April 17

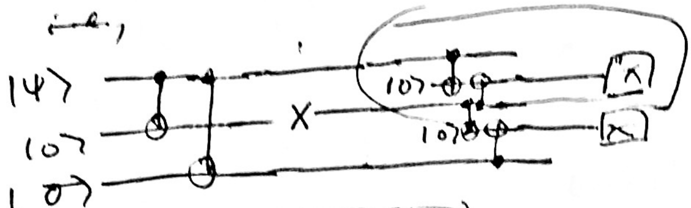
Quantum Error Correction

Last class: quantum repetition code.

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \longrightarrow |\bar{\psi}\rangle = \alpha|000\rangle + \beta|111\rangle$

corrects: 1 error X. ~~many~~

i.e.,



$s_1 = b_1 \oplus b_2$

$s_2 = b_2 \oplus b_3$

Rep. code as a Stabilizer code.

~~Another way to look at rep~~ Repetition code as an Eigen space of commuting Paulis.

$Z|0\rangle = |0\rangle$
$Z|1\rangle = -|1\rangle$

let $|\bar{\Psi}\rangle = \alpha|000\rangle + \beta|111\rangle$

$$(Z_1 \otimes Z_2 \otimes I)|\bar{\Psi}\rangle = \alpha(Z_1 \otimes Z_2 \otimes I)|000\rangle + \beta(Z_1 \otimes Z_2 \otimes I)|111\rangle$$

$$= \alpha|000\rangle + (-1)^2\beta|111\rangle$$

$$= \alpha|000\rangle + \beta|111\rangle$$

Also: $I Z_2 Z_3 |\bar{\Psi}\rangle = |\bar{\Psi}\rangle$

Corrupted codewords also eigenstates: $XZ = -ZX$

$$Z_1 Z_2 (X_1 |\bar{\Psi}\rangle) = (Z_1 X_1 \otimes Z_2 \otimes I)|\bar{\Psi}\rangle$$

$$= -(X_1 Z_1 \otimes Z_2 \otimes I)|\bar{\Psi}\rangle$$

1st bit

$$= -X_1 (Z_1 Z_2 |\bar{\Psi}\rangle)$$

$$= -X_1 |\bar{\Psi}\rangle$$

$$(Z_2 Z_3)(X_1 |\bar{\Psi}\rangle) = X_1 (Z_2 Z_3 |\bar{\Psi}\rangle) = X_1 |\bar{\Psi}\rangle.$$

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$$Z_1 Z_2 (X_2 |\bar{\Psi}\rangle) = -X_2 |\bar{\Psi}\rangle,$$

2nd bit

$$\cancel{Z_1 Z_2 (X_3 |\bar{\Psi}\rangle)} =$$

$$Z_2 Z_3 (X_2 |\bar{\Psi}\rangle) = -X_2 |\bar{\Psi}\rangle$$

$$Z_1 Z_2 X_3 |\bar{\Psi}\rangle = X_3 |\bar{\Psi}\rangle$$

3rd bit

$$Z_2 Z_3 X_3 |\bar{\Psi}\rangle = -X_3 |\bar{\Psi}\rangle$$

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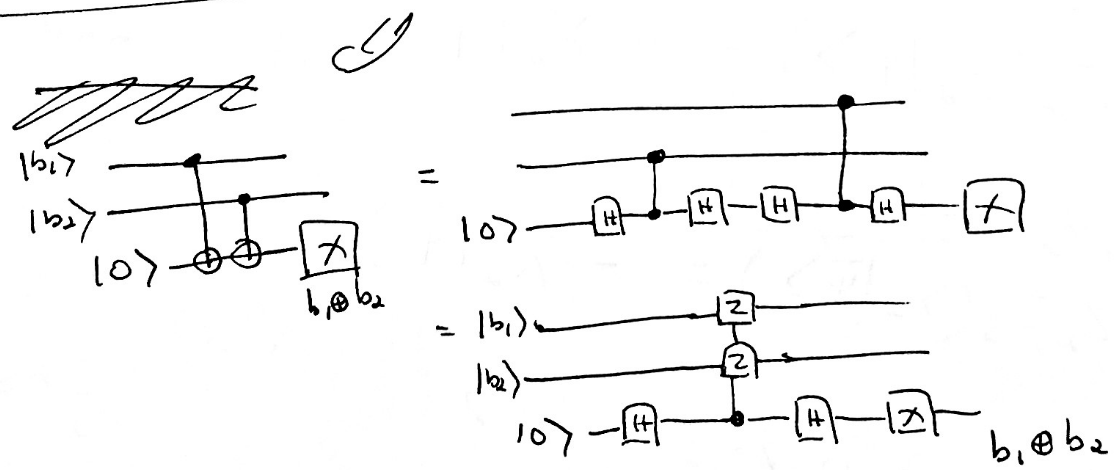

This picture can be generalized to measure more complicated parities.

let A be an n-qubit ~~string of~~ tensor product of X's and Z's:

e.g., $X_0 Z_1 Z_2 X_3$.

