CSCI 3434: Theory of Computation

Lecture 02: Regular Languages and Finite Automata

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Alphabet, Strings, and Languages

- An alphabet $\Sigma = \{a, b, c\}$ is a finite set of letters/symbols.
- A string over an alphabet Σ is finite sequence of symbols, e.g.
  - sequences $cab, baa,$ and aaa are some strings over $\Sigma = \{a, b, c\}$
  - sequences $\epsilon, 0, 1, 00,$ and 01 are some strings over $\Sigma = \{0, 1\}$
- Σ^* is the set of all strings over Σ, e.g. $aabbaa \in \Sigma^*$,
- Naturally, A language L is a collection/set of strings over some alphabet, i.e. $L \subseteq \Sigma^*$ e.g.,
  - $L_{even} = \{w \in \Sigma^* \ : \ w$ is of even length$\}$
  - $L_{\{a^n b^n\}} = \{w \in \Sigma^* \ : w$ is of the form $a^n b^n$ for $n \geq 0\}$

Programs to Accept Languages

- We say that a program accepts a language, if for every string input, it returns yes, if the string is in the language and no otherwise.
- Let's write programs to accept languages L_{even} and $L_{\{a^n b^n\}}$.
- Let's consider more fancy languages:
  - Language of all three colorable graphs
  - Language of all prime numbers
  - Language of all sorted arrays
  - Language of all true theorems in a given logic
- Can we write programs to accept all the languages?
  - Decidable and Undecidable Languages

Existence of undecidable languages.

Proof.

1. Set of all programs over some instruction:
   - $P = \{\epsilon, 0, 1, 00, 01, 10, 11, \dots\}$ over binary alphabet $\Sigma = \{0, 1\}$.
2. Set of all strings over alphabet $\Sigma = \{0, 1\}$:
   - $S = \{\epsilon, 0, 1, 00, 01, 10, 11, \dots\}$
3. Set of all languages over alphabet $\Sigma = \{0, 1\}$ are 2^S e,g,
   - $L_1 = \{0, 1\}$, $L_2 = \{00, 11, 11, 10\}$, $L_3 = \emptyset$
4. From *Cantor's theorem*, we know that $|2^S| > |S|$.
5. There must be some languages for which one can not write acceptor programs, i.e. undecidable languages.

Programming Exercise-I

String Matching Problem $MATCH(A, B)$

Input: Lists of strings $A = \langle s_1, s_2, \dots, s_n \rangle$ and $B = \langle t_1, t_2, \dots, t_n \rangle$ of equal length.

Output: YES if there is a sequence of combining elements that produces same string for both lists. NO, otherwise.

Formally, decide whether there exists a finite sequence (of any length)

$$1 \leq i_1, i_2, \dots, i_m \leq n$$

such that

$$s_{i_1} s_{i_2} \dots s_{\mathrm{i_n}} = t_{i_1} t_{i_2} \dots t_{i_n}.$$

Programming Exercise-I

Example 1:

Input: $A = \langle 110, 0011, 0110 \rangle$ and $B = \langle 110110, 00, 110 \rangle$.

Output: YES.

Since sequence 2, 3, 1 gives the same strings

$$s_2 s_3 s_1 = 00110110110 \text{ and } t_2 t_3 t_1 = 00110110110$$

Example 2:

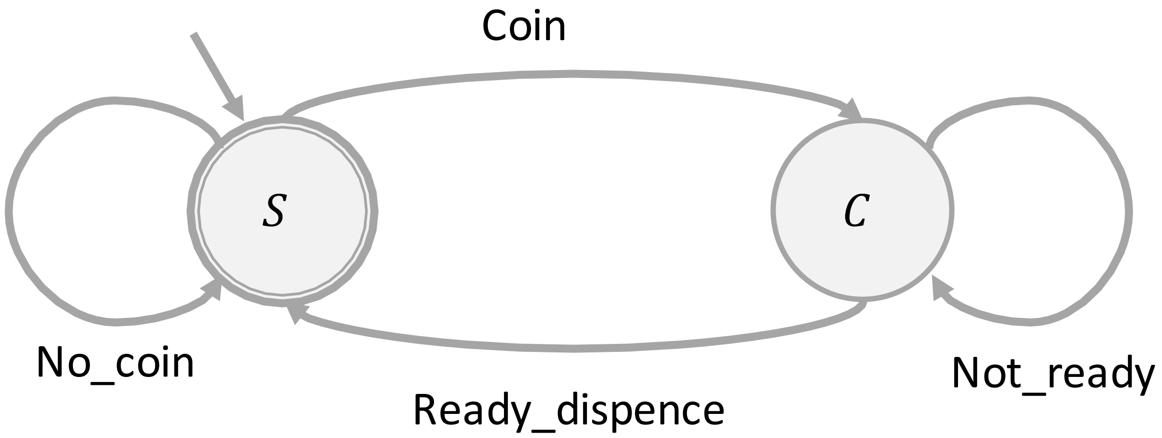
$A = \langle 0011, 11, 1101 \rangle$ and $B = \langle 101, 011, 110 \rangle$.

