CS 208: Automata Theory and Logic

Closure Properties for Regular Languages

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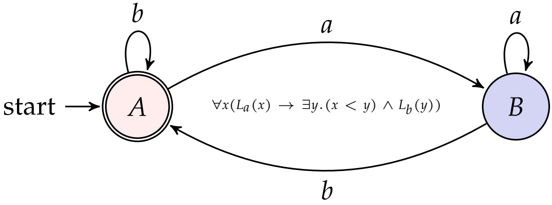

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5. Closure (Kleene Closure, or Star):
   $A^* = \{w_1 w_2 \ldots w_k : k \geq 0 \text{ and } w_i \in A\}$. In other words:

   $$A^* = \cup_{i \geq 0} A^i$$

   where $A^0 = \emptyset$, $A^1 = A$, $A^2 = AA$, and so on.

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Define the notion of a set being closed under an operation (say, $\mathbb{N}$ and $\times$).

Theorem

The class of regular languages is closed under union, intersection, complementation, concatenation, and Kleene closure.

Closure under Union

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Proof.

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- Let $M_1 = (S_1, \Sigma, \delta_1, s_1, F_1)$ and $M_2 = (S_2, \Sigma, \delta_2, s_2, F_2)$ be DFA for L_1 and L_2.

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- DFA Construction: (the Product Construction)
  We claim the DFA $M = (S_1 \times S_2, \Sigma, \delta, (s_1, s_2), F)$ where
  - $\delta((s_1, s_2), a) = (\delta_1(s_1, a), \delta_2(s_2, a))$ for all $s_1 \in S_1$, $s_2 \in S_2$, and $a \in \Sigma$,
  - $F = (F_1 \times S_2) \cup (S_1 \times F_2)$.

  accepts $L_1 \cup L_2$ i.e. $L(M) = L(M_1) \cup L(M_2)$.

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- $L_1 \cup L_2$ is regular since there is a DFA accepting this language.

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Closure under Union via RegEx

Lemma

The class of regular languages is closed under union.

Proof.

- Prove for arbitrary regular languages L_1 and L_2 that $L_1 \cup L_2$ is a regular languages.
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The class of regular languages is closed under complementation.

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- Prove for arbitrary regular language L that $\overline{L}$ is a regular languages.
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- DFA based via product construction,
- Using De Morgan's laws.

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- L^* is regular since there is a REGEX E^* accepting this language.

□

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- We can apply homomorphism to languages as well, for a homomorphism h and a language $L \subseteq \Sigma^*$ we define $h(L) \subseteq \Gamma^*$ as

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- We define inverse-homomorphism of a language $L \in \Gamma^*$ as

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Theorem

The class of regular languages is closed under homomorphism, and inverse-homomorphism.

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Lemma

The class of regular languages is closed under homomorphism.

Proof.

– Prove for arbitrary regular language L and homomorphism h that $h(L)$ is a regular languages. Let E be REGEX accepting L.

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– Prove for arbitrary regular language L and homomorphism h that $h(L)$ is a regular languages. Let E be REGEX accepting L.
– REGEX Construction: We claim the REGEX E_h defined inductively as

$$\begin{aligned} E_h &= \varepsilon && \text{if } E = \varepsilon \\ E_h &= \emptyset && \text{if } E = \emptyset \\ E_h &= h(a) && \text{if } E = a \\ E_h &= F_h + G_h && \text{if } E = F + G \\ E_h &= F_h . G_h && \text{if } E = F.G \\ E_h &= (F_h)^* && \text{if } E = F^* \end{aligned}$$

accepts $h(L)$, i.e. $L(E_h) = h(L(E))$.

□

Closure under Homomorphism

- Proof of correctness: Prove that $L(E_h) = h(L(E))$.
  - if $E = \varepsilon$, then

$$
\begin{aligned}
LHS &= L(E_h) = L(h(\varepsilon)) = L(\varepsilon) = \{\varepsilon\} \\
RHS &= h(L(E)) = h(L(\varepsilon)) = h(\{\varepsilon\}) = \{\varepsilon\}.
\end{aligned}
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  - Similarly for $E = \emptyset$.

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  - Similarly for $E = \emptyset$.
  - if $E = a$, then

$$\begin{aligned} LHS &= L(E_h) = L(h(a)) = \{h(a)\} \\ RHS &= h(L(E)) = h(L(a)) = h(\{a\}) = \{h(a)\}. \end{aligned}$$

  - if $E = F + G$, then

$$\begin{aligned} L(h(E)) &= L(h(F+G)) = L(h(F) + h(G)) = L(h(F)) \cup L(h(G)) \\ h(L(E)) &= h(L(F+G)) = h(L(F)) \cup h(L(G)). \end{aligned}$$

    From inductive hypothesis both of these expression are equal.
  - Other inductive cases are similar, and hence omitted.

Closure under Inverse-Homomorphism

Lemma

The class of regular languages is closed under homomorphism.

Proof.

Let $\mathcal{A} = (S, \Gamma, \delta, s_0, F)$ be a DFA accepting L and $h : \Sigma \to \Gamma^*$ be an arbitrary homomorphism. We show that the DFA $h^{-1}(\mathcal{A}) = (S', \Sigma, \delta', s'_0, F')$ defined below accepts $h^{-1}(L)$.

- $S' = S$, $s'_0 = s_0$, $F' = F$
- $\delta'(s, a) = \hat{\delta}(s, h(a))$

It is an easy induction over w that $\hat{\delta}'(s, w) = \hat{\delta}(s, h(w))$. Now, since accepting states of $\mathcal{A}$ and $h^{-1}(\mathcal{A})$ are the same, $h^{-1}(\mathcal{A})$ accepts w iff $\mathcal{A}$ accepts $h(w)$. □

Practice Questions

1. Quotient Language. For $a \in \Sigma$ and $L \subseteq \Sigma^*$ we define

$$\begin{aligned} L/a &= \{w \,:\, wa \in L\}. \\ a/L &= \{w \,:\, aw \in L\}. \\ L.a &= \{wa \,:\, w \in L\}. \\ a.L &= \{aw \,:\, w \in L\}. \end{aligned}$$

2. $\min(L)$ is the set of strings w such that $w \in L$ and no proper prefix of w is in L.
3. $\max(L)$ is the set of strings such that $w \in L$ and no proper extension $wx \in L$.
4. $\text{INIT}(L)$ is the set of strings w such that for some x we have that $wx \in L$.
5. $\text{HALF}(L)$ is the set of strings w such that for some string x of same size as w we have that $wx \in L$.