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Inexact Search is “Good Enough”

Advanced Machine Learning for NLP

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MATHEMATICAL TREATMENT

Preliminaries: algorithm, separability

- Structured perceptron maintains set of "wrong features"

$$\Delta\vec{\Phi}(x,y,z) \equiv \vec{\Phi}(x,y) - \vec{\Phi}(x,z) \tag{1}$$

- Structured perceptron updates weights with

$$\vec{w} \leftarrow \vec{w} + \Delta\vec{\Phi}(x,y,z) \tag{2}$$

- Dataset D is linearly separable under features Φ with margin δ if

$$\vec{u} \cdot \Delta\vec{\Phi}(x,y,z) \geq \delta \quad \forall x,y,z \in D \tag{3}$$

given some oracle unit vector u.

Violations vs. Errors

- It may be difficult to find the highest scoring hypothesis
- It's okay as long as inference finds a **violation**

$$\vec{w} \cdot \Delta\vec{\Phi}(x, y, z) \leq 0 \tag{4}$$

- This means that y might not be answer algorithm gives (i.e., wrong)

Limited number of mistakes

- Define diameter R as

$$R = \max_{(x,y,z)} ||\Delta\vec{\Phi}(x,y,z)|| \tag{5}$$

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- Weight vector $\vec{w}$ grows with each error
- We can prove that $||\vec{w}||$ can't get too big
- And thus, algorithm can only run for limited number of iterations k where it updates weights
- Indeed, we'll bound it from two directions

$$k^2\delta^2 \leq ||w^{(k+1)}||^2 \leq kR^2 \tag{6}$$

Lower Bound

Lower Bound

$$k^2\delta^2 \leq ||w^{(k+1)}||^2$$

$$(7)$$

Lower Bound

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$$k^2\delta^2 \leq ||w^{(k+1)}||^2$$

$$\vec{w}^{(k+1)} = w^{(k)} + \Delta\vec{\Phi}(x,y,z) \quad (7)$$

$$(8)$$

Update equation

Lower Bound

$$k^2\delta^2 \leq ||w^{(k+1)}||^2$$

$$\vec{w}^{(k+1)} = w^{(k)} + \Delta\vec{\Phi}(x, y, z) \quad (7)$$

$$\vec{u} \cdot \vec{w}^{(k+1)} = \vec{u} \cdot w^{(k)} + \vec{u} \cdot \Delta\vec{\Phi}(x, y, z) \quad (8)$$

$$(9)$$

Multiply both sides by $\vec{u}$

Lower Bound

$$k^2\delta^2 \leq ||w^{(k+1)}||^2$$

$$\vec{w}^{(k+1)} = w^{(k)} + \Delta\vec{\Phi}(x,y,z) \tag{7}$$

$$\vec{u}\cdot\vec{w}^{(k+1)} = \vec{u}\cdot w^{(k)} + \vec{u}\cdot\Delta\vec{\Phi}(x,y,z) \tag{8}$$

$$\vec{u}\cdot\vec{w}^{(k+1)} \geq \vec{u}\cdot w^{(k)} + \delta \tag{9}$$

Definition of margin

Lower Bound

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By induction, $\vec{u}\cdot\vec{w}^{(k+1)} \geq k\delta$ (Base case: $\vec{w}^0 = \vec{0}$)

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$$||\vec{u}|| \; ||\vec{w}^{(k+1)}|| \geq \vec{u} \cdot \vec{w} \geq k\delta \tag{8}$$

For any vectors, $||\vec{a}|| \; ||\vec{b}|| \geq a \cdot b$

Lower Bound

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$\vec{u}$ is a unit vector

Lower Bound

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$$||\vec{w}^{(k+1)}|| \geq k\delta \tag{9}$$

$$||\vec{w}^{(k+1)}||^2 \geq k^2\delta^2 \tag{10}$$

Square both sides, and we're done!

Upper Bound

Upper Bound

$$||\vec{w}^{(k+1)}||^2 \leq kR^2 \tag{11}$$

$$\tag{12}$$

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Update rule

Upper Bound

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Law of cosines

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$$||\vec{w}^{(k+1)}||^2 \leq ||\vec{w}^{(k)}||^2 + R^2 + 2w^{(k)} \cdot \Delta\vec{\Phi}(x,y,z) \tag{14}$$

Definition of diameter

Upper Bound

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$$||\vec{w}^{(k+1)}||^2 \leq ||\vec{w}^{(k)}||^2 + R^2 + 0 \tag{15}$$

If violation

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$$||\vec{w}^{(k+1)}||^2 \leq ||\vec{w}^{(k)}||^2 + R^2 + 0 \tag{15}$$

$$||\vec{w}^{(k+1)}||^2 \leq kR^2 \tag{16}$$

Induction!

Putting it together

- Sandwich:

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- What does this mean?

Putting it together

- Sandwich:

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- Solve for k:

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- What does this mean?
- Limited number of errors (updates)
    - Larger diameter increases errors (worst possible mistake)
    - Larger margin decreases errors (bigger separation from wrong answer)
- Finding the largest violation wrong answer is best (but any violation okay)

In Practice

Harder the search space, the more max violation helps