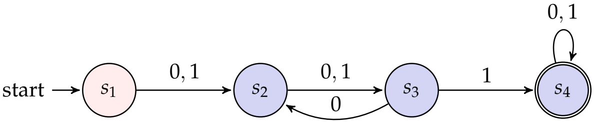
CSCI 3434: Theory of Computation

Lecture 5: Pumping Lemma

Ashutosh Trivedi



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In-Class Exercise (Time: 10 mins)

Find a DFA for the following languages:

- The set of strings having an equal number of 0's and 1's
- The set of strings with an equal number of occurrences of 01 and 10.

Some languages are not regular!

Let's do mental computations again.

- The language $\{0^n 1^n \ : \ n \geq 0\}$
- The set of strings having an equal number of 0's and 1's
- The language $\{ww \ : \ w \in \{0,1\}^*\}$
- The language $\{w\overline{w} \ : \ w \in \{0,1\}^*\}$
- The language $\{0^i 1^j \ : \ i > j\}$
- The language $\{0^i 1^j \ : \ i \leq j\}$
- The language of palindromes of $\{0,1\}$

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How do we prove that a language is not regular?

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Theorem (Pumping Lemma for Regular Languages)

For every regular language L there exists a constant p (that depends on L)

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Why?

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Why? Think: Regular expressions, DFAs

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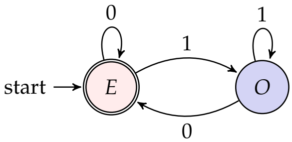
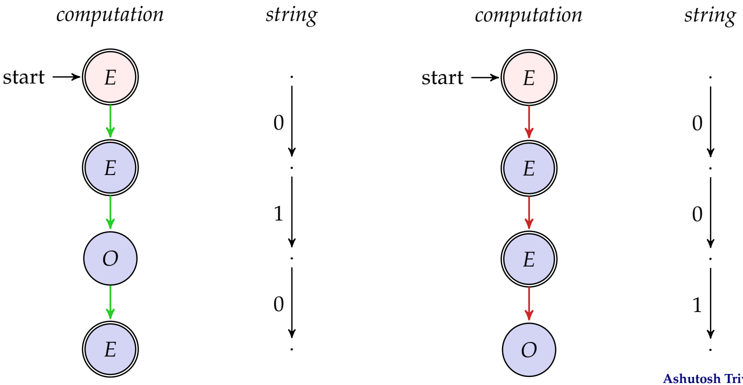
Why? Think: Regular expressions, DFAs Formalize our intuition!

If L is a regular language, then there exists a constant (pumping length) p such that for every string $w \in L$ s.t. $|w| \geq p$ there exists a division of w in strings x, y, and z s.t. $w = xyz$ such that

1. $|y| > 0$,
2. $|xy| \leq p$, and
3. *for all* $i \geq 0$ *we have that* $xy^i z \in L$.

A simple observation about DFA





A simple observation about DFA

Image source: Wikipedia

- Let $A = (S, \Sigma, \delta, s_0, F)$ be a DFA.
- For every string $w \in \Sigma^*$ of the length greater than or equal to the number of states of A, i.e. $|w| \geq |S|$, we have that
- the unique computation of A on w re-visits at least one state.

Pumping Lemma: Proof

Theorem (Pumping Lemma for Regular Languages)

If L is a regular language, then there exists a constant p such that for every string $w \in L$ s.t. $|w| \geq p$ there exists a division of w in strings $x, y,$ and z s.t. $w = xyz$ such that $|y| > 0$, $|xy| \leq p$, and for all $i \geq 0$ we have that $xy^iz \in L$.

Pumping Lemma: Proof

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Proof.

– Let A be the DFA accepting L and p be the set of states in A.

Pumping Lemma: Proof

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Proof.

- Let A be the DFA accepting L and p be the set of states in A.
- Let $w = (a_1a_2 \ldots a_k) \in L$ be any string of length $\geq p$.

Pumping Lemma: Proof

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Proof.

- Let A be the DFA accepting L and p be the set of states in A.
- Let $w = (a_1 a_2 \ldots a_k) \in L$ be any string of length $\geq p$.
- Let $s_0 \xrightarrow{a_1} s_1 \xrightarrow{a_2} s_2 \ldots \xrightarrow{a_k} s_k$ be the run of w on A.

Pumping Lemma: Proof

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- Let i be the index of first state that the run revisits and let j be the index of second occurrence of that state, i.e. $s_i = s_j$,
- Let $x = a_1a_2 \ldots a_i$ and $y = a_{i+1} \ldots a_j$, and $z = a_{j+1} \ldots a_k$.

Pumping Lemma: Proof

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Proof.

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- Let $w = (a_1a_2 \dots a_k) \in L$ be any string of length $\geq p$.
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- Let i be the index of first state that the run revisits and let j be the index of second occurrence of that state, i.e. $s_i = s_j$,
- Let $x = a_1a_2 \dots a_i$ and $y = a_{i+1} \dots a_j$, and $z = a_{j+1} \dots a_k$.
- notice that $|y| > 0$ and $|xy| \leq n$

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Proof.

