Foundations of Network and Computer Security

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Lecture #6
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Announcements

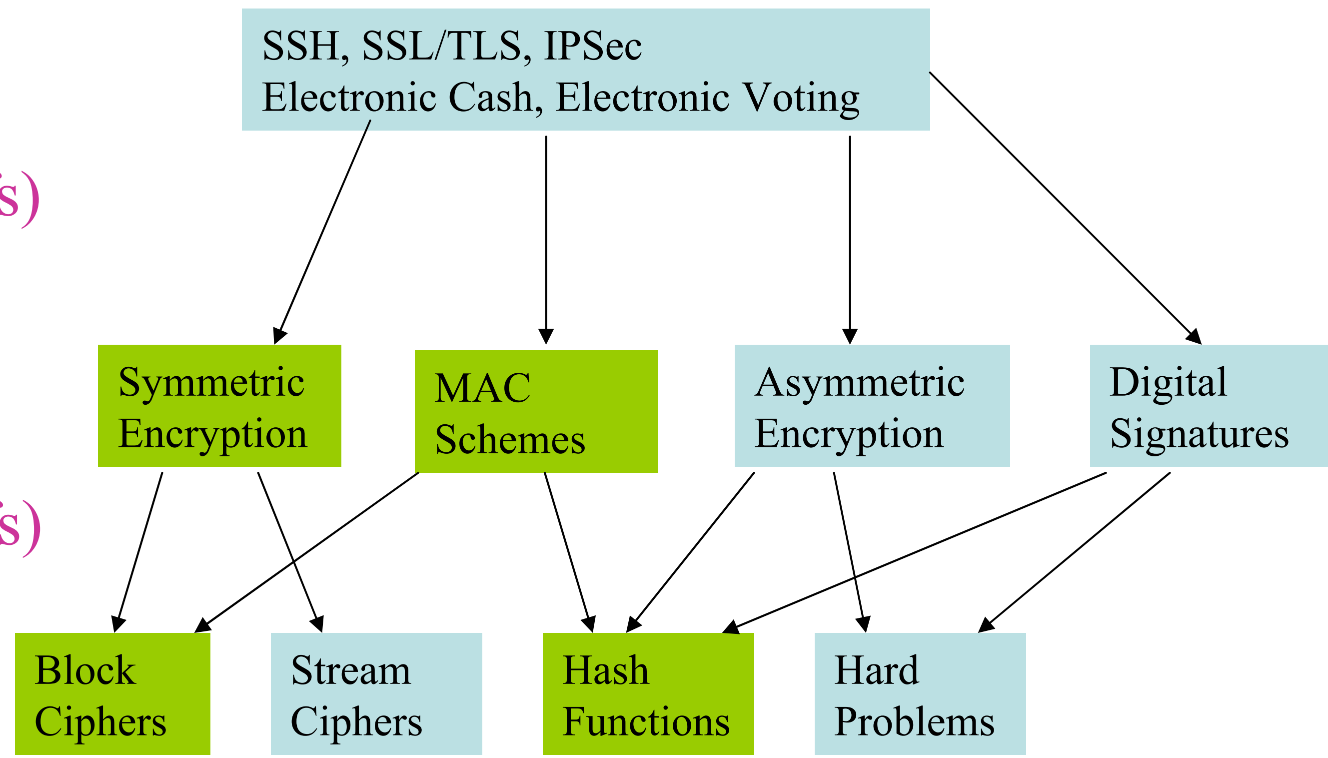
- Quiz #1 later today
- Still some have not signed up for class mailing list
  – Perhaps people still in class but are intending to drop?!
- Please do this by end of today

The Big (Partial) Picture

Second-Level Protocols
(Can do proofs)

First-Level Protocols
(Can do proofs)

Primitives



(No one knows how to prove security; make assumptions)

Symmetric vs. Asymmetric

- Thus far we have been in the symmetric key model
  - We have assumed that Alice and Bob share some random secret string
  - In practice, this is a big limitation
    - Bootstrap problem
    - Forces Alice and Bob to meet in person or use some mechanism outside our protocol
    - Not practical when you want to buy books at Amazon
- We need the Asymmetric Key model!

