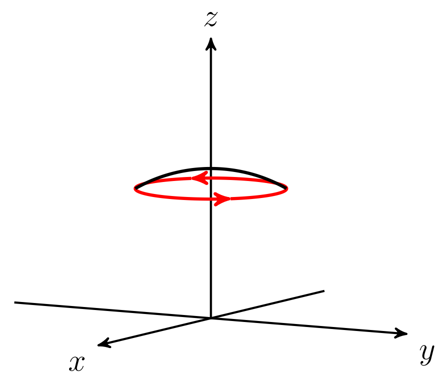
$$\nabla \times \mathbf{F} \cdot n \, d\sigma = \left\langle 0, x^2, y^2 \right\rangle \cdot \langle 1, 1, 1 \rangle \; dA = \left(x^2 + y^2 \right) \; dA$$

$$\iint_R x^2 + y^2 \, dA = \int_0^{2\pi} \int_0^3 r^3 dr d\theta = \int_0^{2\pi} \left. \frac{r^4}{4} \right]_0^3 d\theta = 2\pi \frac{81}{4} = \frac{81\pi}{2}$$

Example: Given $\mathbf{F} = (xz)\,\mathbf{i} + (yz)\,\mathbf{j} + (xy)\,\mathbf{k}$, compute $\displaystyle\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma$, where S is the portion of the sphere $x^2 + y^2 + z^2 = 4$ above the xy-plane and inside the cylinder $x^2 + y^2 = 1$.

The picture on the left shows the hemisphere and the cylinder together. The picture on the right shows the portion of the hemisphere inside the cylinder along with it's bounding curve.





Solution 1: First we try to compute the surface integral directly, using the hemisphere as S. We have

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ xz & yz & xy \end{vmatrix} = \langle x - y, x - y, 0 \rangle \qquad \nabla g = \langle 2x, 2y, 2z \rangle \qquad |\nabla g \cdot \mathbf{k}| = |2z| = 2z$$

Then

$$\iint_R \langle x - y, x - y, 0 \rangle \cdot \frac{\langle x, y, z \rangle}{z}\, dA = \iint_R \frac{x^2 - y^2}{z}\, dA = \iint_R \frac{x^2 - y^2}{\sqrt{4 - x^2 - y^2}}\, dA$$

This is **UGLY**!!

Solution 2: Let's use Stokes's Theorem to write the surface integral as a line integral around the bounding curve of the surface. We have

$$\mathbf{r}(t) = \left\langle \cos t, \sin t, \sqrt{3} \right\rangle \quad \text{for} \quad 0 \le t \le 2\pi$$

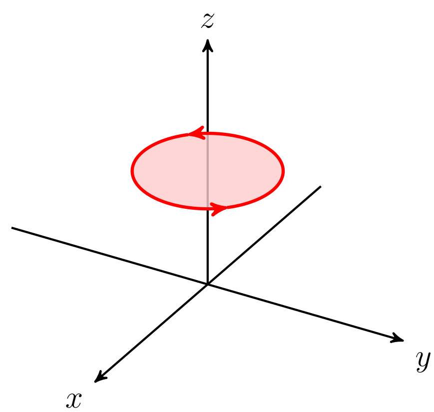
$$d\mathbf{r} = \langle -\sin t, \cos t, 0 \rangle\ dt$$

$$\mathbf{F}(\mathbf{r}(t)) = \left\langle \sqrt{3}\cos t, \sqrt{3}\sin t, \sin t \cos t \right\rangle$$

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \int_0^{2\pi} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} -\sqrt{3}\sin t \cos t + \sqrt{3}\sin t \cos t\, dt = \int_0^{2\pi} 0\, dt = 0$$

Solution 3: Let's compute the surface integral over a simpler surface with the same boundary curve C. The simplest option is to choose S to be the surface $g = z = \sqrt{3}$ inside the cylinder $x^2 + y^2 = 1$.

The desired surface is a disc in the $z = \sqrt{3}$ plane, which looks as follows:



Then

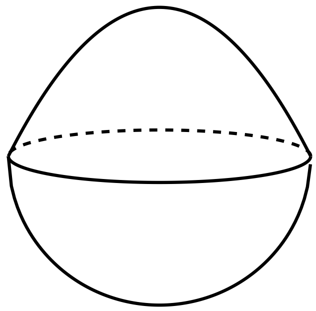
$$\nabla g = \mathbf{k} \qquad |\nabla g \cdot \mathbf{k}| = 1$$

$$\iint_R \langle x - y, x - y, 0 \rangle \cdot \langle 0, 0, 1 \rangle \; dA = \iint_R 0 \, dA = 0$$

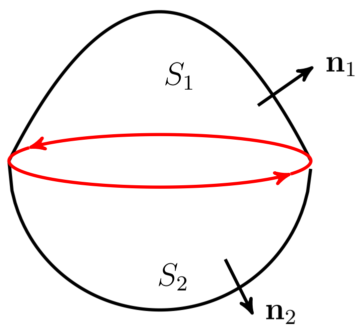
Stokes's Theorem and Closed Surfaces

Fact: If S is a closed oriented surface then $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = 0$

Consider the closed surface shown below



We don't know how to compute the surface integrals over surfaces that are not one-to-one. Instead we can break up the surface S into a top surface S_1 and a bottom surface S_2 where the top and bottom surface meet along the counterclockwise oriented curve C.



