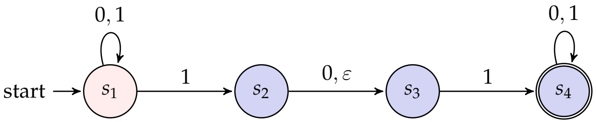
CSCI 3434: Theory of Computation

Lecture 3: Nondeterminism

Ashutosh Trivedi



Department of Computer Science
UNIVERSITY OF COLORADO BOULDER

Recursive Definitions and Structural Induction

Regular Languages: Nondeterminism

Recursive Definitions

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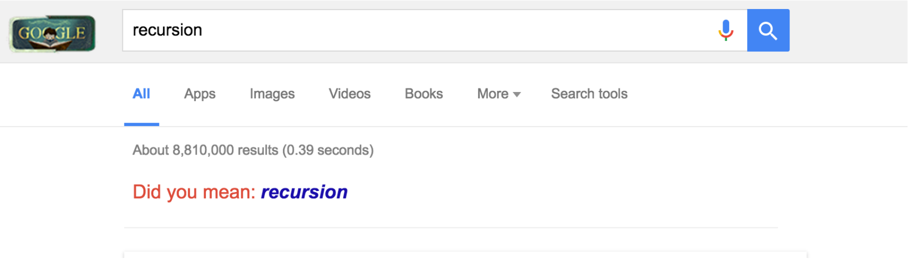

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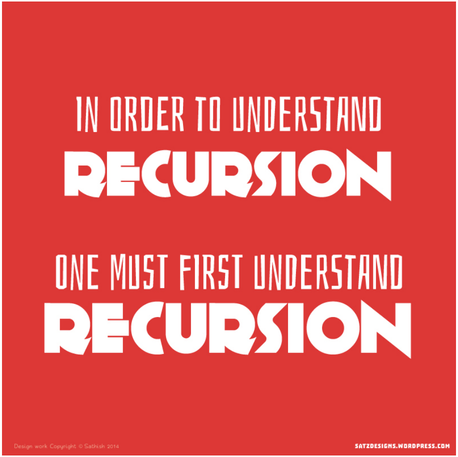

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2. Defining an object in terms of itself.

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- Definitions of the factorial function and Fibonacci sequence
- Definition of a singly-linked list or trees.

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- Every expression defined has an equal number of left and right parenthesis.
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- Other examples

Recursive Definitions and Structural Induction

Regular Languages: Nondeterminism

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- A language L over some alphabet Σ is a set of strings, i.e. $L \subseteq \Sigma^*$.
- Some examples:
  - $L_{\text{even}} = \{w \in \Sigma^* : w \text{ is of even length}\}$
  - $L_{a^*b^*} = \{w \in \Sigma^* : w \text{ is of the form } a^n b^m \text{ for } n, m \geq 0\}$
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- Deterministic finite state automata define languages that require finite resources (states) to recognize.

Definition (Regular Languages)

We call a language regular if it can be accepted by a deterministic finite state automaton.

Why they are "Regular"

- A number of widely different and equi-expressive formalisms precisely capture the same class of languages:
  - Deterministic finite state automata
  - Nondeterministic finite state automata (also with ε-transitions)
  - Kleene's regular expressions, also appeared as Type-3 languages in Chomsky's hierarchy [Cho59].
  - Monadic second-order logic definable languages [B60̈, Elg61, Tra62]
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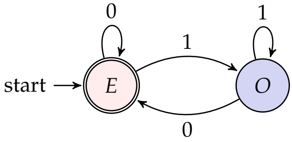
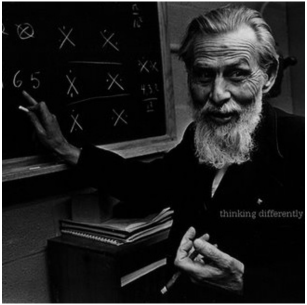
Today we show that:

Theorem (DFA=NFA=ε-NFA)

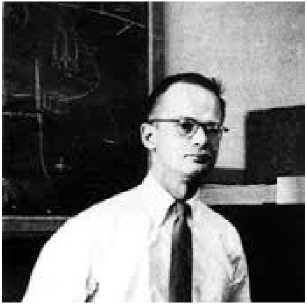
A language is accepted by a deterministic finite automaton if and only if it is accepted by a non-deterministic finite automaton.