Asymmetric Cryptography

- In this model, we no longer require an initial shared key
  - First envisioned by Diffie in the late 70’s
  - Some thought it was impossible
  - MI6 purportedly already knew a method
  - Diffie-Hellman key exchange was first public system
    - Later turned into El Gamal public-key system
  - RSA system announced shortly thereafter

But first, a little math…

- A <u>group</u> is a nonempty set G along with an operation # : $G \times G \to G$ such that for all a, b, c $\in$ G
  - (a # b) # c = a # (b # c) (associativity)
  - $\exists$ e $\in$ G such that e # a = a # e = a (identity)
  - $\exists$ $a^{-1} \in$ G such that a # a^{-1} = e (inverses)
- If $\forall$ a,b $\in$ G, a # b = b # a we say the group is “commutative” or “abelian”
  - All groups in this course will be abelian

Notation

- We'll get tired of writing the # sign and just use juxtaposition instead
  - In other words, a # b will be written ab
  - If some other symbol is conventional, we'll use it instead (examples to follow)
- We'll use power-notation in the usual way
  - a^b means $aaaa\cdots a$ repeated b times
  - a^{-b} means $a^{-1}a^{-1}a^{-1}\cdots a^{-1}$ repeated b times
  - Here $a \in G$, $b \in Z$
- Instead of e we'll use a more conventional identity name like 0 or 1
- Often we write G to mean the group (along with its operation) and the associated set of elements interchangeably

Examples of Groups

- Z (the integers) under + ?
- Q, R, C, under + ?
- N under + ?
- Q under $\times$?
- Z under $\times$?
- 2×2 matrices with real entries under $\times$?
- Invertible 2×2 matrices with real entries under $\times$?

- Note all these groups are infinite
  - Meaning there are an infinite number of elements in them
- Can we have finite groups?

Finite Groups

- Simplest example is G = {0} under +
  - Called the “trivial group”
- Almost as simple is G = {0, 1} under addition mod 2
- Let’s generalize
  - Z_m is the group of integers modulo m
  - $Z_m = \{0, 1, \ldots, m-1\}$
  - Operation is addition modulo m
  - Identity is 0
  - Inverse of any $a \in Z_m$ is m-a
  - Also abelian

The Group Z_m

- An example
  - Let m = 6
  - $Z_6 = \{0,1,2,3,4,5\}$
  - 2+5 = 1
  - 3+5+1 = 3 + 0 = 3
  - Inverse of 2 is 4
    - 2+4 = 0
- We can always pair an element with its inverse

a :	0	1	2	3	4	5
a^{-1} :	0	5	4	3	2	1

- Inverses are always unique
- An element can be its own inverse
  - Above, 0 and 0, 3 and 3

Another Finite Group

- Let $G = \{0,1\}^n$ and operation is $\oplus$
  - A group?
  - What is the identity?
  - What is the inverse of $a \in G$?
- We can put some familiar concepts into group-theoretic notation:
  - Caesar cipher was just $P + K = C$ in Z_{26}
  - One-time pad was just $P \oplus K = C$ in the group just mentioned above

Multiplicative Groups

- Is {0, 1, …, m-1} a group under multiplication mod m?
  - No, 0 has no inverse
- Ok, toss out 0; is {1, …, m-1} a group under multiplication mod m?
  - Hmm, try some examples…
    - m = 2, so G = {1} ✓
    - m = 3, so G = {1,2} ✓
    - m = 4, so G = {1,2,3} oops!
    - m = 5, so G = {1,2,3,4} ✓

Multiplicative Groups (cont)

- What was the problem?
  - 2,3,5 all prime
  - 4 is composite (meaning "not prime")

- Theorem: $G = \{1, 2, \ldots, m\text{-}1\}$ is a group under multiplication mod m iff m is prime

  Proof:

  $\leftarrow$: suppose m is composite, then $m = ab$ where $a,b \in G$ and $a, b \neq 1$. Then $ab = m = 0$ and G is not closed

  $\rightarrow$: follows from a more general theorem we state in a moment

The Group Z_m^*

- $a,b \in N$ are relatively prime iff $\gcd(a,b) = 1$
  - Often we'll write (a,b) instead of $\gcd(a,b)$
- Theorem: $G = \{a : 1 \leq a \leq m-1, (a,m) = 1\}$ and operation is multiplication mod m yields a group
  - We name this group Z_m^*
  - We won't prove this (though not too hard)
  - If m is prime, we recover our first theorem