<u>Then</u>: A is 1) <u>Hermitian</u>: $A^\dagger = A$.

2) $A^2 = \mathbb{I}$

<u>so</u>: $A^2 = \mathbb{I} \Rightarrow$ eigenvalues of A are ± 1.

$P_0 = \dfrac{\mathbb{I} + A}{2}$ projects onto $+1$ eigenspace

$P_1 = \dfrac{\mathbb{I} - A}{2}$ projects onto -1 eigenspace.

$P_0 + P_1 = \mathbb{I}$, so

$|\psi\rangle = (P_0 + P_1)|\psi\rangle = P_0|\psi\rangle + P_1|\psi\rangle$

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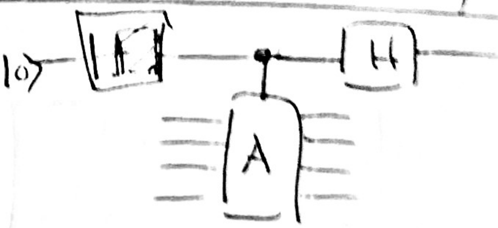
$cA = |0\rangle\langle 0| \otimes \mathbb{I} + |1\rangle\langle 1| \otimes A$,



$(H \otimes \mathbb{I})\, cA\, (H \otimes \mathbb{I})\, |0\rangle|\psi\rangle$

$= (H \otimes \mathbb{I})\, cA \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|\psi\rangle \right)$

$= (H \otimes \mathbb{I}) \left[\frac{1}{\sqrt{2}} |0\rangle|\psi\rangle + |1\rangle A|\psi\rangle \right]$

$= \frac{1}{2}\left[(|0\rangle + |1\rangle)|\psi\rangle + (|0\rangle - |1\rangle)A|\psi\rangle \right]$

$= |0\rangle \left(\frac{\mathbb{I} + A}{2} \right)|\psi\rangle + |1\rangle \left(\frac{\mathbb{I} - A}{2} \right)|\psi\rangle$

$= |0\rangle P_0|\psi\rangle + |1\rangle P_1|\psi\rangle$

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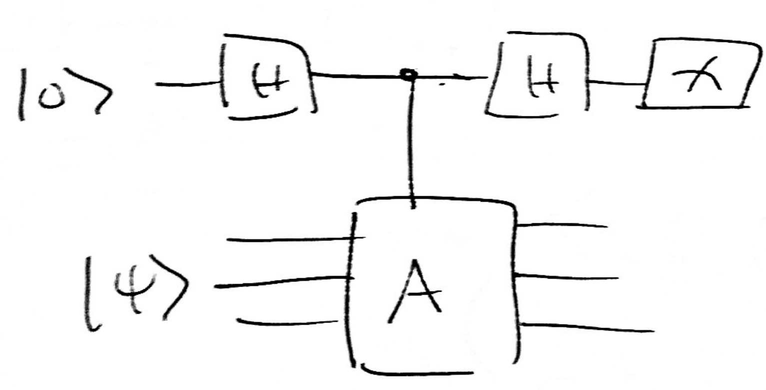
$$\left(\frac{1-A}{2}\right)^2 = \frac{1-2A+A^2}{4} = \frac{1-A}{2}$$

Start Here. Recap.

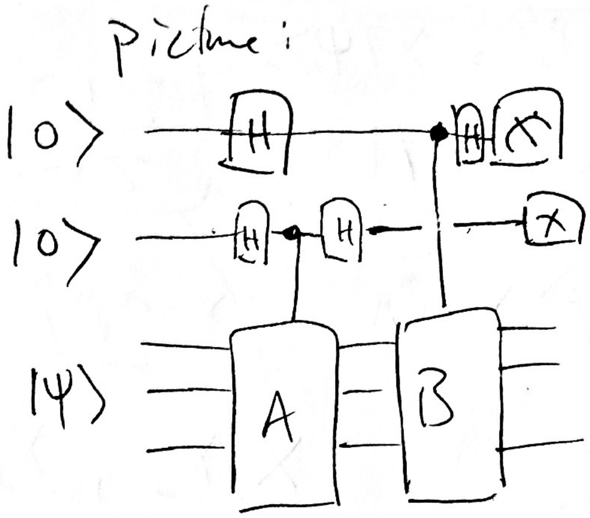


$$0 \Rightarrow \quad |0\rangle P_0^A|\psi\rangle, \quad |1\rangle P_1^A|\psi\rangle$$

"Measuring A" projects into eigen space of A.