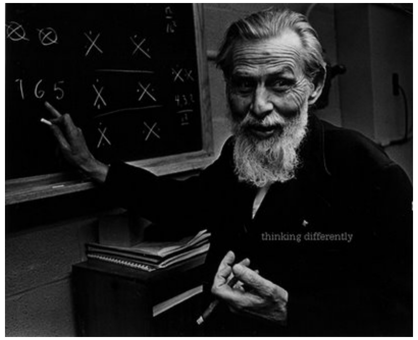
Example 3:

$A = \langle 100, 0, 1 \rangle$ and $B = \langle 1, 100, 0 \rangle$.

Finite State Automata

Finite State Automata





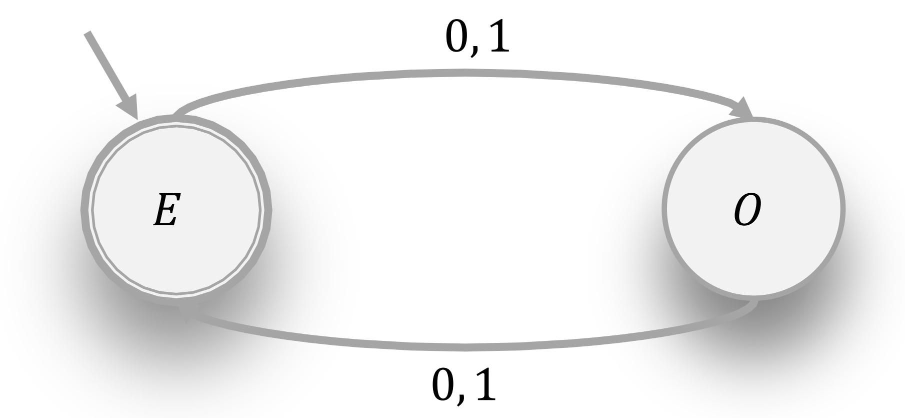
- Introduced first by two neuro-psychologists Warren S. McCullough and Walter Pitts in 1943 as a model for human brain.
- Finite automata can naturally model microprocessors and even software programs working on variables with bounded domain
- Capture so-called regular languages that occur in many different fields (regular expression, monadic second-order logic, algebra)
- Nice theoretical properties
- Applications in digital circuit/protocol verification, compilers, pattern recognition, etc.

Calculemus!

- Let us observe our mental process while we compute the following:
  - Recognize language of strings of an even length.
  - Recognize language of binary strings with an even number of 0's.
  - Recognize language of binary strings with an odd number of 0's.
  - Recognize language of strings containing your identikey.
  - Recognize language of binary (decimal) strings multiple of 2.
  - Recognize language of binary (decimal) strings multiple of 3.
  - Recognize language of binary strings with equal number of 0's and 1's.
  - Recognize language of binary strings of the form $0^n 1^n$
  - Recognize language of binary strings with a prime number of 1's

