Fooling Sets and Introduction to Non-deterministic Finite Automata

Lecture 6

Proving that a language is not regular

- Given a language, we saw how to prove it is regular (union, intersection, concatenation, complement, reversal…)

- How to prove it is not regular?

Proving that a language is not regular

- Pick your favorite language L (= let L be an arbitrary language)

- For any strings x,y (x,y not necessarily in L) we define the following equivalence:

$$\boxed{x \equiv_L y}$$

- Means for EVERY string $z \in \Sigma^*$ we have

$$\boxed{xz \in L \text{ if and only if } yz \in L}$$

Proving that a language is not regular

- Conversely,

$$\boxed{x \not\equiv_L y}$$

- Means for SOME string $z \in \Sigma^*$ we have

either $xz \in L$ and $yz \notin L$

or $xz \notin L$ and $yz \in L$

We say z distinguishes x from y

(take z, glue it to x and y and see what belongs to L)

Example

- Pick your favorite language
- e.g. L = {strings with even zeroes and odd ones}
- Pick x= 0011 and y = 01. None of them in L!
- Can we find distinguishing suffix z?

z=1:

xz=00111 in L

yz =011 not in L

z=0:

xz=00110 not in L

yz =010 in L

z=ε:

xz=0011 not in L

yz =01 not in L

Example

- L = {strings with even zeroes and odd ones}
- Pick x= 0011 and y = 01. None of them in L!
- Can we find distinguishing suffix z?

Bad choice for z!

z=1:

xz=00111 in L

yz =011 not in L

z=0:

xz=00110 not in L

yz =010 in L

z=ε:

xz=0011 not in L

yz =01 not in L

Why do I care?

- I can learn something about the equivalence relation by looking at every DFA that accepts L.

- Assume that after the DFA reads x and y it ends up at the same state:

$$\delta^*(s,x) = \delta^*(s,y) \Rightarrow x \equiv_L y$$

Proof: For any z,

$$\delta^*(s,xz) = \delta^*(s,yz) \Rightarrow$$
$$\delta^*(s,xz) \in A \Leftrightarrow \delta^*(s,yz) \in A$$

Why do I care?

- This implication can be turned around:

In ANY DFA for L

$$x \not\equiv y \Rightarrow \delta^*(s,x) \neq \delta^*(s,y)$$

$$\Rightarrow |Q| \geq 2$$

- For the example before, we found two strings not equivalent.
- Any DFA for the language has AT LEAST two distinct states!
- Kind of trivial, cause what DFA has only one state?

Why do I care?

- Pushing it further:

If we can find k strings $x_1, \cdots, x_k$ such that

$$x_i \not\equiv x_j \qquad \forall i \neq j$$

Then, any DFA for L has at least k states

A way of formally proving how “complicated” a language is if it is regular

Our Example

- L = {strings with even zeroes and odd ones}

x_1=00

x_2=01

x_3=001

x_4=000

Our Example

- L = {strings with even zeroes and odd ones}

x_1=00

x_2=01

x_3=001

x_4=000

z=01

x_1z=0001 not in L

x_2z =00001 in L

Our Example

- L = {strings with even zeroes and odd ones}

x_1=00

x_2=01

x_3=001

x_4=000

z=?

Our Example

- L = {strings with even zeroes and odd ones}

x_1=00

x_2=01

x_3=001

x_4=000

z=1

Our Example

- L = {strings with even zeroes and odd ones}

x_1=00

x_2=01

x_3=001

x_4=000

Any DFA for L has AT LEAST 4 states!

What is a DFA for L?

Our Example

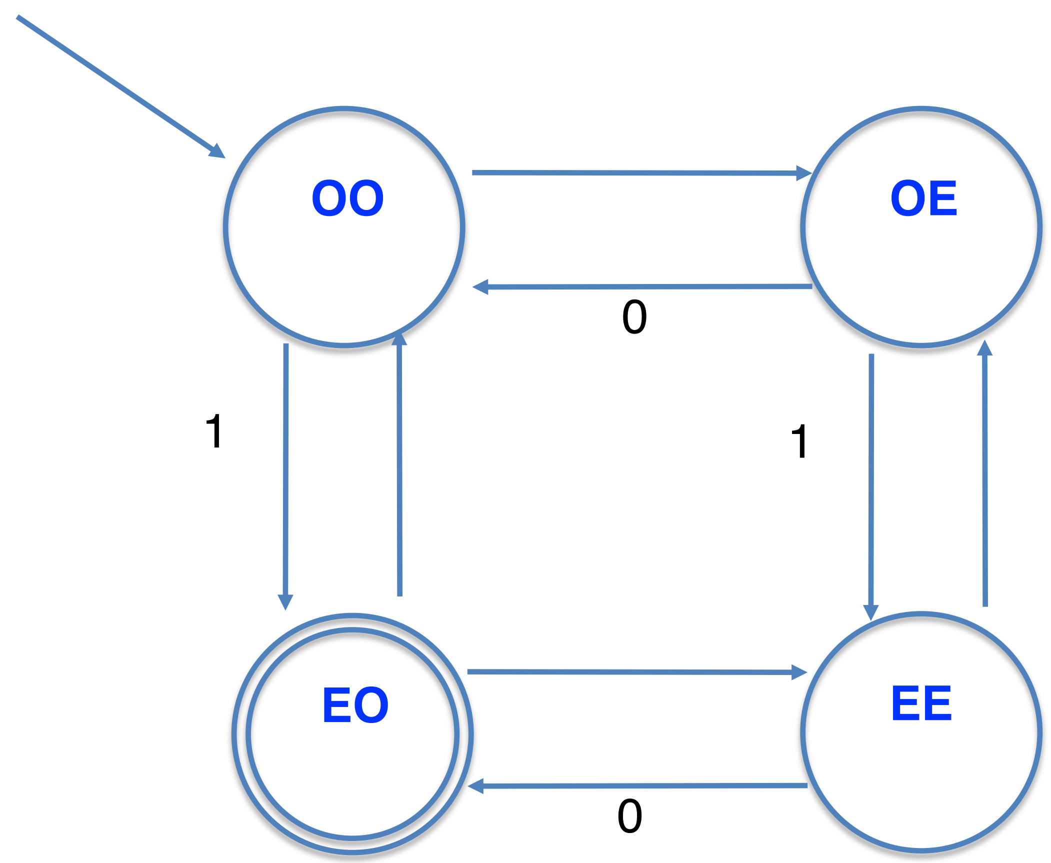
- L = {strings with even zeroes and odd ones}

x_1=00

x_2=01

x_3=001

x_4=000



We proved that this (obvious) DFA is the minimal one!!!