Finite State Automata



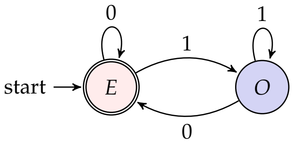


Warren S. McCullough



Walter Pitts

Deterministic Finite State Automata (DFA)



A finite state automaton is a tuple $\mathcal{A} = (S, \Sigma, \delta, s_0, F)$, where:

- S is a finite set called the states;
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- $\delta : S \times \Sigma \to S$ is the transition function;
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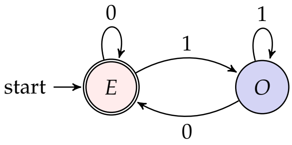

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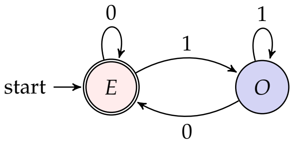

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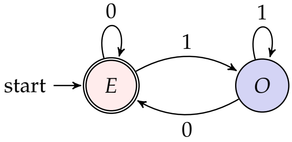

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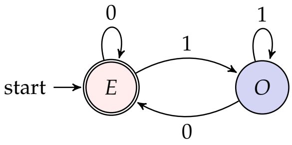
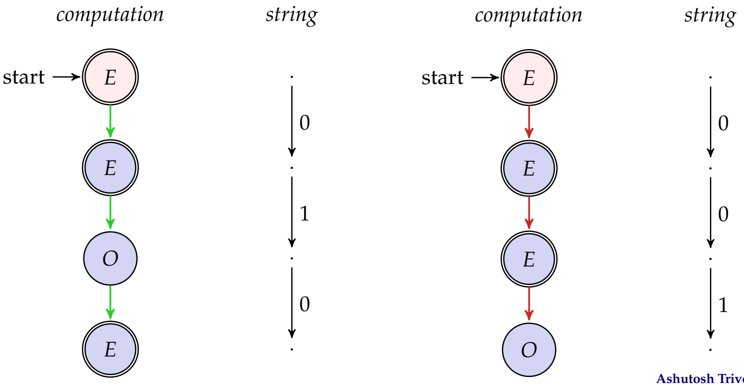
Computation or Run of a DFA





Deterministic Finite State Automata

Semantics using extended transition function:

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Semantics using accepting computation:

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$$s_0, a_1 s_1, a_2, \ldots, a_{n-1}, s_n$$

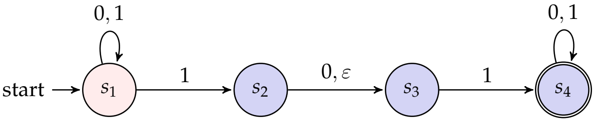
  where s_0 is the starting state, and $\delta(s_{i-1}, a_i) = s_{i+1}$.
- A string w is accepted by a DFA $\mathcal{A}$ if the last state of the unique computation of $\mathcal{A}$ on w is an accept state, i.e. $s_n \in F$.
- Language of a DFA $\mathcal{A}$

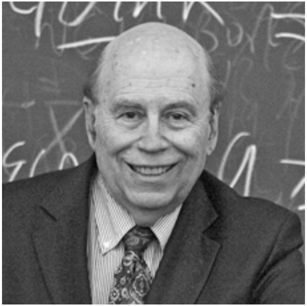
$$L(\mathcal{A}) = \{w \,:\, \text{string } w \text{ is accepted by DFA } \mathcal{A}\}.$$