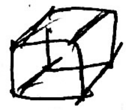
GOT TO HERE

picture:



if A & B commute
$AB = BA$.
can measure A & B "at the same time"
i.e., ~~does~~ order doesn't matter

$$H_1 H_2 (c_1A \; c_2B) H_1 H_2 |0\rangle|0\rangle|\psi\rangle$$



$$= \frac{1}{2}\sum_{x_1, x_2} |x_1\rangle|x_2\rangle P^A_{x_1} P^B_{x_2} |\psi\rangle$$

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Repetition code:

~~start in (2-d) +1 eigenstatesspace of~~

encode into +1 eigenspace of Z_1Z_2, Z_2Z_3,

then simultaneously measure Z_1Z_2 ~~the~~ and Z_2Z_3

(they commute)

$\cancel{Z_1Z_2|\psi\rangle}$

$\cancel{Z_1Z_2}$

if Z_1Z_2 commutes with error, Z_1Z_2 remains $+1$

if Z_1Z_2 anticommutes, $Z_1Z_2 \to -1$

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Noise: $|\psi\rangle \xrightarrow{p} X|\psi\rangle$

$\xrightarrow{p} Y|\psi\rangle$

$\xrightarrow{p} Z|\psi\rangle$

$\xrightarrow{1-3p} |\psi\rangle$

$Z|+\rangle = |-\rangle$
$Z|-\rangle = |+\rangle$

X-repetition code.

X: measure Z-like ~~ops~~ operators $Z_1 Z_2, \ldots$

Z error: measure X-like: $X_1 X_2$, $X_2 X_3$.

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \longrightarrow |\bar{\psi}\rangle = \alpha|+\rangle|+\rangle|+\rangle + \beta|-\rangle|-\rangle|-\rangle$$

$$Z_1|\bar{\Psi}\rangle = \alpha|-\rangle|+\rangle|+\rangle + \beta|+\rangle|-\rangle|-\rangle$$

$$X_1X_2Z_1|\bar{\Psi}\rangle = X_1X_2(\alpha|-\rangle|+\rangle|+\rangle + \beta|+\rangle|-\rangle|-\rangle)$$
$$= -(\alpha|-\rangle|+\rangle|+\rangle + \beta|+\rangle|-\rangle|-\rangle)$$

$$= -Z_1|\bar{\Psi}\rangle$$

$$X_2X_3Z_1|\bar{\Psi}\rangle = Z_1|\bar{\Psi}\rangle.$$

~~[illegible]~~ X-rep. code.

$X_1X_2 = 1 \quad X_2X_3 = 1 \Rightarrow$ in code, leave alone.

$X_1X_2 = -1, \quad X_2X_3 = 1 \Rightarrow$ Z on qubit 1. flip it back

$-1 \quad -1 \quad$ Z on qubit 2. Z_2

$1 \quad -1 \quad$ Z on qubit 3.

Pattern of error commuting with stabilizers tells us about error.

$\alpha|0\rangle+\beta|1\rangle$ let's say we use 3-rep. code to correct single X errors:

$$|\bar\Psi\rangle = \alpha|000\rangle + \beta|111\rangle = \alpha|\bar 0\rangle + \beta|\bar 1\rangle \cancel{Y = iXZ}$$

still subjected to Y and Z errors.

$$Z_1|\bar\Psi\rangle = \alpha|000\rangle - \beta|111\rangle$$
$$= \alpha|\bar 0\rangle - \beta|\bar 1\rangle$$ "logical Z".

$\bar Z|\bar 0\rangle = |\bar 0\rangle$, $Z|\bar 1\rangle = -|\bar 1\rangle$. $\bar Z = Z_1$

what is logical $\bar X$? $\bar X|\bar 0\rangle = |\bar 1\rangle$
$\bar X|\bar 1\rangle = |\bar 0\rangle$

$\bar X = X_1X_2X_3$

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error model on code:

$$\begin{tikzcd} |\bar\Psi\rangle \arrow[r, "\bar p"] \arrow[dr, "1-\bar p"'] & \bar Z|\bar\Psi\rangle \\ & |\bar\Psi\rangle \end{tikzcd}$$

$$|\bar{\bar\Psi}\rangle = \alpha|\bar +\rangle_1|\bar +\rangle_2|\bar +\rangle_3 + \beta|\bar -\rangle_1|\bar -\rangle_2|\bar -\rangle_3$$

$\bar X_1\bar X_2$, $\bar X_2\bar X_2$

1 qubit into 9 qubits

Next time: how this works.