Our Example

- L = {strings with even zeroes and odd ones}

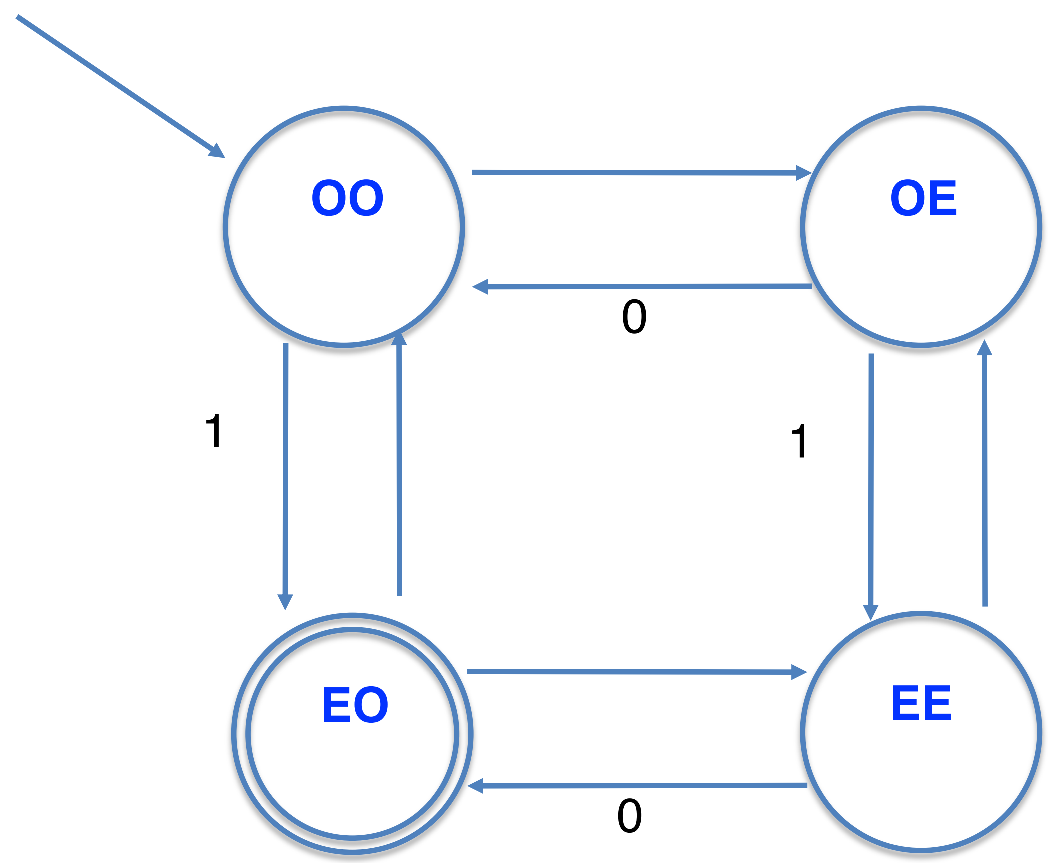
x_1=00

x_2=01

x_3=001

x_4=000

Fooling set.



Proving that a language is not regular

- Suppose I can find an infinite fooling set for L.
- Infinite set of strings $\{x_1, x_2, \ldots\}$ such that

$$x_i \not\equiv x_j \qquad \forall i \neq j$$

- Then every DFA for L has at least infinite number of distinct states
- L not regular!

Proving that a language is not regular

- Example: L={0^n1^n| $n \geq 0$ }= {ε,01,0011,…}

- Claim: This is a fooling set: F={0^n| $n \geq 0$ }

Proof: Let x, y two arbitrary different strings in F.

Therefore x $\neq$ y.

Proving that a language is not regular

- Example: L={$0^n 1^n$| $n \geq 0$ }= {ε,01,0011,…}
- Claim: This is a fooling set: F={0^n| $n \geq 0$ }

Proof: Let x, y two arbitrary different strings in F.

x=0^i for some integer i

y=0^j for some different integer j

z= 1^i

Therefore x $\neq$ y.

Proving that a language is not regular

- Example: L={0^n1^n| $n \geq 0$ }= {ε,01,0011,…}
- Claim: This is a fooling set: F={0^n| $n \geq 0$ }

Proof: Let x, y two arbitrary different strings in F.

x=0^i for some integer i

y=0^j for some different integer j

z= 1^i

xz=0^i 1^i in L

yz=0^j 1^i not in L

Therefore x $\not\equiv$ y.

Proving that a language is not regular

- To prove that L is not Regular:
  - Find some infinite set F
  - Prove for any two strings x and y in F there is a string z such that xz is in L XOR yz is in L.
- How to come up with those fooling sets?
- Be clever :)
- Think of what information you have to keep track of in a DFA for L.

What to keep track of?

- Example: $L=\{0^n1^n\}= \{\varepsilon,01,0011,\ldots\}$
- Is a string in L? What do I have to keep track of?
- I need to keep track of the number of zeroes.
- So, every number of zeroes is intuitively a different state (different equivalence class).
- Fooling set is a set of strings that exercises all possible values that I need to keep track in my head.
- Sometimes easier to narrow it down.

What to keep track of?

- Another Example: $L=\{ww^R | w \in \Sigma^* \}$= even length palindromes

- What is a fooling set?

- I have to remember the whole string w.

Attempt 1:

$F=\Sigma^*$

$x=0000$

$y=00$

Attempt 2:

$F=\{?\}$

What to keep track of?

- Another Example: $L=\{ww^R | w \in \Sigma^* \}$= even length palindromes

- What is a fooling set?

- I have to remember the whole string w.

Attempt 1:

$F=\Sigma^*$

$x=0000$

$y=00$

Attempt 2:

$F=0^*1$

$x=0^i1$

$y=0^j1$

What to keep track of?

- Another Example: $L=\{ww^R | w \in \Sigma^* \}$= even length palindromes
- What is a fooling set?
- I have to remember the whole string w.

$F=0^*1$

$x=0^i1$

$y=0^j1$

What z (exercise)?

What to keep track of?

- Another Example: $L=\{ww^R | w \in \Sigma^* \}$= even length palindromes
- What is a fooling set?
- I have to remember the whole string w.

$F=0^*1$

$x=0^i1$

$y=0^j1$

$z= 10^i$

What to keep track of?

- Another Example: $L=\{w|w=w^R\}$= all palindromes
- What is a fooling set?

$F=0^*1$

$x=0^i1$

$y=0^j1$

$z= 10^i$

What to keep track of?

- Another Example: $L=\{w|w=w^R\}$= all palindromes
- What is a fooling set : SAME!

$F=0^*1$

$x=0^i1$

$y=0^j1$

$z= 10^i$

What to keep track of?