Proposition

Both semantics define the same language. *Proof by induction.*

Nondeterministic Finite State Automata

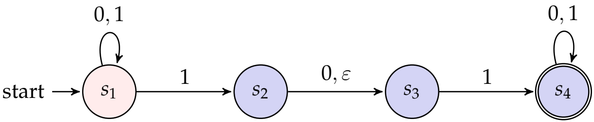




Michael O. Rabin

Dana Scott

Non-deterministic Finite State Automata



A non-deterministic finite state automaton (NFA) is a tuple $\mathcal{A} = (S, \Sigma, \delta, s_0, F)$, where:

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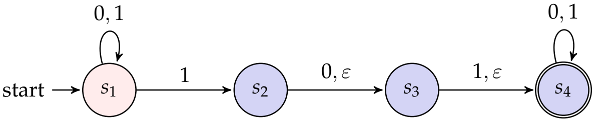

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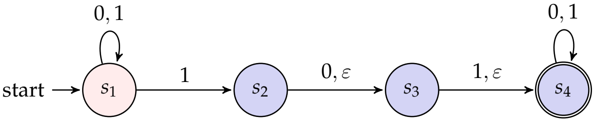

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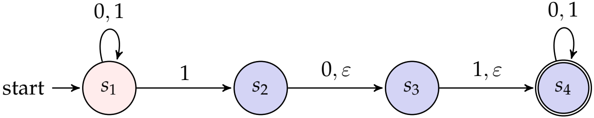

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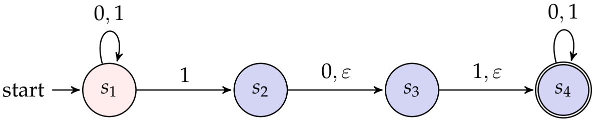

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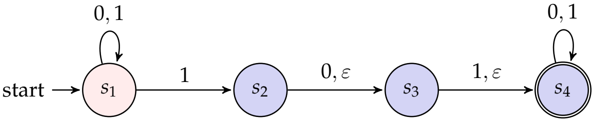

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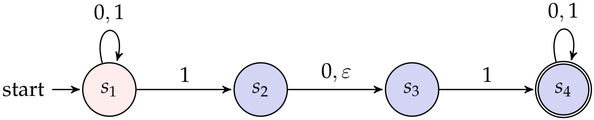
Non-deterministic Finite State Automata



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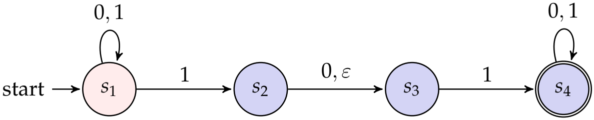

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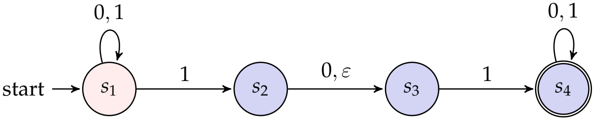

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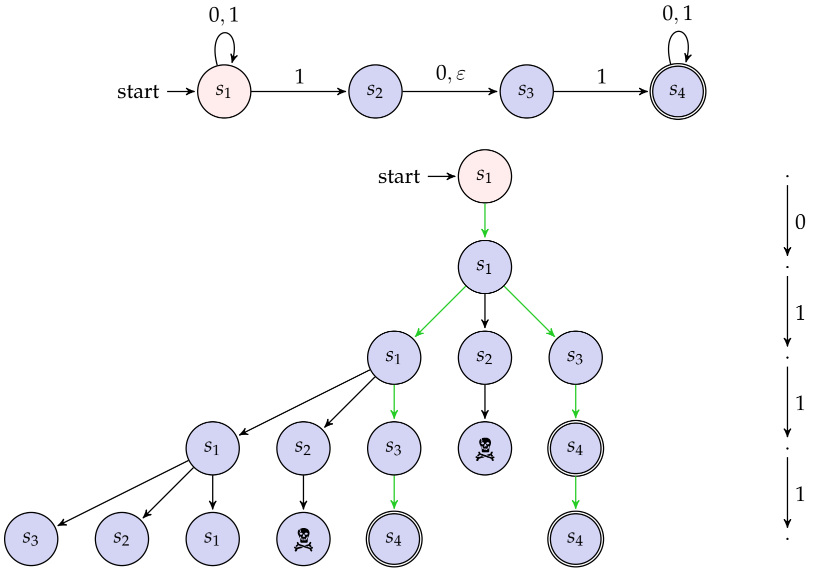
Computation or Run of an NFA



