Quantum Error Correction

Why? quantum states are delicate. Any interaction with its environment will change the state of a QC.

\( |\psi\rangle \)

Recall

\[ |\psi\rangle + |\psi\rangle \]

\[ |\psi\rangle \]

\[ |\psi\rangle \]

Small interaction small change.

interacts with uncontrolled dofs changes the state.

2 level system in lab.

Also: gates are imperfect,

CNOTs \(< 99\% \)

\(< 99.9\% \)

\(< 99.9\% \) 10^-5

contrast: \(< 10^{-15} \) for classical computers.

99.9\% not good enough 10^{-15} is. Why?

how good is good enough?

DISCUSS (pause to discuss.)

Answer: you should set the right error to control.
initial state

separate into time steps:

ideal: \( 10^{\infty} \rightarrow 1\phi_1 \rightarrow (\phi_2) \rightarrow \cdots \rightarrow |\phi_T\rangle \)

real: \( 10^{\infty} \rightarrow |\phi_1\rangle \rightarrow |\phi_2\rangle \rightarrow \cdots \rightarrow |\phi_T\rangle \)

good enough if \( |\phi_T\rangle \approx |\phi_t\rangle \)

\[
\text{let } \begin{align*}
|\phi_t\rangle &= U_t |\phi_{t-1}\rangle \\
\text{remark: } &\text{ but do } \tilde{U}_t \text{ instead.}
\end{align*}
\]

\[
|\tilde{\phi}_1\rangle = |\phi_1\rangle + |E_1\rangle
\]

\[
|\tilde{\phi}_2\rangle = \tilde{U}_2 (|\tilde{\phi}_1\rangle) = \tilde{U}_2 |\phi_1\rangle + |E_1\rangle
\]

\[
= |\phi_2\rangle + |E_2\rangle + \tilde{U}_2 |E_1\rangle
\]

\[
|\tilde{\phi}_T\rangle = |\phi_T\rangle + |E_T\rangle + \tilde{U}_T |E_{T-1}\rangle
\]

\[
+ \tilde{U}_T \tilde{U}_{T-1} |\phi_{T-1}\rangle + \cdots
\]

\[
+ \tilde{U}_T \tilde{U}_{T-2} \cdots \tilde{U}_2 |E_1\rangle
\]
Triangle inequality:
\[ \| \ket{E_t} - \ket{E_t'} \| \leq \| \ket{E_t} \| + \| \ket{E_{t-1}} \| + \cdots + \| \ket{E_1} \| \]
\[ T \leq \text{okay as long as } T \ll 1. \]
\[ \Rightarrow \epsilon \ll \frac{1}{T} \]

For QC, \( \epsilon \approx 0.01 \)
\[ \Rightarrow T \approx 100 \]

For classical, \( \epsilon \approx 10^{-5} \)
need \( T \ll 10^{-5} \)

5 GHz = \( 5 \times 10^9 \) gates per second
\[ \Rightarrow 10^6 \text{ seconds till an error (if you ask!)} \]

Classical Error Correcting Codes.

\[ \text{cell tower.} \]

In communication scenarios, error correction is critical.
Simplest classical error correcting code: repetition.

message  codeword
0       000
1       111

A lie
How does this help?

Question: you're Bob. You get message 011. What do you decide to? 0 or 1?

011: could be either from 000 or 111.
2 cases:
- either 1'st bit got flipped, or
- second 2 bits got flipped.

more likely:
\[ \text{Pr(Flip 1'\text{st}, leave 2'nd + 3'rd)} = p(1-p)^2 \]
less likely:
\[ \text{Pr(Flip 2'nd leave 1'\text{st}, flip next 2)} = (1-p)p^2. \]

So, adopt majority vote decoding:

| 000 | 000 |
| 001 | 000 |
| 010 | 000 |
| 100 | 000 |

011 | 011 |
101 | 011 |
110 | 110 |
111 | 110 |
Does it help.

1 bit: error with prob $p$.

3 bit rep code: error if 2 or more bits are flipped.

$$\Pr(2 \text{ or more bits flipped})$$

$$= \Pr(1+2 \text{ flipped, } 3 \text{ not})$$
$$+ \Pr(1+3 \text{ flipped, } 2 \text{ not})$$
$$+ \Pr(2+3 \text{ flipped, } 1 \text{ not})$$
$$+ \Pr(1+2+3 \text{ flipped})$$

$$= p^3(1-p)$$
$$+ p^2(1-p)$$
$$+ p^3$$
$$+ p^3$$

$$= 3p^2(1-p) + p^3$$

"Logical" error rate $< 3p^2$.

if $p$ small enough, $3p^2 \ll p$.

"3 bit code: block length 3, 1 logical bit"
How to an
Encoding + decoding circuits.

Decide

Encode

m

b₁

b₂

b₃

Decide

b₁

b₂

b₃

S₁

S₂

S₃

S₁ = 0 if b₁ = b₂
S₁ = 1 if b₁ ≠ b₂
S₂ = 0 if b₂ = b₃
S₂ = 1 if b₂ ≠ b₃

S₃ = 0 if b₃ = b₁
S₃ = 1 if b₃ ≠ b₁

Activity

Deciding:

rule to what is it?

Discuss, get back, terms of S₁+S₂?

How to original?

2. Is 1st bit right?

<table>
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<tr>
<th>S₁ = 0</th>
<th>leave it</th>
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<tbody>
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<td>S₂ = 0</td>
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<th>flip 2</th>
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<tr>
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alternative description of repetition code

Parity Check Matrix:

\[ M = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \]

\( \mathbb{C} \{0,1\}^3 \) is in \( \mathbb{R}_3 \) iff

\( \mathbb{C} \{0,1\} \) is in \( \mathbb{R}_3 \) "syndromes".

Next Time:

- can we do this for Q codes?
  - seems hard, since \( 14 \rightarrow 14' \rightarrow 14'' \)
  - no closing predicate repetition
- how do we avoid collapsing logical qubit?
- HW due in 2 weeks