- Another Example: $L=\{w|w=w^R\}$= all palindromes over the alphabet {0,1,a,b,c,d,e,f}
- What is a fooling set : SAME!

$F=0^*1$

$x=0^i1$

$y=0^j1$

$z= 10^i$

Proving that a language is not regular

Language is regular if and only if there is no infinite fooling set.

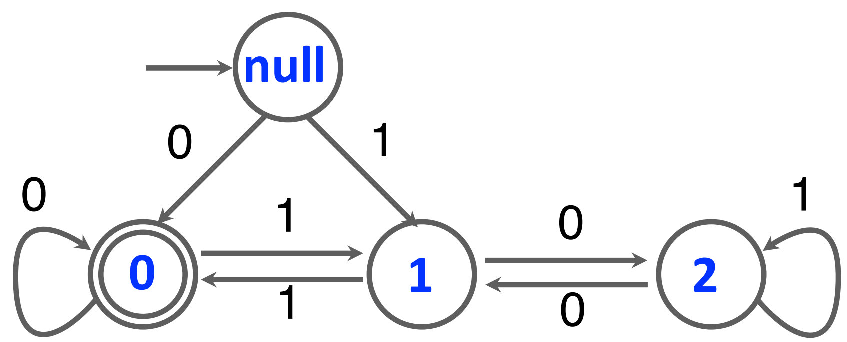
Nondeterminism

- Aka Magic.

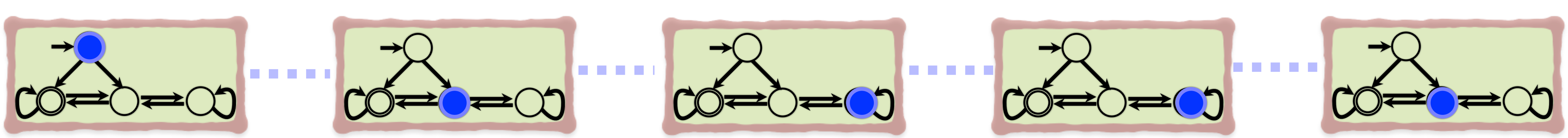
Tracking Computation

current state and remaining input

A computation's *configuration* evolves in each time-step



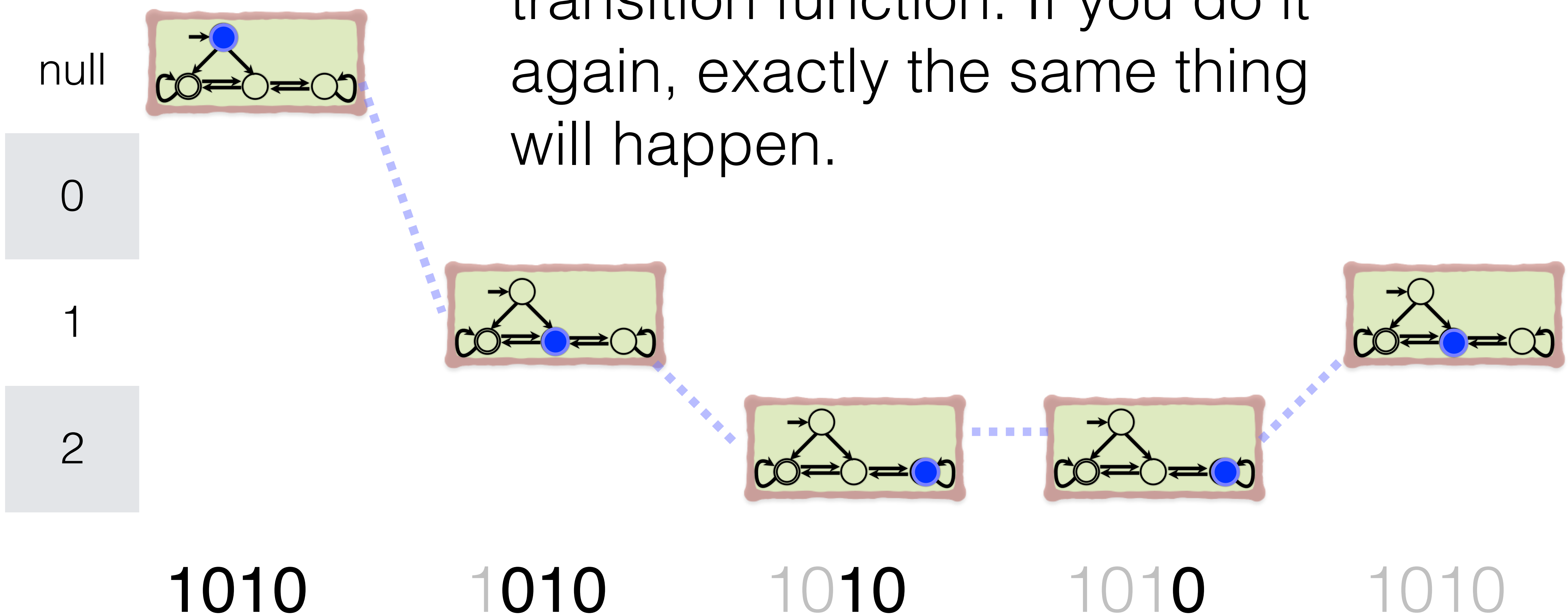
on input 1010



1010 1010 1010 1010 1010

Deterministic Computation

Deterministic: Each step is fully determined by the configuration of the previous step and the transition function. If you do it again, exactly the same thing will happen.