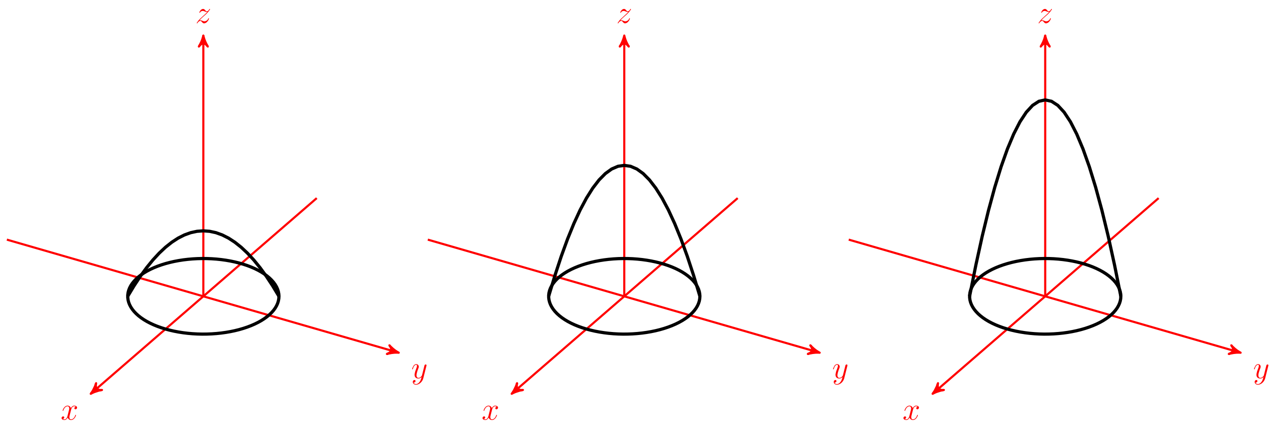
Then

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma + \iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma$$

We'd like to use Stokes's Theorem to relate each of the surface integrals to a line integral around the bounding curves of S_1 and S_2 (which happen to be the same curve). But, remember that to use Stokes's Theorem we must have the bounding curve oriented counterclockwise with respect to the unit outward pointing normal. Notice that for S_1 the curve, as shown in the picture, is oriented counterclockwise w.r.t. $\mathbf{n}_1$. But, for S_2 the curve is oriented clockwise w.r.t. $\mathbf{n}_2$. This means that when we use Stokes's Theorem to rewrite the sufrace integral over S_2 we must integrate over the curve traversed backwards. So, we have

$$\iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma + \iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \oint_C \mathbf{F} \cdot d\mathbf{r} + \oint_{-C} \mathbf{F} \cdot d\mathbf{r} = \oint_C \mathbf{F} \cdot d\mathbf{r} - \oint_C \mathbf{F} \cdot d\mathbf{r} = 0$$

Example: Let S be the paraboloid $z = a\,(1 - x^2 - y^2)$, for $z \geq 0$, where $a > 0$ is a constant. Let $\mathbf{F} = \langle x - y, y + z, z - x \rangle$. For what value(s) of a (if any) does $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma$ have its maximum value?



Using Stoke's Theorem we can see that the value of the surface integral does not depends on the height of the surface. Since the surface has the same bounding curve for any value of a we can write the sufrace integral as

$$\oint_C \mathbf{F} \cdot dr = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma$$

where C is the unit circle $x^2 + y^2 = 1$. Let's use the line integral formulation to compute the surface integral. We parameterize the curve C as $\mathbf{r} = \langle \cos t, \sin t, 0 \rangle$ for $0 \leq t \leq 2\pi$. Then $r'(t) = \langle -\sin t, \cos t, 0 \rangle$ and we have

$$\begin{aligned}
\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma &= \int_0^{2\pi} \langle \cos t - \sin t, \sin t, -\cos t \rangle \cdot \langle -\sin t, \cos t, 0 \rangle \; dt \\
&= \int_0^{2\pi} -\sin t \cos t + \sin^2 t + \sin t \cos t \, dt \\
&= \int_0^{2\pi} \sin^2 t \, dt = \pi
\end{aligned}$$

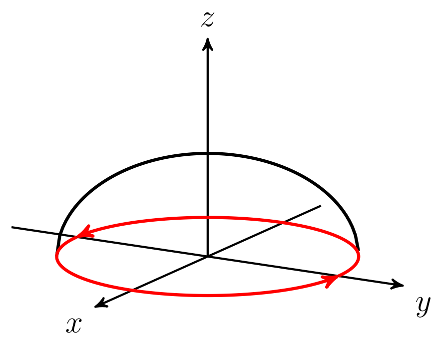
Stokes's Theorem and Conservative Fields

Recall that if a vector field $\mathbf{F}$ is conservative, then the circulation around any closed curve C is zero. Last time we proved this using the Fundamental Theorem of Line Integrals. Stokes's Theorem gives us an alternative proof. Recall that a fields is conservative if it satisfies $\nabla \times \mathbf{F} = \mathbf{0}$. Then from Stokes's Theorem we have

$$\oint_C \mathbf{F} \cdot dr = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \iint_S \mathbf{0} \cdot \mathbf{n}\, \mathbf{d}\sigma = \mathbf{0}$$

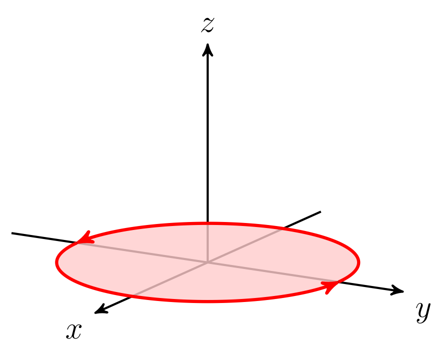
Example: The goal is to evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma$, where $\mathbf{F} = \langle yz, -xz, xy \rangle$ and S is the surface of the upper half of the ellipsoid $x^2 + y^2 + 2z^2 = 1$ for $z \geq 0$.