- Let A be the DFA accepting L and p be the set of states in A.
- Let $w = (a_1a_2 \ldots a_k) \in L$ be any string of length $\geq p$.
- Let $s_0 \xrightarrow{a_1} s_1 \xrightarrow{a_2} s_2 \ldots \xrightarrow{a_k} s_k$ be the run of w on A.
- Let i be the index of first state that the run revisits and let j be the index of second occurrence of that state, i.e. $s_i = s_j$,
- Let $x = a_1a_2 \ldots a_i$ and $y = a_{i+1} \ldots a_j$, and $z = a_{j+1} \ldots a_k$.
- notice that $|y| > 0$ and $|xy| \leq n$
- Also, notice that for all $i \geq 0$ the string xy^iz is also in L.

□

Applying Pumping Lemma

Theorem (Pumping Lemma for Regular Languages)

$L \in \Sigma^*$ *is a regular language*
$\implies$
there exists $p \geq 1$ *such that*
for all strings $w \in L$ *with* $|w| \geq p$ *we have that*
there exists $x, y, z \in \Sigma^*$ *with* $w = xyz$*,* $|y| > 0$*,* $|xy| \leq p$ *such that*
for all $i \geq 0$ *we have that*
$xy^iz \in L$.

Applying Pumping Lemma

Theorem (Pumping Lemma for Regular Languages)

$L \in \Sigma^*$ is a regular language
$\Longrightarrow$
there exists $p \geq 1$ such that
for all strings $w \in L$ with $|w| \geq p$ we have that
there exists $x, y, z \in \Sigma^*$ with $w = xyz$, $|y| > 0$, $|xy| \leq p$ such that
for all $i \geq 0$ we have that
$xy^iz \in L$.

Pumping Lemma (Contrapositive)

For all $p \geq 1$ we have that
there exists a string $w \in L$ with $|w| \geq p$ such that
for all $x, y, z \in \Sigma^*$ with $w = xyz$, $|y| > 0$, $|xy| \leq p$ we have that
there exists $i \geq 0$ such that
$xy^iz \notin L$
$\Longrightarrow$
$L \in \Sigma^*$ is not a regular language.

Applying Pumping Lemma

Pumping Lemma (Contrapositive)

For all $p \geq 1$ *we have that*
there exists *a string* $w \in L$ *with* $|w| \geq p$ *such that*
for all $x, y, z \in \Sigma^*$ *with* $w = xyz$, $|y| > 0$, $|xy| \leq p$ *we have that*
there exists $i \geq 0$ *such that*
$xy^iz \notin L$
$\implies L \in \Sigma^*$ *is not a regular language.*

How to show that a language L is non-regular.

1. Let p be an arbitrary number (pumping length).
2. (Cleverly) Find a representative string w of L of size $\geq p$.
3. Try out all ways to break the string into xyz triplet satisfying that $|y| > 0$ and $|xy| \leq n$. If the step 3 was clever enough, there will be finitely many cases to consider.
4. For every triplet show that for some i the string xy^iz is not in L, and hence it yields contradiction with pumping lemma.

Applying Pumping Lemma I

Theorem

Prove that the language $L = \{0^n 1^n\}$ *is not regular.*

Applying Pumping Lemma I

Theorem

Prove that the language $L = \{0^n 1^n\}$ is not regular.

Proof.

1. State the contrapositive of Pumping lemma.
2. Let p be an arbitrary number.
3. Consider the string $0^p 1^p \in L$. Notice that $|0^p 1^p| \geq p$.

Applying Pumping Lemma I

Theorem

Prove that the language $L = \{0^n 1^n\}$ is not regular.

Proof.

1. State the contrapositive of Pumping lemma.
2. Let p be an arbitrary number.
3. Consider the string $0^p 1^p \in L$. Notice that $|0^p 1^p| \geq p$.
4. Only way to break this string in xyz triplets such that $|xy| \leq p$ and $y \neq \varepsilon$ is to choose $y = 0^k$ for some $1 \leq k \leq p$.

Applying Pumping Lemma I

Theorem

Prove that the language $L = \{0^n 1^n\}$ is not regular.

Proof.

1. State the contrapositive of Pumping lemma.
2. Let p be an arbitrary number.
3. Consider the string $0^p 1^p \in L$. Notice that $|0^p 1^p| \geq p$.
4. Only way to break this string in xyz triplets such that $|xy| \leq p$ and $y \neq \varepsilon$ is to choose $y = 0^k$ for some $1 \leq k \leq p$.
5. For each such triplet, there exists an i (say $i = 0$) such that $xy^i z \notin L$.
6. Hence L is non-regular.

□

Applying Pumping Lemma II

Theorem

Prove that the language

$$M = \{w \ : \ w \text{ has an equal number of } 0 \text{ and } 1\}$$

is not regular.