9 qubit code:

$\alpha|0\rangle + \beta|1\rangle \to \alpha|+++\rangle + \beta|---\rangle$ (get Z-types.)

$\downarrow$

$\alpha|\mp\rangle|\mp\rangle|\mp\rangle + \beta|=\rangle|=\rangle|=\rangle$ (get X types)

1 logical qubit
9-qubit code

$$|\overline{\Psi}\rangle = \alpha\left[\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \otimes \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \otimes \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)\right]$$
$$+ \beta\left[\frac{1}{\sqrt{2}}(|000\rangle - |111\rangle) \otimes \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle) \otimes \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle)\right]$$

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Stabilizers.

$\overline{X}_1 \overline{X}_2 I_3 =$
$I \ \overline{X} \ \overline{X}$

XXXXXX III
III XXX XXX

ZZI III III
IZZ III III
III ZZI III
III IZZ III
III III ZZI
III III IZZ

need to check: for each 1 qubit Pauli, we get a different pattern of (anti-)commutation.
$\Leftrightarrow$ different syndrome outcome.

$P_i|\overline{\Psi}\rangle$ will be eigenstate of the stabilizers
the eigenvalues will be the syndrome.

If each error has a different syndrome, we can undo the P_i after measuring

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$\bar{X}\ \bar{X}\ I$

XXX XXX III
III XXX XXX
ZZI III III
IZZ III III
III ZZI III
III IZZ III
III III ZZI
III III IZZ

Stabilizers, single qubit Pauli.

P_i ,

$P_i|\bar{\Psi}\rangle$ is a simultaneous eigenstate.

For each pattern of eigenvalues (aka syndrome) we need to show there is a ~~[illegible]~~ correction unitary to apply that gives us $|\bar{\Psi}\rangle$ back.

X-type

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9
XXX XXX III									
III XXX XXX									
ZZI ~~ZZ~~I III									
IZZ I~~ZZ~~ II									
III ZZI II									
III IZZ II									
III III ZZ									
III III IZZ									

0 if commute
1 if anti commute

instructor: do two columns,
activity: fill out more

Z - type

	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7	Z_8	Z_9
XXXXXXIII	1	1	1	1	1	1	0	0	0
III XXXXXX	0	0	0	1	1	1	1	1	1

if $\begin{smallmatrix}1\\0\end{smallmatrix}$, apply Z_1 or Z_2, or Z_3.

doesn't matter:

$$Z_1\left(\frac{|000\rangle \pm |111\rangle}{\sqrt{2}}\right) = Z_2\left(\frac{|000\rangle \pm |111\rangle}{\sqrt{2}}\right) = Z_3\left(\frac{|000\rangle \pm |111\rangle}{\sqrt{2}}\right)$$

because Z_1Z_2 & Z_2Z_3 stabilizers.

if X_1, $ZZI\ III\ III \to$ ~~flip~~ apply X_1 —

i.e.

$$Z_1\left(\tfrac{1}{\sqrt{2}}|000\rangle \pm |111\rangle\right)$$

$$= Z_1\left(Z_1Z_2\left(\tfrac{1}{\sqrt{2}}|000\rangle \pm |111\rangle\right)\right)$$

$$= \left[(Z_1)(Z_1Z_2)\right]\left(\tfrac{1}{\sqrt{2}}(|000\rangle \pm |111\rangle)\right)$$

$$= Z_2\left(\tfrac{1}{\sqrt{2}}(|000\rangle \pm |111\rangle)\right)$$

QECC Big Picture.

1) encode $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ into $|\bar\psi\rangle = \alpha|\bar 0\rangle + \beta|\bar 1\rangle$

$|\bar 0\rangle$ and $|\bar 1\rangle$ "logical ~~to~~ 0 and 1",
span the space of simultaneous +1
eigenstates of some commuting operators:
$S_1 \ldots S_{n-1}$ (Paulis!)

2) an error happens, gives $E_i|\bar\psi\rangle$.

3) These are eigenvectors of S_i:
$S_j E_i |\bar\psi\rangle = (-1)^{\omega(S_j,E_i)} E_i S_j |\bar\psi\rangle = (-1)^{\omega(S_j,E_i)} E_i |\bar\psi\rangle$

4) ~~the [illegible]~~ Measure $\{S_j\}$ for $j = 1 \ldots n-1$.
get pattern of ± 1: $(-1)^{\omega(S_j,E_i)}$

5) if each error E_i gets it's own pattern, you ~~have a~~ can correct the errors: just apply $E_i^\dagger$ to reverse it!

Next class: Some good error correcting codes