The surface S looks as follows:



We could of course compute the curl of F and integrate it directly over the surface of the ellipsoid. Thankfully, Stokes's Theorem gives us two better options.

Option 1: The ellipsoid is not a particularly easy surface to integrate over. Recall that we can integrate $(\nabla \times \mathbf{F}) \cdot \mathbf{n}$ over any surface that has the same bounding curve as S with the same orientation. Instead of the ellipsoid, we choose to integrate over the disc in the xy-plane bounded by the circle $x^2 + y^2 = 1$.



The outward pointing normal vector is clearly $\mathbf{n} = \mathbf{k}$, so we have

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA$$

The curl of $\mathbf{F}$ is given by

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial x & \partial y & \partial z \\ yz & -xz & xy \end{vmatrix} = \langle 2x, 0, -2z \rangle$$

Then

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \iint_R \langle 2x, 0, -2z \rangle \cdot \mathbf{k}\, dA = -2 \iint_R z \cdot \mathbf{k}\, dA = -2 \iint_R 0 \cdot \mathbf{k}\, dA = 0$$

Option 2: The other option is we can use Stokes's Theorem to write the surface integral as a line integral around the unit circle. Parameterizing the circle we have $\mathbf{r} = \langle \cos t, \sin t, 0\rangle$ and $\mathbf{r}' = \langle -\sin t, \cos t, 0\rangle$ which gives

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \int_0^{2\pi} \langle 0, 0, \sin t \cos t\rangle \cdot \langle -\sin t, \cos t, 0\rangle \; dt = \int_0^{2\pi} 0\, dt = 0$$

Example: The French physicist Andre-Marie Ampere discovered that an electrical current I in a wire produces a magnetic field $\mathbf{B}$. A special case of Ampere's Law relates the current to the magnetic field through the equation

$$\oint_C \mathbf{B} \cdot \mathbf{dr} = \mu \mathbf{I}$$

where μ is a physical constant and C is a closed curve through which the wire passes. Assume that the current I is given in terms of the current density $\mathbf{J}$ as

$$I = \iint_S \mathbf{J} \cdot \mathbf{n}\, d\sigma$$

where S is an oriented surface with C as a boundary. Use Stokes's Theorem to show that an equivalent form of Ampere's Law is the following partial differential equation:

$$\nabla \times \mathbf{B} = \mu \mathbf{J}$$

We have by Stokes's Theorem

$$\mu I = \mu \iint_S \mathbf{J} \cdot \mathbf{n}\, d\sigma = \oint_C \mathbf{B} \cdot \mathbf{dr} = \iint_{\mathbf{S}} (\nabla \times \mathbf{B}) \cdot \mathbf{n}\, \mathbf{d}\sigma$$

Moving the terms to the same side of the equation we have

$$\iint_S [(\nabla \times \mathbf{B}) - \mu \mathbf{J}] \cdot \mathbf{n}\, d\sigma = 0$$

Since this is true for any surface S with a boundary curve C and any normal vector $\mathbf{n}$ it must be the case that $\nabla \times \mathbf{B} = \mu \mathbf{J}$.

Example: Let S be the surface defined by $z = x^2 + y^2$ for $z \leq 4$. Compute the flux of the vector field $\mathbf{F} = \langle -3xz^2, 0, z^3 \rangle$ through the surface S.

We need to compute

$$\text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma$$

Now, we wouldn't normally use Stokes's Theorem to compute fluxes, but if we can find a vector field $\mathbf{G}$ such that $\mathbf{F} = (\nabla \times \mathbf{G})$ then we can use Stokes's Theorem to instead compute

$$\text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iint_S (\nabla \times \mathbf{G}) \cdot \mathbf{n} \, d\sigma = \oint_C \mathbf{G} \cdot dr$$

where C is the boundary curve of S. Recall that when we learned about the curl, we proved the identity $\nabla \cdot \nabla \times \mathbf{G} = 0$. So, if $\mathbf{F}$ can be written as a curl, it must be divergence free. Indeed, for our $\mathbf{F}$ we have

$$\nabla \cdot \mathbf{F} = -3z^2 + 0 + 3z^2 = 0.$$

If we assume that $\mathbf{G} = \langle G_x, G_y, G_z \rangle$ then we need

$$\nabla \times \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial x & \partial y & \partial z \\ G_x & G_y & G_z \end{vmatrix} = \left\langle -3xz^2, 0, z^3 \right\rangle$$

The components of $\mathbf{G}$ must satisfy the following three equations for the curl

$$\left(\frac{\partial G_z}{\partial y} - \frac{\partial G_y}{\partial z} \right) = -3xz^2, \quad \left(\frac{\partial G_z}{\partial x} - \frac{\partial G_x}{\partial z} \right) = 0, \quad \left(\frac{\partial G_y}{\partial x} - \frac{\partial G_x}{\partial y} \right) = z^3$$