Examples of Z_m^*

- Let m = 15
  - What elements are in Z_{15}^*?
    - {1,2,4,7,8,11,13,14}
  - What is 2^{-1} in Z_{15}^*?
    - First you should check that $2 \in Z_{15}^*$
    - It is since (2,15) = 1
  - Trial and error:
    - 1, 2, 4, 7, 8 ✓
  - There is a more efficient way to do this called "Euclid's Extended Algorithm"
    - Trust me

Euler's Phi Function

- Definition: The number of elements of a group G is called the <u>order</u> of G and is written |G|
  - For infinite groups we say $|G| = \infty$
  - All groups we deal with in cryptography are finite
- Definition: The number of integers $i < m$ such that $(i,m) = 1$ is denoted $\phi(m)$ and is called the "Euler Phi Function"
  - Note that $|Z_m^*| = \phi(m)$
  - This follows immediately from the definition of $\phi()$

Evaluating the Phi Function

- What is $\phi(p)$ if p is prime?
  - p-1
- What is $\phi(pq)$ if p and q are distinct primes?
  - If p, q distinct primes, $\phi(pq) = \phi(p)\phi(q)$
  - Not true if p=q
  - We won't prove this, though it's not hard

Examples

- What is $\phi(3)$?
  - $|Z_3^*| = |\{1,2\}| = 2$
- What is $\phi(5)$?
- What is $\phi(15)$?
  - $\phi(15) = \phi(3)\phi(5) = 2 \times 4 = 8$
  - Recall, $Z_{15}^* = \{1,2,4,7,8,11,13,14\}$

LaGrange's Theorem

- Last bit of math we'll need for RSA
- Theorem: if G is any finite group of order n, then $\forall$ a $\in$ G, $a^n = 1$
  - Examples:
    - $6 \in Z_{22}$, 6+6+…+6, 22 times = 0 mod 22
    - $2 \in Z_{15}^*$, $2^8 = 256 = 1 \bmod 15$
    - Consider $\{0,1\}^5$ under $\oplus$
      - $01011 \in \{0,1\}^5$, $01011^{32} = 00000^{16} = 00000$
  - It always works (proof requires some work)

Basic RSA Cryptosystem

- Basic Setup:
  - Alice and Bob do *not* share a key to start with
  - Alice will be the sender, Bob the receiver
    - Reverse what follows for Bob to reply
  - Bob first does key generation
    - He goes off in a corner and computes two keys
    - One key is pk, the "public key"
    - Other key is sk, the "secret key" or "private key"
  - After this, Alice can encrypt with pk and Bob decrypts with sk

Basic RSA Cryptosystem

- Note that after Alice encrypts with pk, she cannot even decrypt what she encrypted
  - Only the holder of sk can decrypt
  - The adversary can have a copy of pk; we don't care

Alice

Bob's Public Key

Bob's Public Key

Adversary

Bob

Bob's Private Key

Key Generation

- Bob generates his keys as follows
  - Choose two large distinct random primes p, q
  - Set $n = pq$ (in Z… no finite groups yet)
  - Compute $\phi(n) = \phi(pq) = \phi(p)\phi(q) = (p-1)(q-1)$
  - Choose some $e \in Z_{\phi(n)}^*$
  - Compute $d = e^{-1}$ in $Z_{\phi(n)}^*$
  - Set pk = (e,n) and sk = (d,n)
    - Here (e,n) is the ordered pair (e,n) and does not mean gcd

Key Generation Notes

- Note that pk and sk share n
  - Ok, so only d is secret
- Note that d is the inverse in the group $Z_{\phi(n)}^*$ and not in Z_n^*
  - Kind of hard to grasp, but we'll see why
- Note that factoring n would leak d
- And knowing $\phi(n)$ would lead d
  - Bob has no further use for p, q, and $\phi(n)$ so he shouldn't leave them lying around

RSA Encryption

- For any message $M \in Z_n^*$
  - Alice has pk = (e,n)
  - Alice computes $C = M^e \bmod n$
  - That's it

- To decrypt
  - Bob has sk = (d,n)
  - He computes $C^d \bmod n = M$
    - We need to prove this

