Introduction To Quantum Computing

PHYS/CSCI 3090

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Homework

- Please note that all 4 questions are required.

Come see us!

- Alexandra Kolla/ Graeme Smith: Friday 3:00-4:00 pm, JILA X317.
- Ariel Shlosberg: Tu/Th 2:00-4:00pm, DUANG2B90 (physics help room)
- Steven Kordonowy: Th 11am-12pm, ECAE 124.

Last class

- Two bit gates
- Quantum bits (qubits)
- a qubit is a system that obeys Axioms

  States

  Dynamics

  Measurements

Today

We will talk more about gates and measurements on multiple qubit systems.

We'll look at more examples of one and two qubit gates and what they can do. This will be useful for studying computation.

Axioms of Quantum Theory

Axiom 1 (states)

The state of a quantum system is a complex unit vector:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \qquad |\alpha|^2 + |\beta|^2 = 1$$

2-dimensional vector: qubit

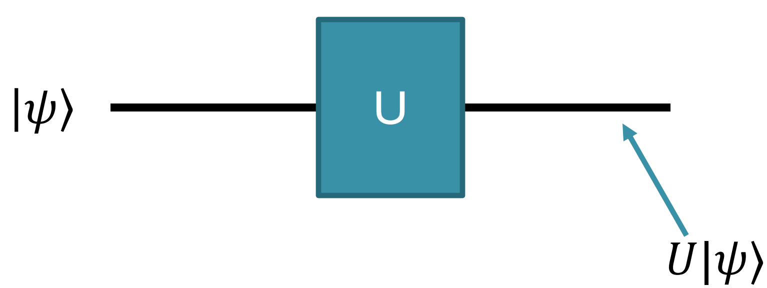
d-dimensional vector: qudit

Axioms of Quantum Theory

Axiom 2 (dynamics)

The evolution of a closed system is described by a unitary matrix $U^t U = I$, where U^t is the conjugate transpose

$$|\psi\rangle \rightarrow U|\psi\rangle$$



Axioms of Quantum Theory

Axiom 3 (Measurements)

Can "measure" a system in any basis for its state space.
If you measure

$$|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$$

in the basis $\{|0\rangle, |1\rangle\}$, you get an outcome "x" with probability $|\alpha_x|^2$.

Furthermore, the state of the system "collapses" to $|x\rangle$.

"Born rule"

Axioms of Quantum Theory

Axiom 4 (Composite Systems)

If

A has a state in span($|a\rangle$), $a = 1 \dots d_a$

B has a state in span($|b\rangle$), b$= 1 \dots d_b$

AB has a state in span($|a\rangle \otimes |b\rangle$)

Tensor products

$$|\psi_A\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle = \begin{bmatrix}\alpha_0\\ \alpha_1\end{bmatrix}$$

$$|\phi_B\rangle = \beta_0|0\rangle + \beta_1|1\rangle = \begin{bmatrix}\beta_0\\ \beta_1\end{bmatrix}$$

$$|\psi_A\rangle \otimes |\phi_B\rangle =$$

$$\alpha_0\beta_0|0\rangle \otimes |0\rangle + \alpha_0\beta_1|0\rangle \otimes |1\rangle + \alpha_1\beta_0|1\rangle \otimes |0\rangle + \alpha_1\beta_1|1\rangle \otimes |1\rangle$$

$$= \alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle$$

$$\begin{bmatrix}\alpha_0\\ \alpha_1\end{bmatrix} \otimes \begin{bmatrix}\beta_0\\ \beta_1\end{bmatrix} = \begin{bmatrix}\alpha_0\beta_0\\ \alpha_0\beta_1\\ \alpha_1\beta_0\\ \alpha_1\beta_1\end{bmatrix}$$

Tensor products

$$|\psi_A\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle = \begin{bmatrix}\alpha_0\\ \alpha_1\end{bmatrix}$$

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Copy of 2nd vector

Another Copy of 2nd vector

Tensor products

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$$|\phi_B\rangle = \beta_0|0\rangle + \beta_1|1\rangle = \begin{bmatrix}\beta_0\\ \beta_1\end{bmatrix}$$

$$\begin{bmatrix}\alpha_0\\ \alpha_1\end{bmatrix} \otimes \begin{bmatrix}\beta_0\\ \beta_1\end{bmatrix} = \begin{bmatrix}\alpha_0\beta_0\\ \alpha_0\beta_1\\ \alpha_1\beta_0\\ \alpha_1\beta_1\end{bmatrix}$$

First entry
Of first
vector

Copy of 2nd
vector

Another
Copy of 2nd
vector

Second entry
of second vector

Concept test: tensor products

$$|\psi\rangle = |+\rangle \otimes |0\rangle$$

(Recall $|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$)

What is $|\psi\rangle$ written as a vector?