A little guess and check shows that $\mathbf{G} = \langle 0, xz^3, 0 \rangle$ works. Then we have

$$\text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \oint_C \left\langle 0, xz^3, 0 \right\rangle \cdot dr$$

The boundary curve C is the circle $x^2 + y^2 = 4$ in the $z = 4$ plane which can be parametrized by

$$\mathbf{r} = \langle 2\cos t, 2\sin t, 4 \rangle \quad \Rightarrow \quad \mathbf{r}' = \langle -2\sin t, 2\cos t, 0 \rangle$$

Then we have

$$\text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \int_0^{2\pi} (2\cos t)(4)^3 (2\cos t) \; dt = 256 \int_0^{2\pi} \cos^2 t \, dt = 256\pi$$

13.9 The Divergence Theorem

Recall that the Flux form of Green's Theorem gave us a way to compute the flux of a planar flow $\mathbf{F} = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}$ through a closed curve by integrating the divergence of $\mathbf{F}$ over the interior of the curve:

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_R \nabla\cdot\mathbf{F}\,dA$$

Recall that the divergence $\nabla\cdot\mathbf{F} = \text{div}\,\mathbf{F}$ at a point (x,y,z) can be interpreted as a **flux density**. In the case that $\mathbf{F}$ is a planar flow we can interpret this flux density as a rate per unit **area** that fluid is flowing into or out of a region. When we added up all the little divergences in the interior of the curve (via integration) we obtained the total flux across the boundary of the bounding curve C.

Today we will extend this concept to non-planar flows of the form $\mathbf{F} = P(x,y,z)\,\mathbf{i} + Q(x,y,z)\,\mathbf{j} + R(x,y,z)\,\mathbf{k}$ and flux through surfaces. The generalization of the Flux form of Green's Theorem for a closed planar curve is called the **Divergence Theorem**:

The Divergence Theorem: Let $\mathbf{F}$ be a fluid veloicty field with continuous first partial derivatives, S be a closed, simple surface with outward pointing normal vector $\mathbf{n}$, and E be the interior region of S. Then

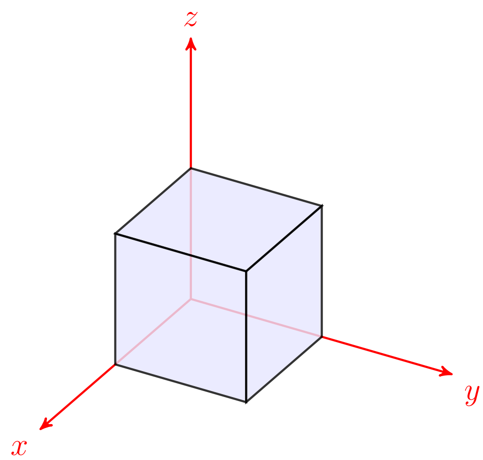
$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iiint_E \nabla\cdot\mathbf{F}\,dV$$

This says that the rate at which fluid is leaving a closed surface S is equal to the integral of div $\mathbf{F}$ over the interior of the region enclosed by S.

This should make intuitive sense. When we integrate div $\mathbf{F}$ over the region E we are implicitly breaking up the region E into little volume chunks, evaluating div $\mathbf{F}$ on each of the chunks, and adding up the contributions of each chunk. Since div $\mathbf{F}$ on each chunk gives us the net volume-flux through that little chunk, adding them all up gives the net volume flux in the entire region.

Example 1: Verify the Divergence Theorem by finding the flux of the field $\mathbf{F} = \langle xy, yz, xz\rangle$ through the surface of the cube cut from the 1st octant by $x = y = z = 1$.

The surface S looks like



We first compute the flux directly via $\iint_S \mathbf{F}\cdot\mathbf{n}d\sigma$. The surface of the cube is the union of each of its six faces. We can make life easier by realizing that on the three sides lying in the three coordinate planes we have $\mathbf{F}\cdot\mathbf{n} = 0$, indicating that flux across those surfaces is 0. We now need to compute the flux through front, top, and side faces lying off the coordinate planes. We then have

$$\iint_S \mathbf{F}\cdot\mathbf{n}d\sigma = \iint_{Front} \mathbf{F}\cdot\mathbf{n}d\sigma + \iint_{Top} \mathbf{F}\cdot\mathbf{n}d\sigma + \iint_{Side} \mathbf{F}\cdot\mathbf{n}d\sigma$$

Front: The front face lies in the plane $x = 1$. Plugging this into the vector field we have $\mathbf{F} = \langle y, yz, z\rangle$ along the front face. The outward pointing unit vector is $\mathbf{n} = \mathbf{i}$. Finally, since the front face is parallel to the yz plane we have $d\sigma = dydz$. Then the flux integral becomes

$$\iint_{Front} \mathbf{F}\cdot\mathbf{n}d\sigma = \int_0^1\int_0^1 \langle y, yz, z\rangle\cdot\langle 1,0,0\rangle\; dydz = \int_0^1\int_0^1 y\,dydz = \frac{1}{2}$$