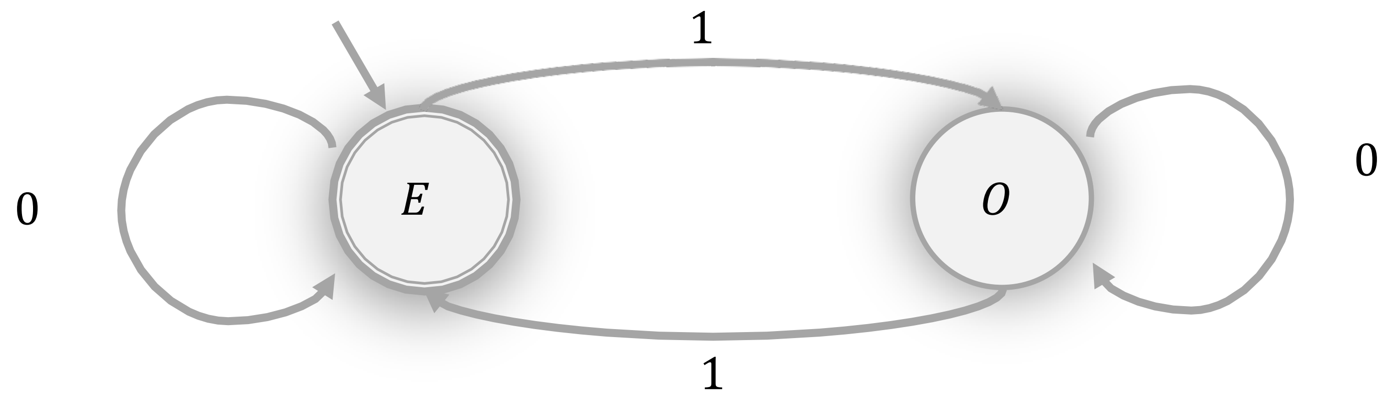
Finite State Automata: Examples

1. Automaton accepting strings of even length:



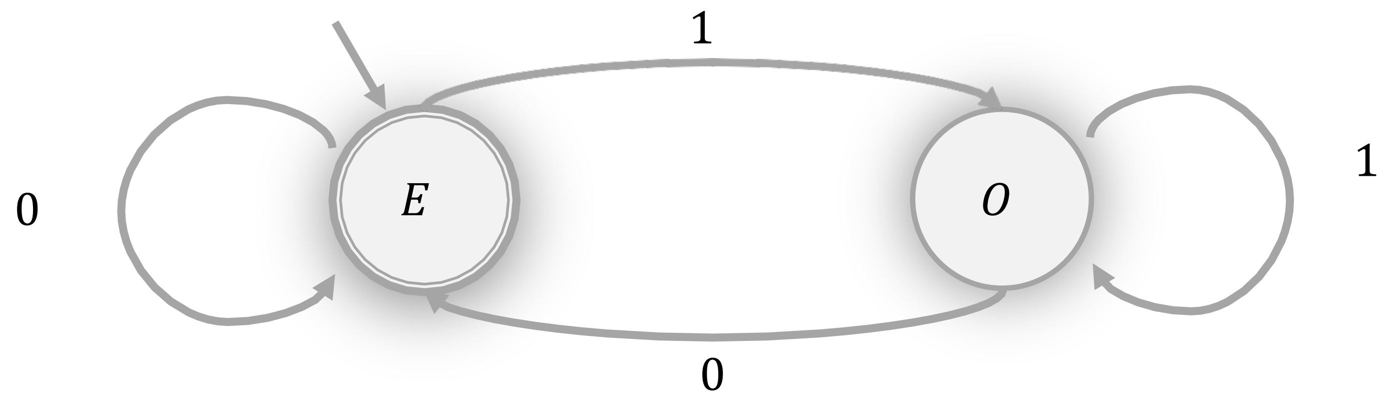
Finite State Automata: Examples

2. Automaton accepting strings with an even number of 1’s:

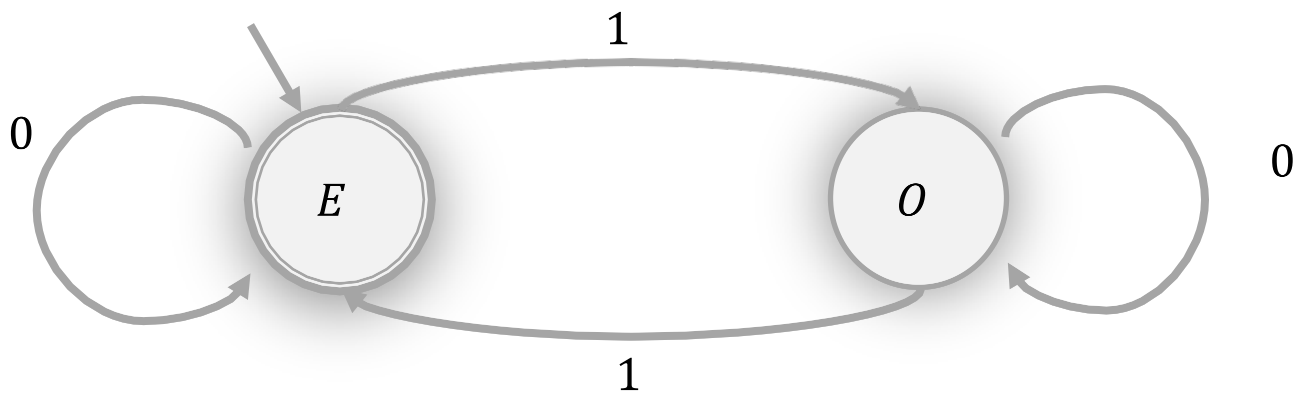


Finite State Automata: Examples

3. Automaton accepting binary strings characterizing an even number:



Finite State Automata: Definition



A finite state automaton is a tuple $(S, \Sigma, \delta, s_0, F)$, where:

- S is a finite set called the states,
- Σ is a finite set called the alphabet,
- $\delta : S \times \Sigma \to S$ is the transition function,
- $s_0 \in S$ is the start state, and
- $F \subseteq S$ is the set of accept states.

Example: The automaton in the figure above can be represented as $(S, \Sigma, \delta, s_0, F)$, where

- $S = \{E, O\}$, $\Sigma = \{0,1\}$, $s_0 = E$, $F = \{E\}$,
- and transition function δ is such that
  - $\delta(E, 0) = E$,
  - $\delta(E, 1) = O$, and
  - $\delta(O, 0) = O$,
  - $\delta(O, 1) = E$.