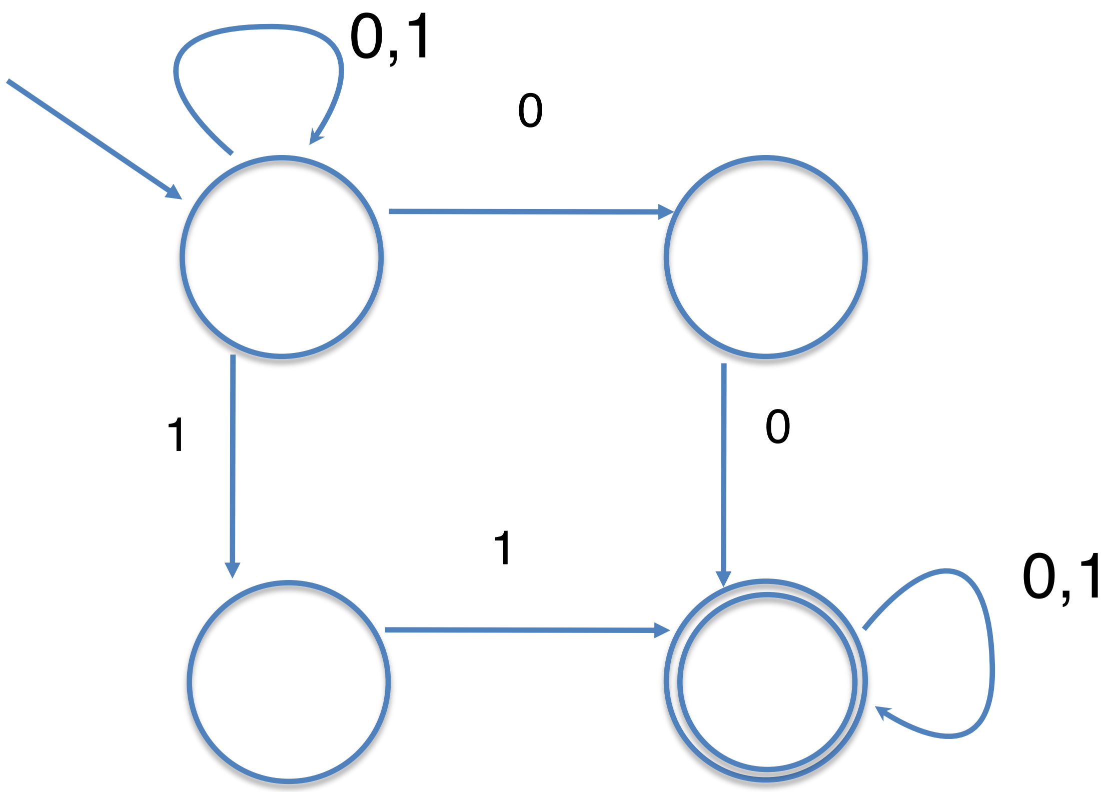


Nondeterminism

- Determinism: opposite of free will
- Nondeterminism: you suddenly have choices!

Non-Deterministic FA

What can be non-deterministic about an FA?



What language?

- At a given state, on a given input, a set of "next-states"
- set could be empty, could be all states…

NFA : Formally

$$\text{DFA} : M = (\Sigma, Q, \delta, s, A)$$

Σ: alphabet Q: state space s: start state A: set of accepting states

$$\delta : Q \times \Sigma \rightarrow Q$$

$$\delta(q, a) = \text{ a state}$$

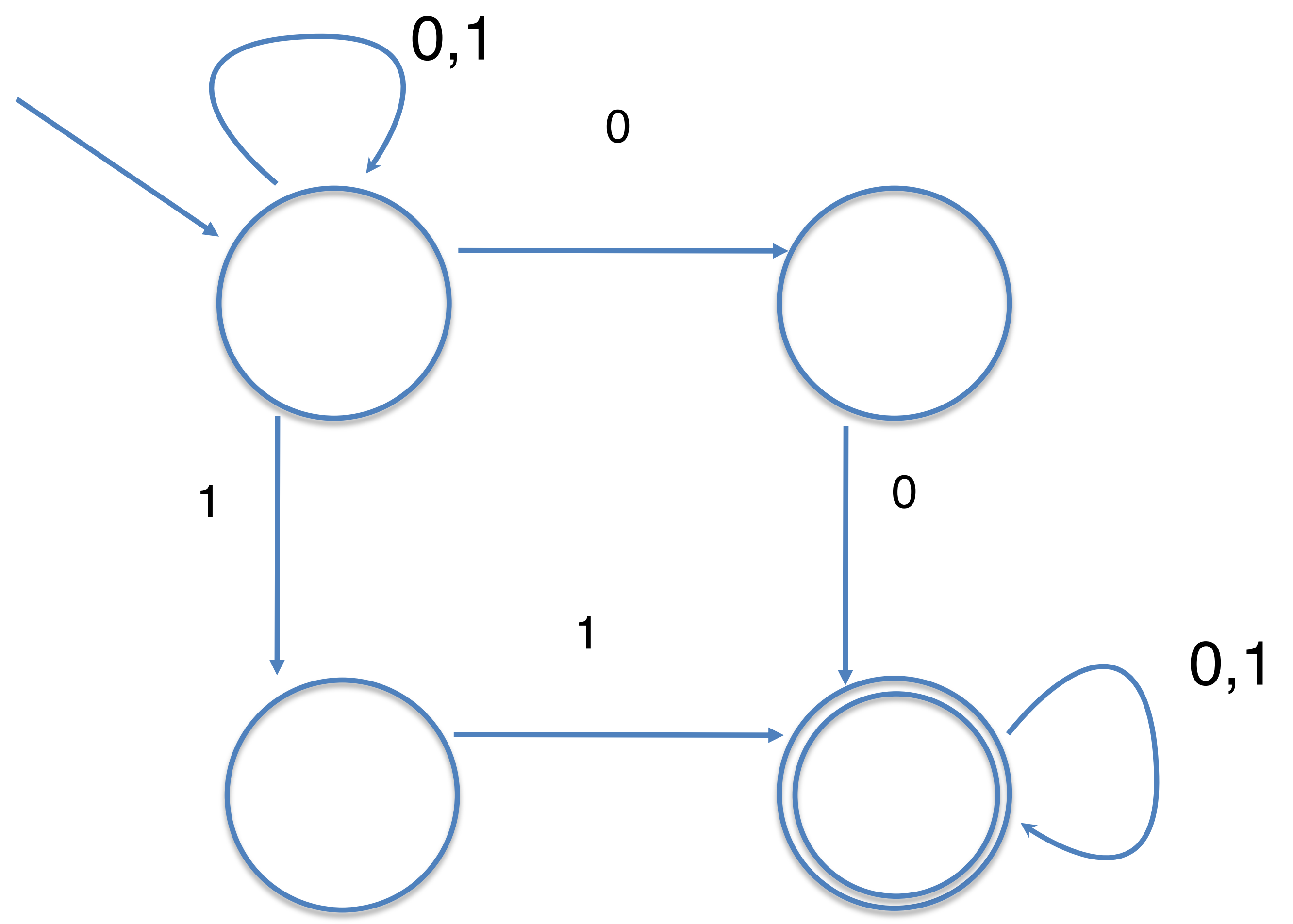
$$\text{NFA} : N = (\Sigma, Q, \delta, s, A)$$