Top: The top face lies in the plane $z = 1$. Plugging this into the vector field we have $\mathbf{F} = \langle xy, y, x\rangle$ along the top face. The outward pointing unit vector is $\mathbf{n} = \mathbf{k}$. Finally, since the top face is parallel to the xy plane we have $d\sigma = dxdy$. Then the flux integral becomes

$$\iint_{Top} \mathbf{F}\cdot\mathbf{n}d\sigma = \int_0^1\int_0^1 \langle xy, y, x\rangle\cdot\langle 0,0,1\rangle\; dydz = \int_0^1\int_0^1 x\,dxdy = \frac{1}{2}$$

Side: The side face lies in the plane $y = 1$. Plugging this into the vector field we have $\mathbf{F} = \langle x, z, xz \rangle$ along the side face. The outward pointing unit vector is $\mathbf{n} = \mathbf{j}$. Finally, since the side face is parallel to the xz plane we have $d\sigma = dxdz$. Then the flux integral becomes

$$\iint_{Side} \mathbf{F} \cdot \mathbf{n} d\sigma = \int_0^1 \int_0^1 \langle x, z, xz \rangle \cdot \langle 0, 1, 0 \rangle \; dydz = \int_0^1 \int_0^1 z \, dxdz = \frac{1}{2}$$

Summing the flux across the three nonzero faces we find

$$\iint_S \mathbf{F} \cdot \mathbf{n} d\sigma = \iint_{Front} \mathbf{F} \cdot \mathbf{n} d\sigma + \iint_{Top} \mathbf{F} \cdot \mathbf{n} d\sigma + \iint_{Side} \mathbf{F} \cdot \mathbf{n} d\sigma = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}$$

OK, that was tedious. Instead of computing the flux directly using surface multiple integrals, we can use the Divergence Theorem to compute it via a single volume integral. We first need to compute $\nabla \cdot \mathbf{F} = \text{div}\mathbf{F}$.

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(xz) = y + z + x$$

We then have

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma &= \iiint_E \nabla \cdot \mathbf{F} \, dV \\
&= \int_0^1 \int_0^1 \int_0^1 x + y + z \, dxdydz \\
&= \int_0^1 \int_0^1 \frac{1}{2} + y + z \, dydz \\
&= \int_0^1 1 + z \, dz = \frac{3}{2}
\end{aligned}$$

as expected.

Divergence Theorem Intuition

In our discussions on the Flux Form of Green's Theorem for planar flows, we interpretted div $\mathbf{F}$ as a **flux density** where the units were flux per area. When the vector field $\mathbf{F}$ is nonplanar we can interpret div $\mathbf{F}$ again as a flux density, but this time the units are flux per volume. The Divergence Theorem can firm up this interpretation.

Let $P_0(x, y, z)$ be a point in space and B_a a ball with center at P_0 and small radius a.

Since B_a is small, the vector field $\mathbf{F}$ does not change much in B_a and we have

$$\text{div } \mathbf{F}(P) \approx \text{div } \mathbf{F}(P_0)$$

for all points P in B_a. Then the flux across the surface of B_a (which we'll call S_a) is

$$\begin{aligned} \iint_{S_a} \mathbf{F} \cdot \mathbf{n} \, d\sigma &= \iiint_{B_a} \text{div } \mathbf{F} \, dV \\ &\approx \iiint_{B_a} \text{div } \mathbf{F}(P_0) \, dV \\ &\approx \text{div } \mathbf{F}(P_0) \iiint_{B_a} dV \\ &\approx \text{div } \mathbf{F}(P_0) \times \text{Volume of } B_a \end{aligned}$$

Now, this approximation gets better as $a \to 0$, so we have

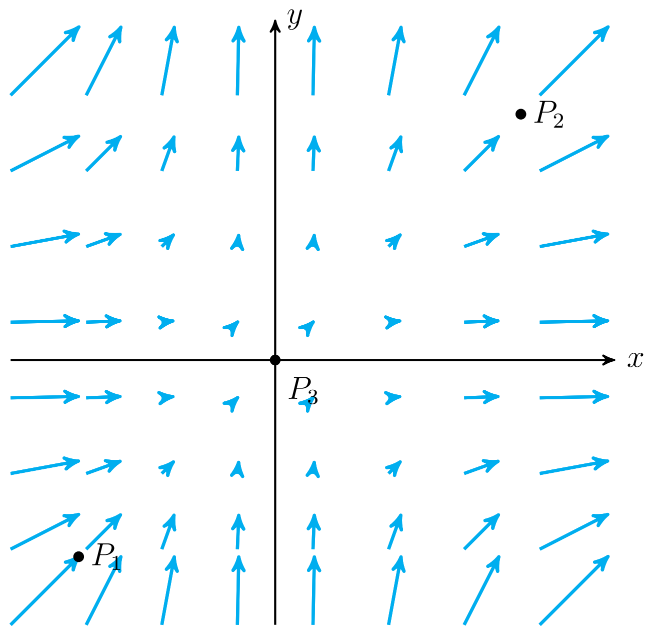
$$\text{div } \mathbf{F}(P_0) = \lim_{a \to 0} \frac{1}{\text{Vol}(B_a)} \iint_{S_a} \mathbf{F} \cdot \mathbf{n} \, d\sigma$$

which has dimensions Flux per unit Volume. In other words, the divergence of $\mathbf{F}$ is a volume flux density.

If div $\mathbf{F}(P_0) > 0$ then the net flow is outward near P_0 and P_0 is called a **source**.

