# Classification: The PAC Learning Framework

Machine Learning: Jordan Boyd-Graber
University of Colorado Boulder

LECTURE 5B

# Content Questions

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# Quiz!

## Admin Questions

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## PAC Learnability: Rectangles

Is the hypothesis class of axis-aligned rectangles PAC learnable?

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A. Blumer, A. Ehrenfeucht, D. Haussler, and M.K. Warmuth. Learnability and the Vapnik-Chervonenkis dimension. Journal of the ACM (JACM), 36(4):929?965, 1989

# What's the learning algorithm

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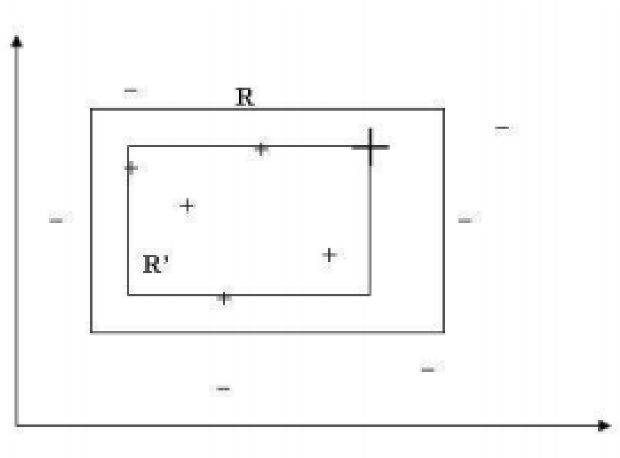


Call this $h_S$, which we learned from data. $h_s \in c$

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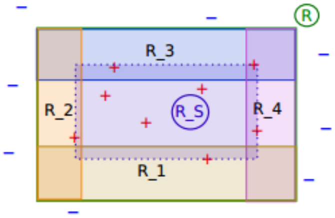

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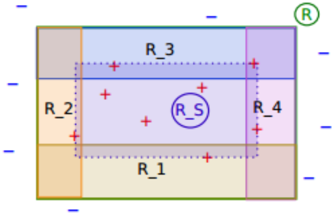


We get a bad $h_S$ only if we have an observation fall in this region. So let's bound this probability.

## Bounds

$$\Pr[error] = \Pr[\Cup_{i=1}^{4} x \notin R_i] \quad (1)$$

$$\leq \sum_{i=1}^{4} \Pr[x \notin R_i] \quad (2)$$

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If we assume that $P(R_i) \geq \frac{\epsilon}{4}$, then

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Solving for $m$ gives

$$m \geq \frac{4 \ln 4/\delta}{\epsilon} \quad (5)$$

Are Boolean conjunctions PAC learnable? Think of every feature as a Boolean variable; in a given example the variable is given the value 1 if its corresponding feature appears in the examples and 0 otherwise. In this way, if the number of measured features is $n$ the concept is represented as a Boolean function $c : \{0,1\} \longmapsto \{0,1\}$. For example we could define a chair as something that has four legs **and** you can sit on **and** is made of wood. Can you learn such a conjunction concept over $n$ variables?

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- After last example, $x_1 \bar{x}_3 \bar{x}_4$

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- Our algorithm can be two specific. It might not say yes when it should.
- We make an error on a literal if we've never seen it before (there are $2n$ literals: $x_1, \bar{x}_1$)

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Let $p(z)$ be the probability that our concept returns a positive example in which literal $z$ is false.

$$R(h) \leq \sum_{z} p(z) \tag{7}$$

A literal $z$ is bad if $p(z) \geq \frac{\epsilon}{2n}$.

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If $h$ has no bad literals, then $h$ will have error less than $\epsilon$.

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$$m \geq \frac{2n}{\epsilon}\left(\ln 2n + \ln \frac{1}{\delta}\right) \quad (8)$$

# 3-DNF

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Not efficiently learnable unless P = NP.