$$\delta : Q \times \Sigma \rightarrow 2^Q = P(Q)$$

$$\delta(q, a) = \{ \text{ a set of states } \}$$

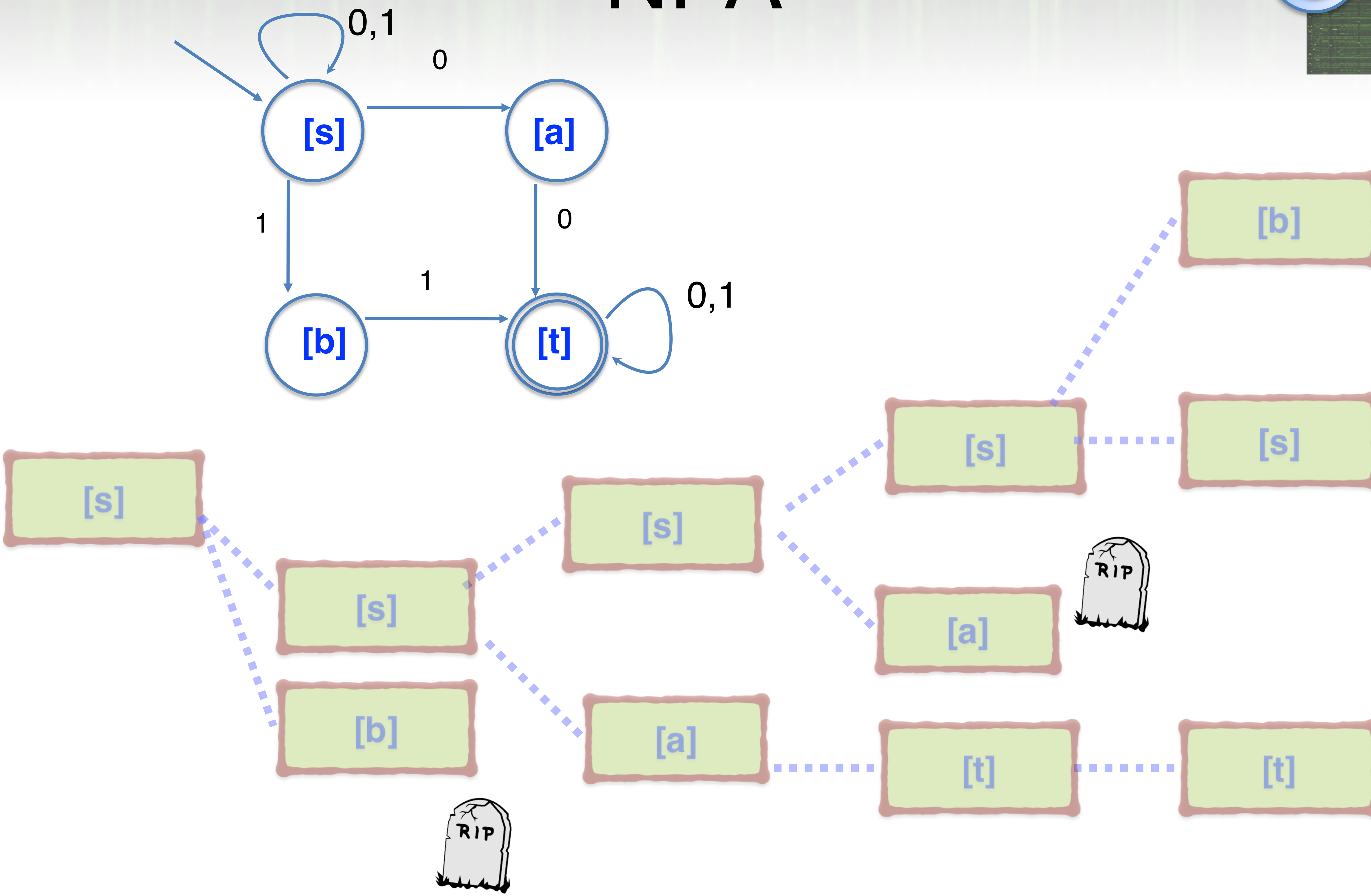
NFA

- Input = 1001



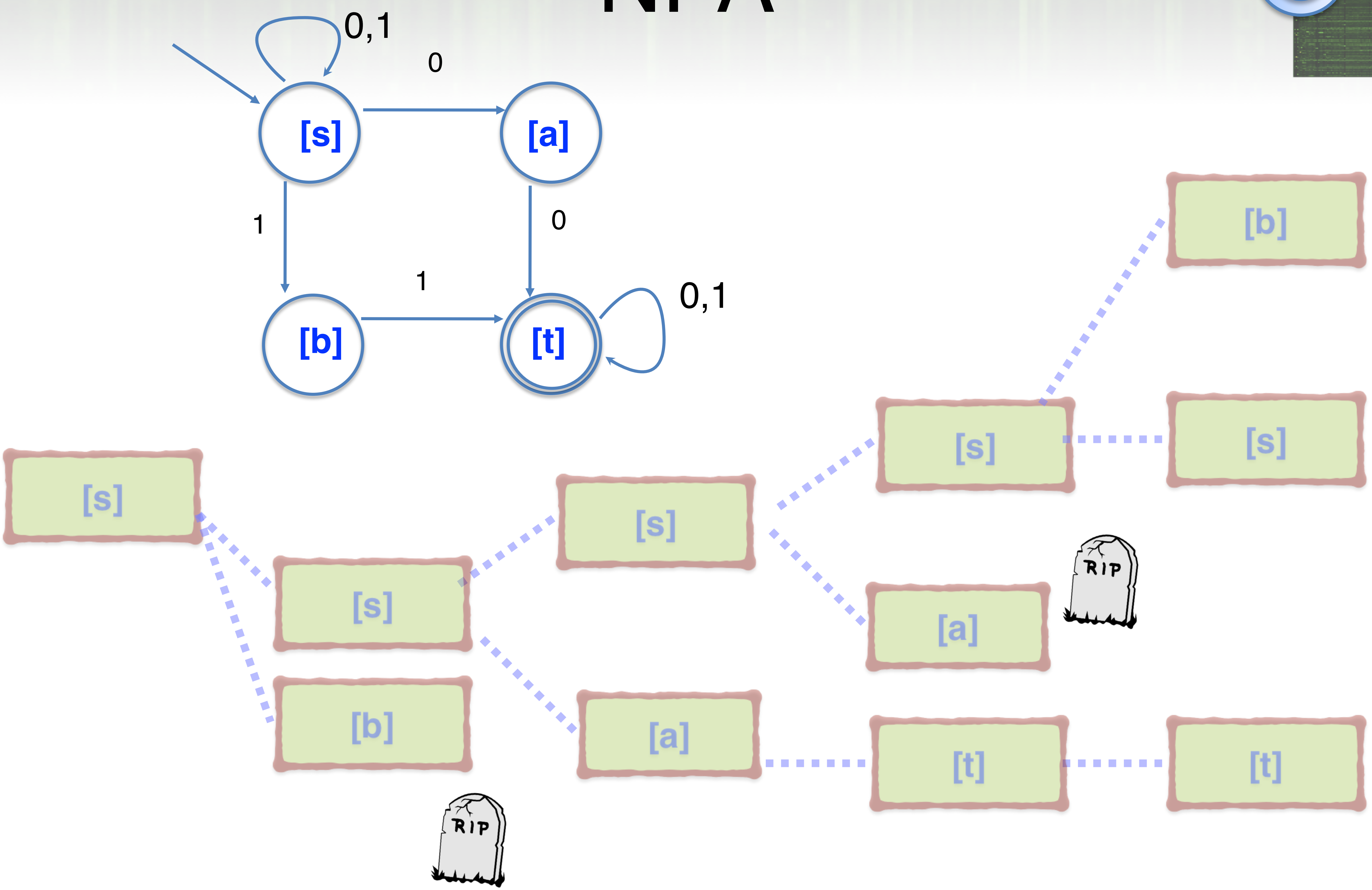
- L ={contains either 00 or 11}

NFA



1001 1001 1001 1001 1001

NFA



ne states are accepting. There needs to be AT LEAST one accep

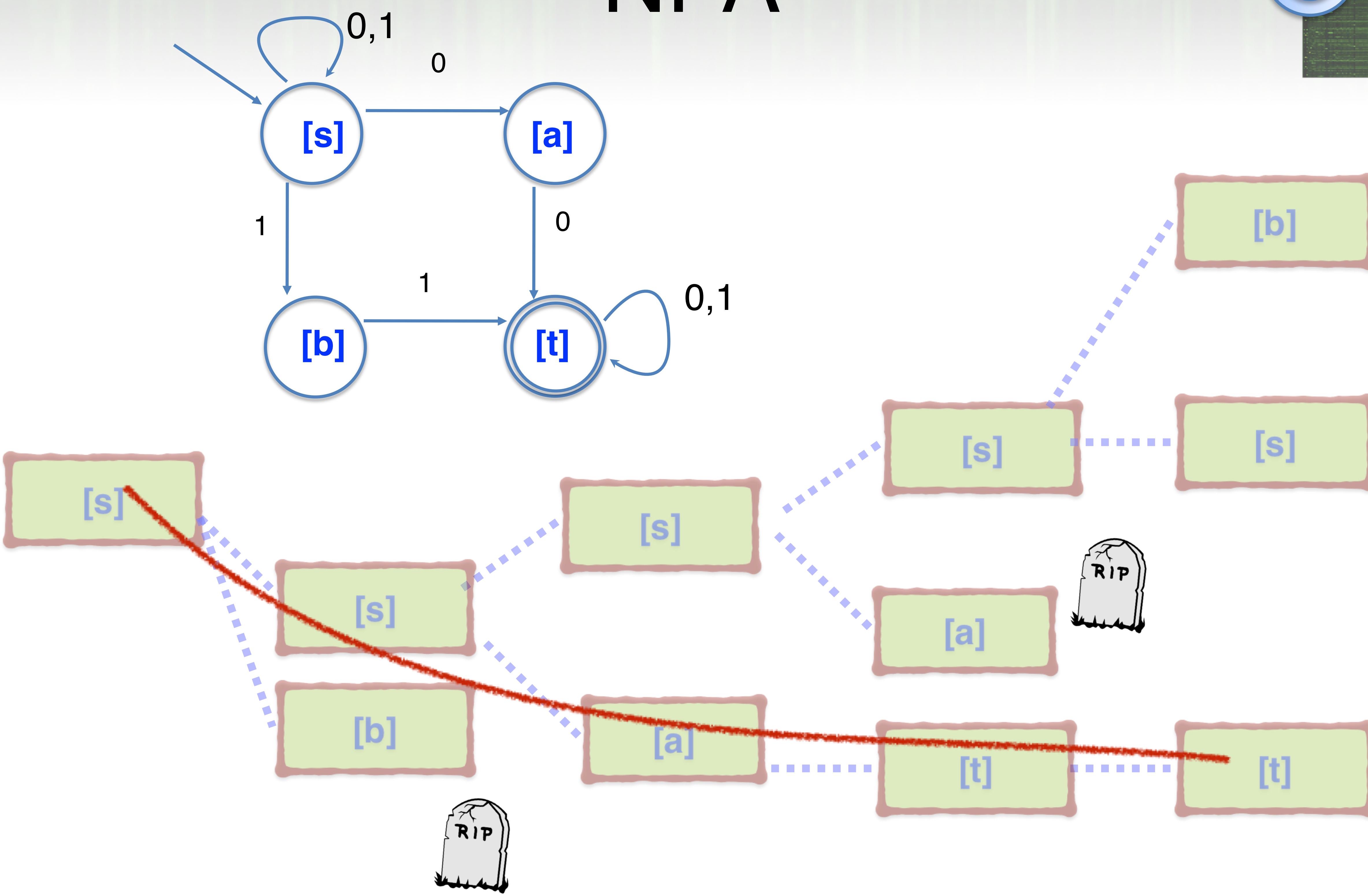
Nondeterminism

- What is non determinism?
- Magic?
- Parallelism?
- Advice?

Nondeterminism

- What is non determinism?
- Suppose I wanted to prove to you that the string 1001 is in L ={contains either 00 or 11}
- We built a DFA with product last time.
- Proof is an accepting computation

NFA



1001 1001 1001 1001 1001

Nondeterminism

- What is non determinism?
- Suppose I wanted to prove to you that the string 1001 is in L ={contains either 00 or 11}
- We built a DFA with product last time.
- Proof is an accepting computation: guide for the reader to how to follow the steps to a given conclusion.