If div $\mathbf{F}(P_0) < 0$ then the net flow is inward near P_0 and P_0 is called a **sink**.

Example: Consider the vector field div $\mathbf{F} = 2x + 2y$ which looks like the following:



We can see from the picture that the indicated points satisfy

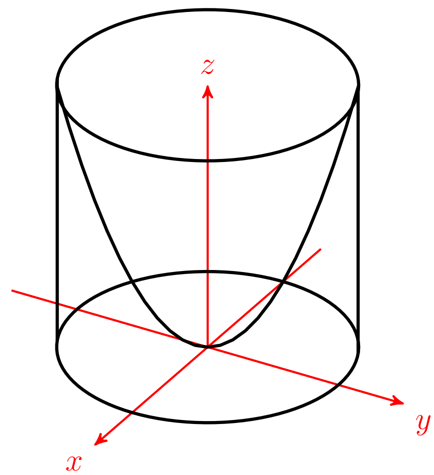
div $\mathbf{F}(P_1) < 0 \quad \Rightarrow \quad P_1$ is a **sink**

div $\mathbf{F}(P_2) > 0 \quad \Rightarrow \quad P_2$ is a **source**

div $\mathbf{F}(P_3) = 0 \quad \Rightarrow \quad P_3$ is **neither**

Example 2: Find the outward flux of the field $\mathbf{F} = \langle y, xy, -z \rangle$ through the surface S bounded on the sides by the cylinder $x^2 + y^2 = 4$, on the top by the paraboloid $z = x^2 + y^2$, and the bottom by $z = 0$.

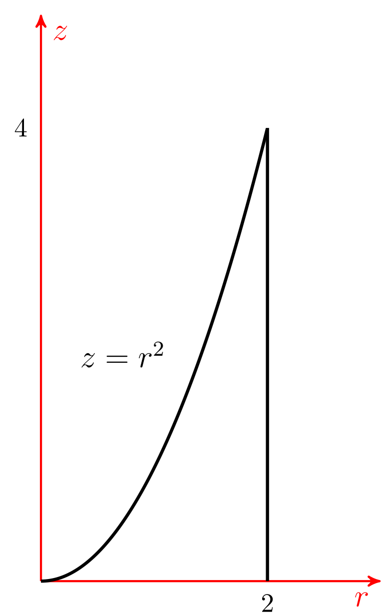
The surface in three dimensions looks like the following



Instead of directly computing the flux through each of the three surfaces making up S we'll use the Divergence Theorem straight away. We have

$$\nabla \cdot \mathbf{F} = 0 + x - 1 = x - 1$$

To integrate over the region E inside of the surface S we should probably use cylindrical coordinates. The solid drawn in the zr-plane looks like



We then have

$$\begin{aligned}
\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma &= \iiint_E x-1\,dV \\
&= \int_0^{2\pi}\int_0^{2}\int_0^{r^2} (r\cos\theta-1)\,r\,dzdrd\theta \\
&= \int_0^{2\pi}\int_0^{2} r^3\left(r\cos\theta-1\right)\,drd\theta \\
&= \int_0^{2\pi} \frac{32}{5}\cos\theta-4\,d\theta = -8\pi
\end{aligned}$$

Example 3: Find the outward flux of the field $\mathbf{F} = \langle x^3, ze^x, 3zy^2\rangle$ through the surface S given by the cylinder $x^2+y^2=1$ capped on the ends by the planes $z=-1$ and $z=2$.

We want to use the Divergence Theorem so we first take the divergence of $\mathbf{F}$. We have

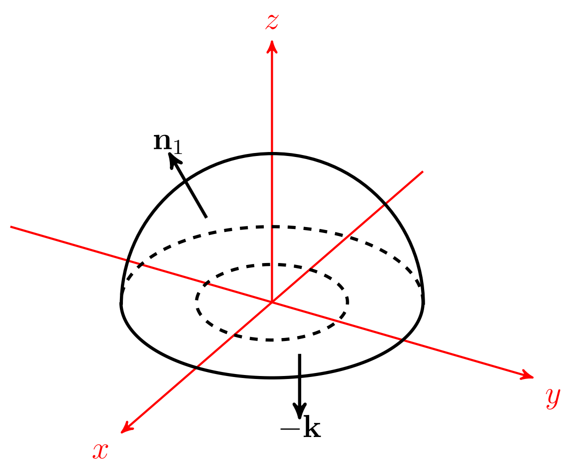
$$\nabla\cdot\mathbf{F} = 3x^2+0+3y^2 = 3x^2+3y^2$$

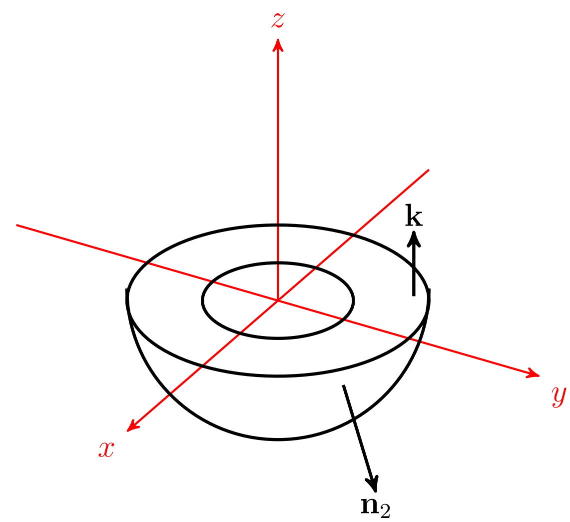
We want to use cylindrical coordinates for the cylinder. The divergence of $\mathbf{F}$ in cylindrical coordinates of of course $\nabla\cdot\mathbf{F} = 3r^2$. Setting up the triple integral we then have

