Classification: Rademacher Complexity

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LECTURE 6B

Slides adapted from Rob Schapire

Content Questions

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Administrivia Questions

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Single Hypothesis

What is the Rademacher complexity of a hypothesis set reduced to a single hypothesis?

(1)

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$$\mathscr{R}_m(H) = \mathbb{E}_{S\sim D^m,\sigma}\left[\sup_{h\in H}\frac{1}{m}\sum_{i=1}^{m}\sigma_i h(z_i)\right] \quad (1)$$

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$$(5)$$

Rademacher Identity 1

Prove

$$\mathscr{R}_m(\alpha H) = |\alpha|\mathscr{R}_m(H)$$

If $\alpha \geq 0$

If $\alpha < 0$

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If $\alpha \geq 0$

$$\sup_{h \in \alpha H} \sum_{i=1}^{m} \sigma_i h(x_i) = \quad (6)$$

$$\sup_{h \in H} \sum_{i=1}^{m} \alpha \sigma_i h(x_i) = \quad (7)$$

$$\alpha \sup_{h \in H} \sum_{i=1}^{m} \sigma_i h(x_i) \quad (8)$$

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If $\alpha < 0$

$$\sup_{h \in \alpha H} \sum_{i=1}^{m} \sigma_i h(x_i) = \quad (9)$$

$$\sup_{h \in H} \sum_{i=1}^{m} \alpha \sigma_i h(x_i) = \quad (10)$$

$$(-\alpha) \sup_{h \in H} \sum_{i=1}^{m} (-\sigma_i) h(x_i) \quad (11)$$

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Since σ_i and $-\sigma$ have the same distribution, $\mathscr{R}_m(\alpha H) = |\alpha|\mathscr{R}_m(H)$

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$$= \frac{1}{m}\mathbb{E}_{\vec{\sigma},S}\left[\sup_{h \in H, h' \in H'} \sum_{i=1}^{m} \sigma_i \left(h(x_i) + h'(x_i)\right)\right] \tag{13}$$

$$\tag{14}$$

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VC Dimension

To show VC dimension of a set of points

- Show that **a** set of d *can* be shattered
- Show that **no** set of $d+1$ can be shattered

Axis Aligned Rectangles

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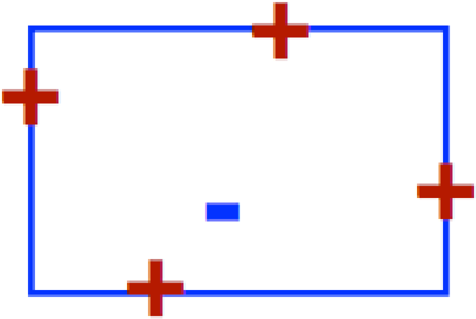

Hyperplanes

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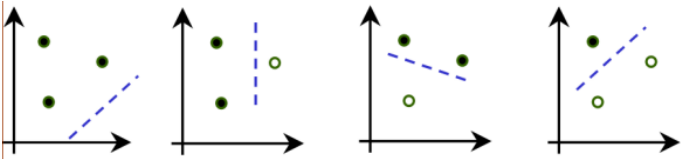

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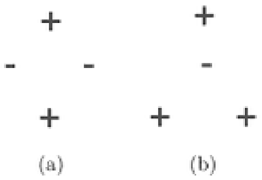

Figure 3.2 Unrealizable dichotomies for four points using hyperplanes in $\mathbb{R}^2$. (a) All four points lie on the convex hull. (b) Three points lie on the convex hull while the remaining point is interior.

Hyperplanes

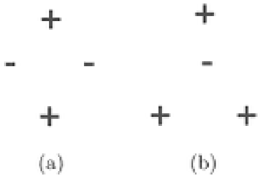

Figure 3.2 Unrealizable dichotomies for four points using hyperplanes in $\mathbb{R}^2$. (a) All four points lie on the convex hull. (b) Three points lie on the convex hull while the remaining point is interior.

In general, the VC dimension of d-dimensional hyperplanes is $d+1$

Finite Subsets

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- If a set has d points, there are 2^d ways to do that
- Each configuration requires a different hypothesis
- Solving for the number of hypotheses gives $\lg|H|$

Next time

- Getting more practical
- SVMs
- Excellent theoretical properties