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- A computation or a run of a NFA on a string $w = a_0 a_1 \dots a_{n-1}$ is a finite sequence

$$s_0, r_1, s_1, r_2, \dots, r_{k-1}, s_n$$

  where s_0 is the starting state, and $s_{i+1} \in \delta(s_{i-1}, r_i)$ and $r_0 r_1 \dots r_{k-1} = a_0 a_1 \dots a_{n-1}$.
- A string w is accepted by an NFA $\mathcal{A}$ if the last state of some computation of $\mathcal{A}$ on w is an accept state $s_n \in F$.
- Language of an NFA $\mathcal{A}$

$$L(\mathcal{A}) = \{w \,:\, \text{string } w \text{ is accepted by NFA } \mathcal{A}\}.$$

Proposition

Both semantics define the same language. *Proof by induction.*

Why study NFA?

NFA are often more convenient to design than DFA, e.g.:

- $\{w : w$ contains 1 in the third last position$\}$.
- $\{w :: w$ is a multiple of 2 or a multiple of 3$\}$.
- Union and intersection of two DFAs as an NFA
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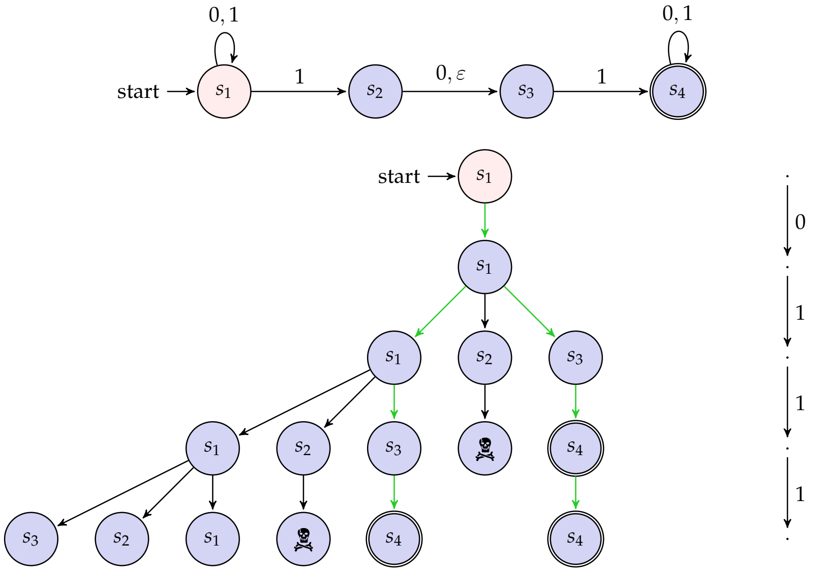
And, surprisingly perhaps:

Theorem (DFA=NFA)

Every non-deterministic finite automaton has an equivalent (accepting the same language) deterministic finite automaton. *Subset construction.*

Computation of an NFA: An observation



ε-free NFA = DFA

Let $\mathcal{A} = (S, \Sigma, \delta, s_0, F)$ be an ε-free NFA. Consider the DFA $Det(\mathcal{A}) = (S', \Sigma', \delta', s_0', F')$ where

- $S' = 2^S$,
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Theorem (ε-free NFA = DFA)

$L(\mathcal{A}) = L(Det(\mathcal{A}))$.