$$\begin{aligned}
\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma &= \iiint_E \nabla\cdot\mathbf{F}\,dV \\
&= \int_0^{2\pi}\int_0^{1}\int_{-1}^{2} \left(3r^2\right) rdzdrd\theta \\
&= 3\int_0^{2\pi}\int_0^{1}\int_{-1}^{2} r^3dzdrd\theta \\
&= 9\int_0^{2\pi}\int_0^{1} r^3drd\theta \\
&= \frac{9}{4}\int_0^{2\pi} d\theta \\
&= \frac{9}{4}\left(2\pi\right) = \frac{9\pi}{2}
\end{aligned}$$

We originally stated the Divergence Theorem only for surfaces surrounding simple regions. It turns out the theorem still holds for certain non-simple regions. Consider computing the flux through the boundary of the surface defined by 2 concentric spheres of radius R_I and R_O, respectively. Note that in this case there are two outward pointing normals. One that points outward from the surface of the larger sphere, and one that points outwards toward the origin for the inner sphere.

We do this by considering what happens when we split the sphere in half across the xy-plane and insert the temporary surface $z = 0$. We then have surfaces S_1 and S_2 which look like





Now consider the flux through each of the two washer-shaped surfaces we created by dividing the sphere in half. Notice that both surfaces occupy the same place in space, and thus their values of $\mathbf{F}$ are the same. But, on the top surface we have $\mathbf{n}_1 = -\mathbf{k}$ and on bottom surface we have $\mathbf{n}_2 = \mathbf{k}$. The contributions to the total flux integral on these surfaces cancel out because we have

$$\mathbf{F} \cdot \mathbf{n}_1 = \mathbf{F} \cdot (-\mathbf{k}) = -(\mathbf{F} \cdot \mathbf{n}_2)$$

We then have, by the Divergence Theorem,

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_{S_1} \mathbf{F} \cdot \mathbf{n}\, d\sigma + \iint_{S_2} \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_{E_1} \nabla \cdot \mathbf{F}\, dV + \iiint_{E_2} \nabla \cdot \mathbf{F}\, dV = \iiint_E \nabla \cdot \mathbf{F}\, dV$$

Thus the Divergence Theorem holds for regions defined by concentric spheres!

Example 4: Find the outward flux through the surface formed by concentric spheres of radii 1 and 2 by the field $\mathbf{F} = \langle 5x^3 + 12xy^2, y^3 + e^y \sin z, 5z^3 + e^y \cos z \rangle$.

We need to compute the divergence of $\mathbf{F}$

$$\nabla \cdot \mathbf{F} = 15x^2 + 12y^2 + 3y^2 + e^y \sin z + 15z^2 - e^y \sin z = 15x^2 + 15y^2 + 15z^2$$

We should compute the triple integral in sphereical coordinates, so we have

$$\nabla \cdot \mathbf{F} = 15x^2 + 15y^2 + 15z^2 = 15\rho^2$$

Then the volume integral is

$$\begin{aligned}\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_E \nabla \cdot \mathbf{F}\, dV &= \int_0^{2\pi} \int_0^{\pi/2} \int_1^2 \left(15\rho^2\right) \rho^2 \sin\phi\, d\rho d\phi d\theta \\ &= 15 \int_0^{2\pi} \int_0^{\pi/2} \int_1^2 \rho^4 \sin\phi\, d\rho d\phi d\theta = 186\pi\end{aligned}$$

Example 5: Calculate the flux of the field $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ through the open cone $z = \sqrt{x^2+y^2}$ for $0 \le z \le 3$.

Notice that the Divergence Theorem does not directly apply here because the cone is not a closed surface. However, we can be clever and create a closed surface by adding a top to the cone that lies in the plane $z = 3$. If we denote the surface of the cone and the top cap by C and T respectively, we then have the clused surface $S = C \cup T$. Then, using the Divergence Theorem we have

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_C \mathbf{F} \cdot \mathbf{n}\, d\sigma + \iint_T \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_E \nabla \cdot \mathbf{F}\, dV$$

Rearranging this expression for the flux through the cone we have

$$\iint_C \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_E \nabla \cdot \mathbf{F}\, dV - \iint_T \mathbf{F} \cdot \mathbf{n}\, d\sigma$$

So we can compute the flux through the cone by computing the triple integral of the divergence of $\mathbf{F}$ over the region E and then subtracting the flux through the top cap T. We have

$$\nabla \cdot \mathbf{F} = 2$$

Updating our expression for the flux through the cone, we have

$$\iint_C \mathbf{F} \cdot \mathbf{n}\, d\sigma = 2 \iiint_E dV - \iint_T \mathbf{F} \cdot \mathbf{n}\, d\sigma$$