State Diagram

Let's draw the state diagram of the following automaton $(S, \Sigma, \delta, s_1, F)$:

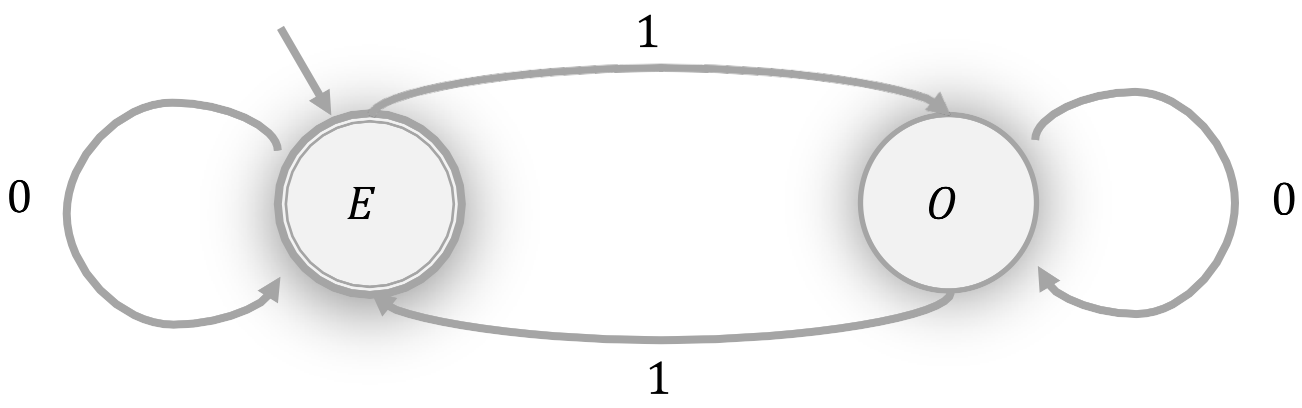
- $S = \{s_1, s_2, s_3\}$
- $\Sigma = \{0,1\}$,
- δ is given in a tabular form below:

S	0	1
s_1	s_1	s_2
s_2	s_3	s_2
s_3	s_2	s_2

- s_1 is the initial state, and
- $F = \{s_2\}$.

Which language does it accept?

Semantics (Meaning) of Finite State Automata



A finite state automaton (DFA) is a tuple $(S, \Sigma, \delta, s_0, F)$, where:

- S is a finite set called the states,
- Σ is a finite set called the alphabet,
- $\delta : S \times \Sigma \to S$ is the transition function,
- $s_0 \in S$ is the start state, and
- $F \subseteq S$ is the set of accept states.

- A computation or a run of a DFA on a string $w = a_0 a_1 \ldots a_{n-1}$ is the finite sequence
$$s_0, a_1, s_1, a_2, \ldots, a_{n-1}, s_n$$
where $s0$ is the starting state, and $\delta(s_{i-1}, a_i) = s_{i+1}$.
- A run is accepting if the last state of the unique computation is an accept state, i.e. $s_n \in F$.
- The Language of a DFA A
$$L(A) = \{w : \text{the unique run of } A \text{ on } w \text{ is accepting}\}.$$
- A language is called regular if it is accepted by a finite state automata.

Examples

- Recognize language of strings of an even length.
- Recognize language of binary strings with an even number of 0’s.
- Recognize language of binary strings with an odd number of 0’s.
- Recognize language of strings containing your identikey.
- Recognize language of binary (decimal) strings multiple of 2.
- Recognize language of binary (decimal) strings multiple of 3.
- Recognize language of binary strings with equal number of 0’s and 1’s. ✖
- Recognize language of binary strings of the form 0^n1^n ✖
- Recognize language of binary strings with a prime number of 1’s ✖
- Recognize language of binary strings that end with a 0.
- Recognize language of binary strings that begin with a 1.

Properties of Regular Languages

- Let A and B be languages (remember they are sets). We define the following operations on them:
  - Union: $A \cup B = \{w : w \in A \text{ or } w \in B\}$
  - Concatenation: $AB = \{wv : w \in A \text{ and } v \in B\}$
  - Closure (Kleene Closure, or Star):

    $\mathrm{A}^* = \{w_1 w_2 \ldots w_k : k \geq 0 \text{ and } w_i \in A\}$.

    Or, $A^* = \cup_{i \geq 0} A_i$ where $A_0 = \emptyset, A_1 = A, A_2 = AA$, and so on.
- Define the notion of a set being closed under an operation (say, $\mathbb{N}$ and $\times$).

Theorem. Regular languages are closed under union, intersection, concatenation, and Kleene star.