By induction, hint $\hat{\delta}(s_0, w) = \hat{\delta}'(\{s_0\}, w)$.

Proof of correctness: $L(\mathcal{A}) = L(Det(\mathcal{A}))$.

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– Base case: Let w be ε. The base case follows immediately from the definition of extended transition functions:

$$\hat{\delta}(s_0, \varepsilon) = s_0 \text{ and } \hat{\delta}'(\{s_0\}, \varepsilon) = \{s_0\}.$$

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$$\hat{\delta}(s_0, xa) = \bigcup_{s \in \hat{\delta}(s_0, x)} \delta(s, a), \text{by definition of } \hat{\delta}.$$

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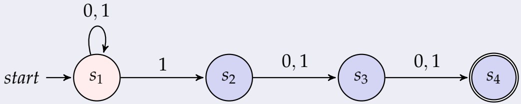
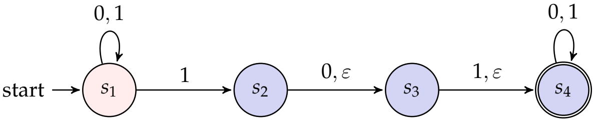
Equivalence of NFA and DFA

Exercise (In class)

Determinize the following automaton:



NFA with ε transitions = DFA



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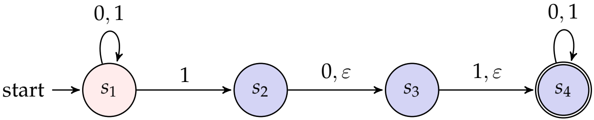
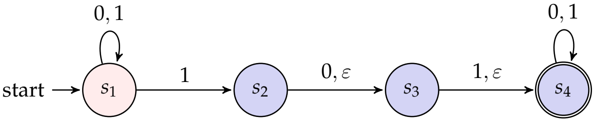


– ε-closure ECLOS(s) of a state s is the set of states that can be reached from s (including itself) via ε-transitions. E.g.

NFA with ε transitions = DFA



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$$\textsc{Eclos}(s_2) = \{s_2, s_3, s_4\} \text{ and } \textsc{Eclos}(s_3) = \{s_3, s_4\}$$

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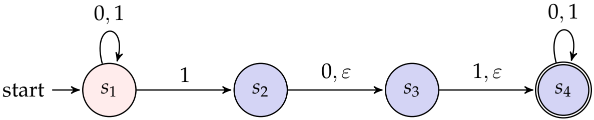

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- $\textsc{Eclos}(R) = \cup_{s \in R} \textsc{Eclos}(R)$. E.g.

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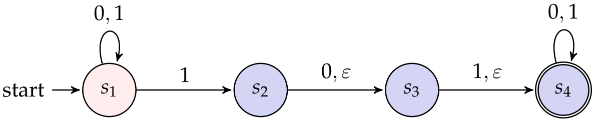

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$$\text{ECLOS}(\{s_1, s_2\}) = \{s_1, s_2, s_3, s_4\}$$

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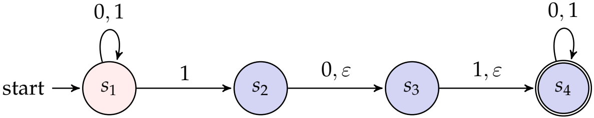

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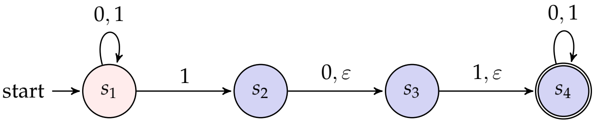

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Theorem (NFA with ε-transitions = DFA)

$L(\mathcal{A}) = L(Det(\mathcal{A}))$. *By induction, hint $\hat{\delta}(s_0, w) = \hat{\delta}'(\{s_0\}, w)$.*

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