The volume of a cone with height 3 and top radius 3 we have

$$2 \iiint_E dV = 2\pi\, 3^2\, \frac{3}{3} = 18\pi$$

Now we need to compute the flux through the top cap of the cone. This should in general be easier than the sides of the cone because it is just a flat plane. However, we can make it a bit easier here by recognizing that the vector field **F** lies completely in the plane. Since **F** is parallel to surface T there can be no flux through it. Thus we have

$$\iint_T \mathbf{F} \cdot \mathbf{n}\, d\sigma = 0$$

and we have

$$\iint_C \mathbf{F} \cdot \mathbf{n}\, d\sigma = 18\pi$$

Example 6: Consider the flux through surface C defined by the cylinder $x^2+y^2=4$ between $z=-2$ and $z=2$ of the field given by

$$\mathbf{F} = \frac{x}{(x^2+y^2+z^2)^{3/2}}\mathbf{i} + \frac{y}{(x^2+y^2+z^2)^{3/2}}\mathbf{j} + \frac{z}{(x^2+y^2+z^2)^{3/2}}\mathbf{k}$$

Show that the flux through the cylinder C is the same as the flux through the sphere S of radius 1 and oriented outward.

We might be tempted to use the Divergence Theorem to compute each of the fluxes simultaneously, but we cannot do this in this case because the vector field **F** is not continuous at the origin so div **F** is undefined there. However, if we form the surface defined by the cylinder with outward pointing normal and the sphere with inward pointing normal we can use the Divergence Theorem because the origin is not in the region. We then have

$$\iint_C \mathbf{F} \cdot \mathbf{n}\, d\sigma - \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_E \nabla \cdot \mathbf{F}\, dV$$

Note that the minus in the left-hand side is necessary because we defined S with outward pointing normal, but to use the Divergence Theorem on the non-simple region E we need the normal pointing inward. Next we compute the divergence of **F**. We have

$$\nabla \cdot \mathbf{F} = \frac{-2x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}}\mathbf{i} + \frac{-2y^2+x^2+z^2}{(x^2+y^2+z^2)^{5/2}}\mathbf{j} + \frac{-2z^2+x^2+y^2}{(x^2+y^2+z^2)^{3/2}}\mathbf{k} = 0$$

Then plugging in our expression for the Divergence Theorem we have

$$\iint_C \mathbf{F} \cdot \mathbf{n}\, d\sigma - \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = 0 \qquad \Rightarrow \qquad \iint_C \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$$

In fact, because $\nabla \cdot \mathbf{F} = 0$, the fluxes through any closed surfaces that contain the origin will be equal.

Example 7: Consider a fluid with density $\rho\left(x, y, z\right)$ and velocity field $\vec{\mathbf{v}}\left(x, y, z\right)$. In class we said that when a fluid has no sources, the divergence of the fluid field at some point must correspond to a change in density (i.e. the fluid is either expanding or being compressed). It turns out, that there is a nifty partial differential equation that models this behavior, called the **continuity equation** for a fluid. It is given by

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \vec{\mathbf{v}}\right) = 0$$

This equation basically says that the change in the density of the fluid must be balanced out by the divergence of the the density times the velocity field. We will derive this equation with some help from the Divergence Theorem. The Divergence Theorem is exceptionally useful when working in systems where some physical quantity is conserved. In this case, since there are no sources or sinks in the system, mass of the fluid is conserved. The equations that describe these conserved quantities are called **conservation laws**. We proceed as follows.

Consider an arbitrary closed surface S with interior region E. The flux of the fluid (in units volume/time) across the surface S is given by

$$\text{Volume Flux} = \iint_S \vec{\mathbf{v}} \cdot \mathbf{n}\, d\sigma$$

where $\mathbf{n}$ is the outward pointing normal of the surface. We can obtain the *mass flux* by multiplying the velocity field by the density. Then we have

$$\text{Mass Flux} = \iint_S \left(\rho \vec{\mathbf{v}}\right) \cdot \mathbf{n}\, d\sigma$$

Now, assuming that there are no sources or sinks inside the region E conservation of mass says the outward mass flux of the fluid through the surface S must be equal to the rate of decrease of the total mass of the fluid in E. In pseudo-equations we have

$$\text{Outward Mass Flux} = \text{Rate of Decrease of Mass in } E$$

or, more mathematically

$$\iint_S \left(\rho \vec{\mathbf{v}}\right) \cdot \mathbf{n}\, d\sigma = -\frac{\partial M}{\partial t}$$

Now, the mass of the fluid in E can be easily computed via

$$M = \iiint_E \rho\, dV$$

Plugging this expression into the conservation of mass equation above we have

$$\iint_S \left(\rho \vec{\mathbf{v}}\right) \cdot \mathbf{n}\, d\sigma = -\frac{\partial}{\partial t} \iiint_E \rho\, dV = -\iiint_E \frac{\partial \rho}{\partial t}\, dV$$

From the Divergence Theorem we have

$$\iint_S (\rho\vec{\mathbf{v}}) \cdot \mathbf{n}\, d\sigma = \iiint_E \nabla \cdot (\rho\vec{\mathbf{v}})\ dV$$

Plugging this expression into the previous conservation equation and collecting all terms on the left-hand side, we have

$$\iiint_E \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\vec{\mathbf{v}})\right]\ dV = 0$$

Now, notice that our surface S and volume E were **completely arbitrary**. That means that the expression above must hold for **any** volume in space. Now, for the integral to be truly zero it must be the case that it's integrand is identically zero. In other words

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\vec{\mathbf{v}}) = 0$$

This is the continuity equation and is used in almost all models of compressible fluids.