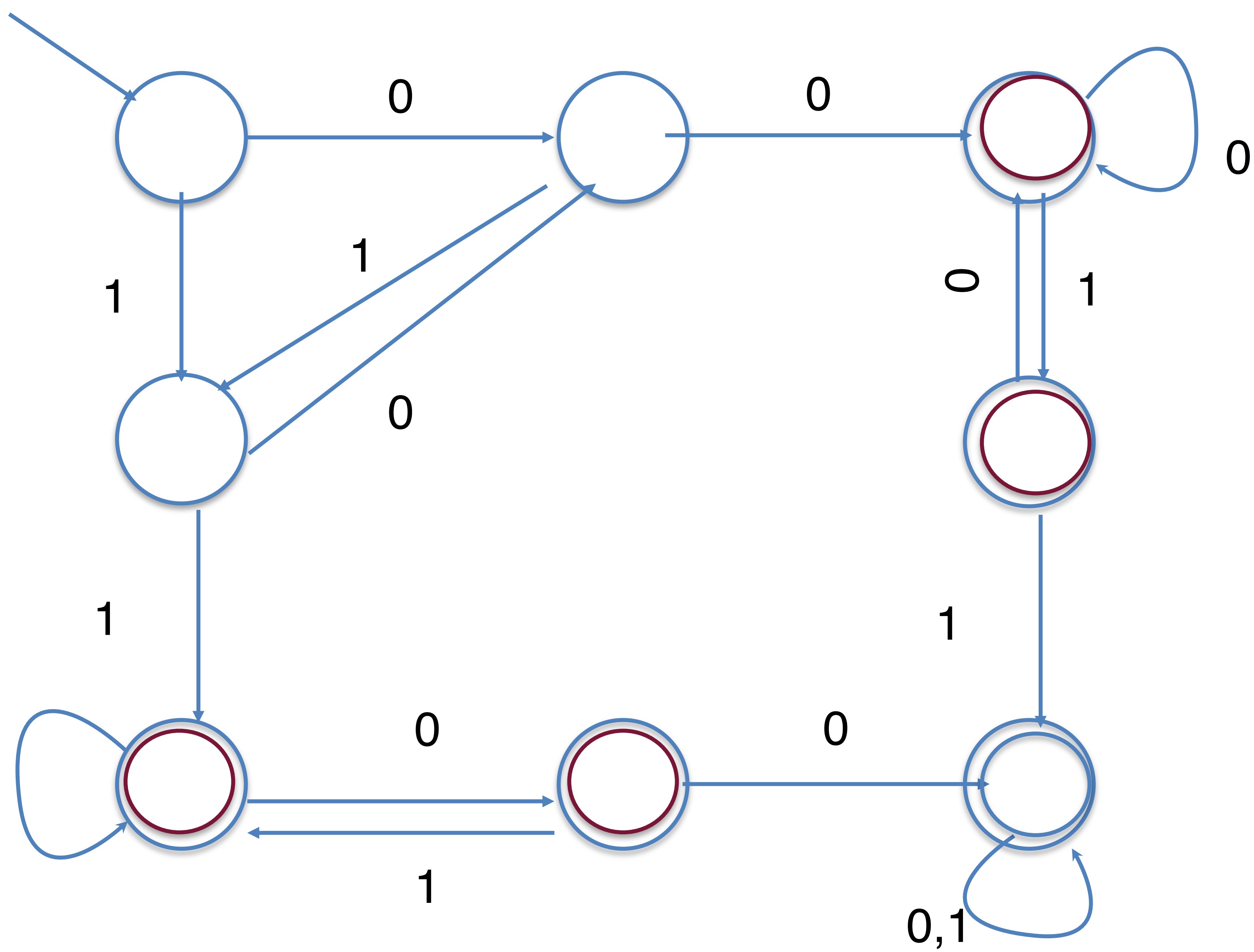
Nondeterminism

- P vs. NP
- Are they the same?
- Easier to give the proof than come up with the proof! (?)

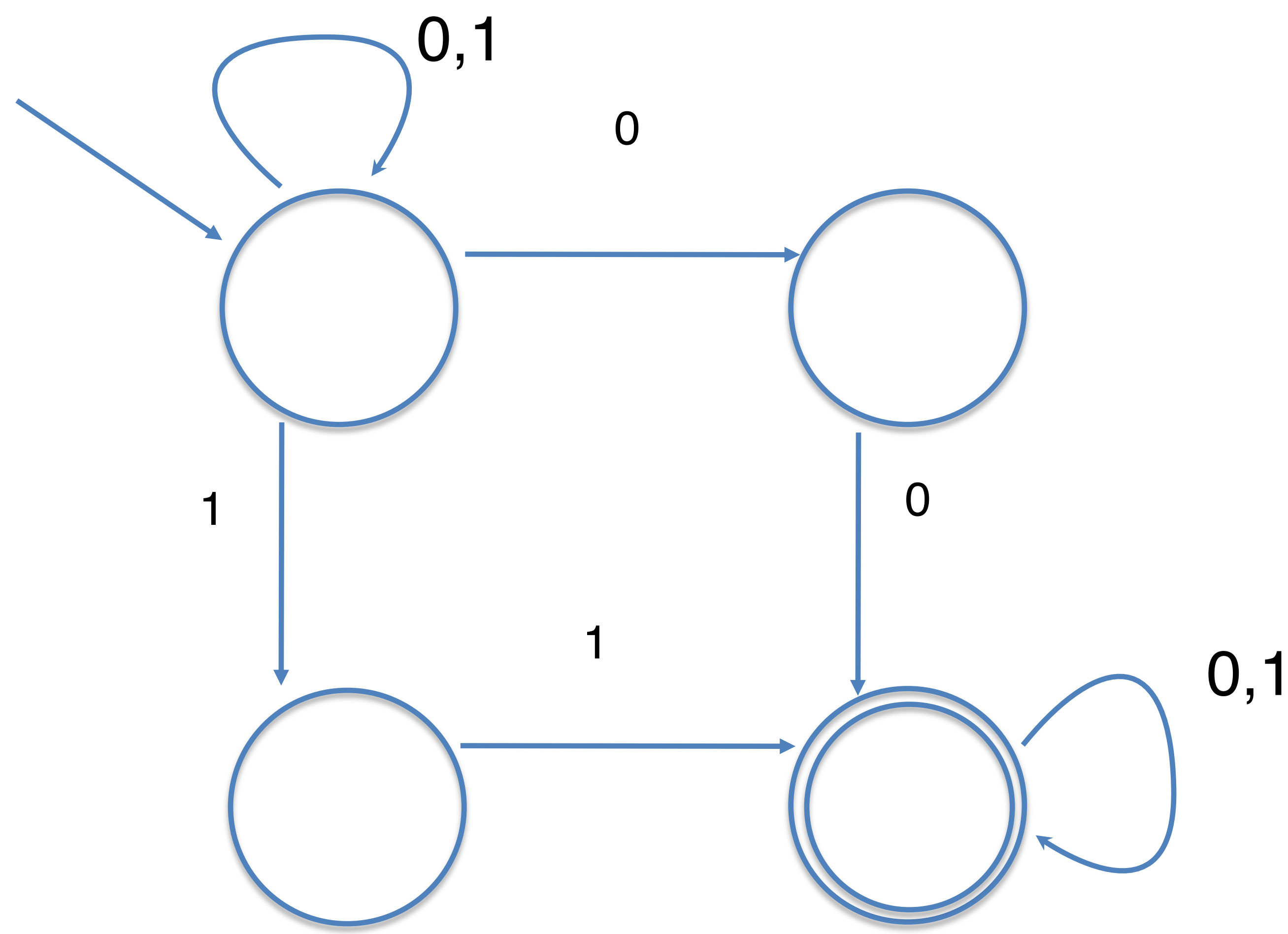
Nondeterminism

- For FSM, nondeterminism does not give you more expressive power!
- Any language that can be accepted by an NFAs can also be accepted by a DFA.
- It is more efficient, last example had 4 states but product construction had 8!