Applying Pumping Lemma II

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Prove that the language

$$M = \{w \,:\, w \text{ has an equal number of } 0 \text{ and } 1\}$$

is not regular.

Proof.

1. Let p be an arbitrary number.
2. Consider the string $0^p1^p \in L$. Notice that $|0^p1^p| \geq p$.

Applying Pumping Lemma II

Theorem

Prove that the language

$$M = \{w \ : \ w \text{ has an equal number of } 0 \text{ and } 1\}$$

is not regular.

Proof.

1. Let p be an arbitrary number.
2. Consider the string $0^p1^p \in L$. Notice that $|0^p1^p| \geq p$.
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Applying Pumping Lemma II

Theorem

Prove that the language

$$M = \{w \ : \ w \text{ has an equal number of } 0 \text{ and } 1\}$$

is not regular.

Proof.

1. Let p be an arbitrary number.
2. Consider the string $0^p1^p \in L$. Notice that $|0^p1^p| \geq p$.
3. Only way to break this string in xyz triplets such that $|xy| \leq p$ and $y \neq \varepsilon$ is to choose $y = 0^k$ for some $1 \leq k \leq p$.
4. For each such triplet, there exists an i (say $i = 0$) such that $xy^iz \notin L$.
5. Hence L is non-regular.

□

Applying Pumping Lemma III

Theorem

Prove that the language

$$M = \{1^{n^2} \ : \ n \geq 0\}$$

is not regular.

Applying Pumping Lemma III

Theorem

Prove that the language

$$M = \{1^{n^2} \ : \ n \geq 0\}$$

is not regular.

Proof.

1. Let p be an arbitrary number.
2. Consider the string $1^{p^2} \in L$. Notice that $|1^{p^2}| \geq p$.

Applying Pumping Lemma III

Theorem

Prove that the language

$$M = \{1^{n^2} \ : \ n \geq 0\}$$

is not regular.

Proof.

1. Let p be an arbitrary number.
2. Consider the string $1^{p^2} \in L$. Notice that $|1^{p^2}| \geq p$.
3. Only way to break this string in xyz triplets such that $|xy| \leq p$ and $y \neq \varepsilon$ is to choose $y = 1^k$ for some $1 \leq k \leq p$.

Applying Pumping Lemma III

Theorem

Prove that the language

$$M = \{1^{n^2} \ : \ n \geq 0\}$$

is not regular.

Proof.

1. Let p be an arbitrary number.
2. Consider the string $1^{p^2} \in L$. Notice that $|1^{p^2}| \geq p$.
3. Only way to break this string in xyz triplets such that $|xy| \leq p$ and $y \neq \varepsilon$ is to choose $y = 1^k$ for some $1 \leq k \leq p$.
4. Now consider $1^l 1^k 1^k 1^{p^2 - l + k}$ (pumping twice) and show that it is not perfect square.
5. Hence L is non-regular.

□

Applying Pumping Lemma IV

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Theorem

Prove that the language

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is not regular.

Proof.

1. Let p be an arbitrary number.
2. Consider the string $0^p 1^{p+1} \in L$. Notice that $|0^p 1^{p+1}| \geq p$.

Applying Pumping Lemma IV

Theorem

Prove that the language

$$M = \{0^i 1^j \ : \ i > j\}$$

is not regular.

Proof.

1. Let p be an arbitrary number.
2. Consider the string $0^p 1^{p+1} \in L$. Notice that $|0^p 1^{p+1}| \geq p$.
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Applying Pumping Lemma IV

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Prove that the language

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1. Let p be an arbitrary number.
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3. Only way to break this string in xyz triplets such that $|xy| \leq p$ and $y \neq \varepsilon$ is to choose $y = 0^k$ for some $1 \leq k \leq p$.
4. All pumping-ups are in the language!

Applying Pumping Lemma IV

Theorem

Prove that the language

$$M = \{0^i 1^j \ : \ i > j\}$$

is not regular.

Proof.

1. Let p be an arbitrary number.
2. Consider the string $0^p 1^{p+1} \in L$. Notice that $|0^p 1^{p+1}| \geq p$.
3. Only way to break this string in xyz triplets such that $|xy| \leq p$ and $y \neq \varepsilon$ is to choose $y = 0^k$ for some $1 \leq k \leq p$.
4. All pumping-ups are in the language!
5. Solution: pump-down.
6. Hence L is non-regular.

□

Proving a language Regular

Proving Regularity

Pumping Lemma is necessary but not sufficient condition for regularity.

Proving a language Regular

Proving Regularity

Pumping Lemma is necessary but not sufficient condition for regularity.

Consider the language

$$L = \{\#a^n b^n \ : \ n \geq 1\} \cup \{\#^k w \ : \ k \neq 1, w \in \{a, b\}^*\}.$$

Verify that this language satisfies the pumping condition, but is not regular!