RSA Example

- Let p = 19, q = 23
  - These aren't large primes, but they're primes!
  - n = 437
  - $\phi(n) = 396$
  - Clearly $5 \in Z^*_{396}$, so set e=5
  - Then d=317
    - $ed = 5 \times 317 = 1585 = 1 + 4 \times 396$ ✓
  - pk = (5, 437)
  - sk = (396, 437)

RSA Example (cont)

- Suppose $M = 100$ is Alice’s message
  - Ensure $(100,437) = 1$ \checkmark
  - Compute $C = 100^5 \bmod 437 = 85$
  - Send 85 to Bob
- Bob receives $C = 85$
  - Computes $85^{317} \bmod 437 = 100$ \checkmark
- We’ll discuss implementation issues later

RSA Proof

- Need to show that for any $M \in Z_n^*$, $M^{ed} = M \bmod n$
  - ed = 1 mod $\phi(n)$ [by def of d]
  - So ed = $k\phi(n) + 1$ [by def of modulus]
  - So working in Z_n^*, $M^{ed} = M^{k\phi(n) + 1} = M^{k\phi(n)} M^1 = (M^{\phi(n)})^k M = 1^k M = M$
    - Do you see LaGrange's Theorem there?
- This doesn't say anything about the security of RSA, just that we can decrypt

Security of RSA

- Clearly if we can factor efficiently, RSA breaks
  - It’s unknown if breaking RSA implies we can factor
- *Basic* RSA is not good encryption
  - There are problems with using RSA as I’ve just described; don’t do it
  - Use a method like OAEP
    - We won’t go into this

Factoring Technology

- Factoring Algorithms
  - Try everything up to sqrt(n)
    - Good if n is small
  - Sieving
    - Ditto
  - Quadratic Sieve, Elliptic Curves, Pollard’s Rho Algorithm
    - Good up to about 40 bits
  - Number Field Sieve
    - State of the Art for large composites

The Number Field Sieve

- Running time is estimated as

$$e^{(1.526+o(1))(\log n)^{1/3}(\log \log n)^{2/3}}$$

- This is super-polynomial, but sub-exponential
  - It's unknown what the complexity of this problem is, but it's thought that it lies between P and NPC, assuming P $\neq$ NP

NFS (cont)

- How it works (sort of)
  - The first step is called “sieving” and it can be widely distributed
  - The second step builds and solves a system of equations in a large matrix and must be done on a large computer
    - Massive memory requirements
    - Usually done on a large supercomputer

The Record

- In Dec, 2003, RSA-576 was factored
  - That's 576 bits, 174 decimal digits
  - The next number is RSA-640 which is

    31074182404900437213507500358885679300373460228427
    27545720161948823206440518081504556346829671723286
    78243791627283803341547107310850191954852900733772
    4822783525742386454014691736602477652346609

  - Anyone delivering the two factors gets an immediate A in the class (and 10,000 USD)

On the Forefront

- Other methods in the offing
  - Bernstein's Integer Factoring Circuits
  - TWIRL and TWINKLE
    - Using lights and mirrors
  - Shamir and Tromer's methods
    - They estimate that factoring a 1024 bit RSA modulus would take 10M USD to build and one year to run
      - Some skepticism has been expressed
  - And the beat goes on…
    - I wonder what the NSA knows

Implementation Notes

- We didn’t say anything about how to *implement* RSA
  - What were the hard steps?!
    - Key generation:
      - Two large primes
      - Finding inverses mode $\phi(n)$
    - Encryption
      - Computing M^e mod n for large M, e, n
  - All this can be done reasonably efficiently

Implementation Notes (cont)

- Finding inverses
  - Linear time with Euclid's Extended Algorithm
- Modular exponentiation
  - Use repeated squaring and reduce by the modulus to keep things manageable
- Primality Testing
  - Sieve first, use pseudo-prime test, then Rabin-Miller if you want to be sure
    - Primality testing is the slowest part of all this
    - Ever generate keys for PGP, GPG, OpenSSL, etc?

Note on Primality Testing

- Primality testing is *different* from factoring
  - Kind of interesting that we can tell something is composite without being able to actually factor it
- Recent result from IIT trio
  - Recently it was shown that deterministic primality testing could be done in polynomial time
    - Complexity was like $O(n^{12})$, though it's been slightly reduced since then
  - One of our faculty thought this meant RSA was broken!
- Randomized algorithms like Rabin-Miller are far more efficient than the IIT algorithm, so we'll keep using those