DFA for L = {w: w contains 00 or 11}



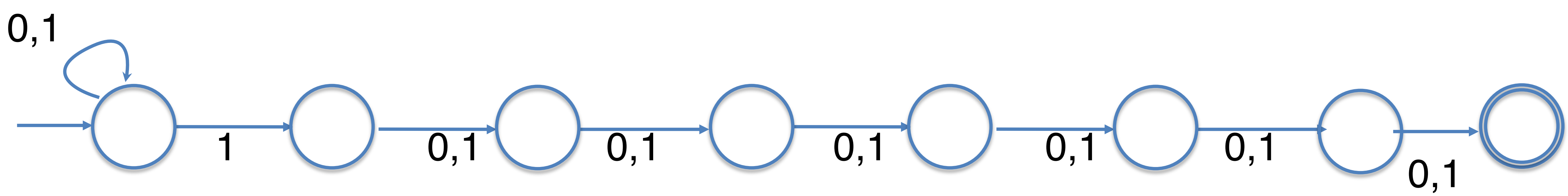
NFA for L = {w: w contains 00 or 11}



NFA : More efficient

Design an NFA to recognize

$L(M) = \{w \mid w : \text{7th character from the end is a 1}\}$



- Minimum DFA for this language would have 2^7 states

at least!

- need to remember the last 7 symbols.

NFA : Formally

- NFA has 5 parts, similar to a DFA : $N = (\Sigma, Q, \delta, s, A)$

Σ: alphabet Q: state space s: start state F: set of accepting states

$$\delta : Q \times \Sigma \rightarrow \mathrm{P}(Q)=2^{Q} \text{ transition function}$$

- Define extended transition function:

$$\delta^*: Q \times \Sigma \rightarrow \mathrm{P}(Q)=2^{Q}$$

$\delta^*(q, w)=$

…….. if w=ε
……… if w=ax

NFA : Formally

- NFA has 5 parts, similar to a DFA : $N = (\Sigma, Q, \delta, s, A)$

Σ: alphabet Q: state space s: start state F: set of accepting states

$$\delta : Q \times \Sigma \rightarrow \text{P}(Q) = 2^{Q} \text{ transition function}$$

- Define extended transition function:

$$\delta^*: Q \times \Sigma^* \rightarrow \text{P}(Q) = 2^{Q}$$

$$\delta^*(q, w) = \begin{cases} \{q\} & \text{if } w = \varepsilon \\ \bigcup_{p \in \delta(q,a)} \delta^*(p, x) & \text{if } w = ax \end{cases}$$

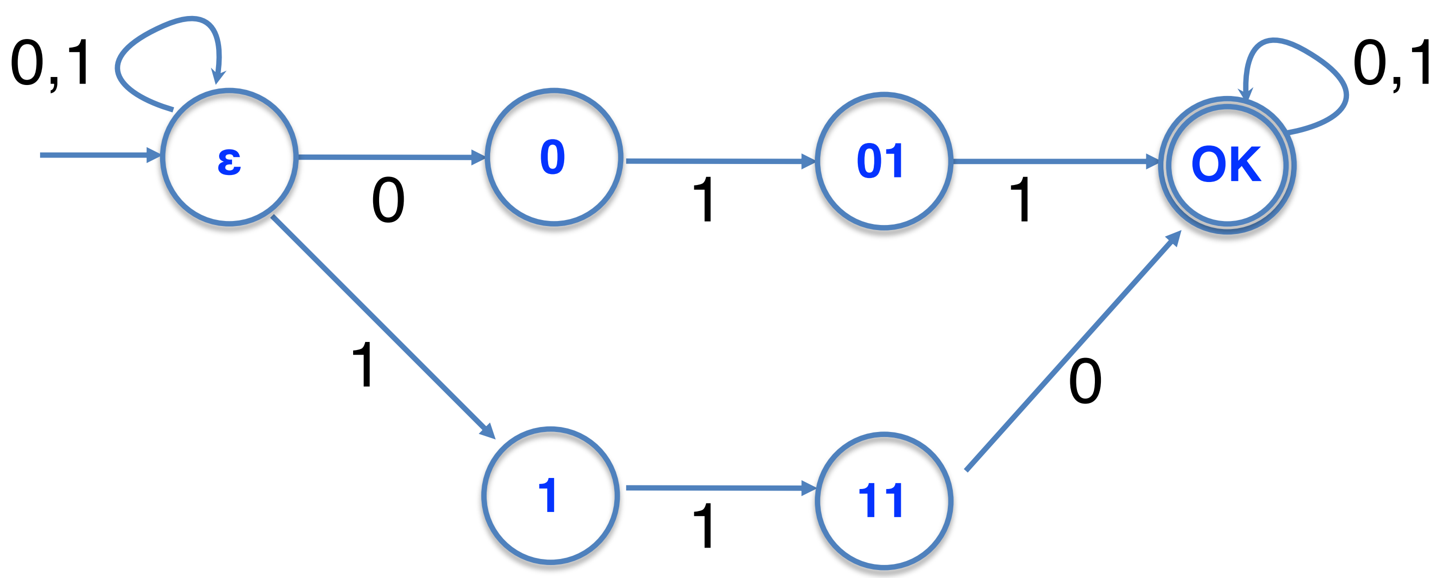
NFA : When does it accept?

NFA accepts a string w if and only if

$$\delta^*(s, w) \cap \mathbf{A} \neq \emptyset$$

NFA : Examples

Design an NFA to recognize
$L(M) = \{w \mid w \text{ contains 011 or 110 }\}$

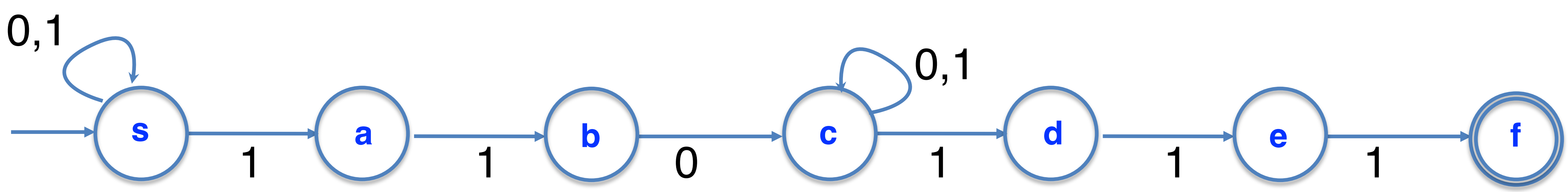


For any input string, if it contains 011 or 110, then there is *some* computation path, that ends in the final state

And vice versa

NFA : Examples

Design an NFA to recognize
$L(M) = \{w \mid w$ has the substring 110 and ends in 111 $\}$



Design an NFA to recognize
$L(M) = \{w \mid w$ has the substring 110 and ends in 000 $\}$