A) $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

B) $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$

C) $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

D) $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

Concept test: tensor products

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C) $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

D) $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

First entry
Of first
vector

Copy of 2nd
vector

$$\begin{bmatrix}\alpha_0\\\alpha_1\end{bmatrix}\otimes\begin{bmatrix}\beta_0\\\beta_1\end{bmatrix}=\begin{bmatrix}\alpha_0\beta_0\\\alpha_0\beta_1\\\alpha_1\beta_0\\\alpha_1\beta_1\end{bmatrix}$$

Another
Copy of 2nd
vector

Second entry
of second vector

$$\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\\0\end{bmatrix}$$

Axioms

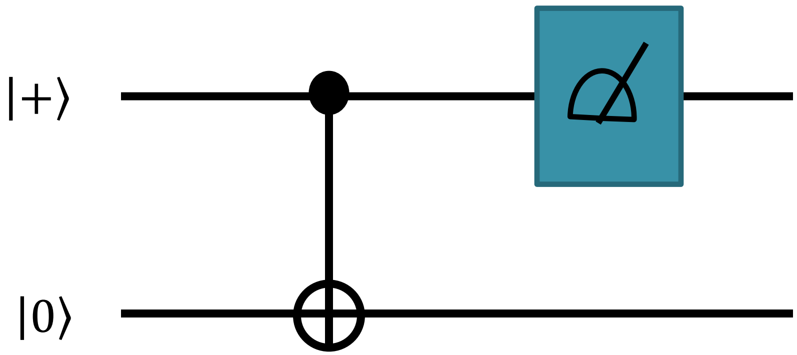
States

Dynamics

Measurements

Composite Systems

What if I want to measure one qubit but not the other?



Is there a rule/axiom to cover this?
Let's fix up axiom 3 to include this kind of thing.

n qubit state

Any state on n qubits can be written as

$$|\psi\rangle = \alpha_0|0\rangle \otimes |\psi_0\rangle + \alpha_1|1\rangle \otimes |\psi_1\rangle,$$

where $|\psi_i\rangle$ are n-1 qubit states and $|\alpha_0|^2 + |\alpha_1|^2 = 1$

Axiom 3' partial measurement

You can "measure" the first qubit of the n qubit state

$$|\psi\rangle = \alpha_0|0\rangle \otimes |\psi_0\rangle + \alpha_1|1\rangle \otimes |\psi_1\rangle,$$

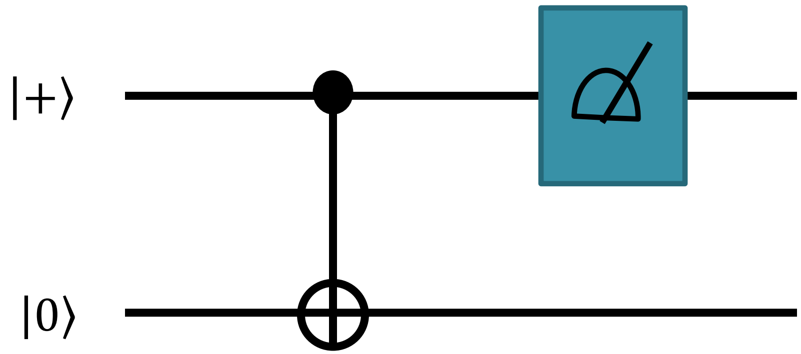
in the $\{|0\rangle, |1\rangle\}$ basis.

You get "x" with probability $|\alpha_x|^2$and getting x "collapses" the state into

$$|x\rangle \otimes |\psi_x\rangle$$

"Genearlized Born rule" or "Born rule"

So what happens here:



Arbitrary one qubit states

- Given a state $|\psi\rangle$, there is always a unitary U such that $|\psi\rangle = U|0\rangle$

- So, you can prepare any one qubit state by starting with $|0\rangle$ and applying a unitary.

Arbitary Two qubit States

Using one qubit gates is not enough to prepare an arbitrary two qubit state

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle.$$

To see this, note that

$$U|0\rangle \otimes V|0\rangle = (\alpha_0|0\rangle + \alpha_1|1\rangle) \otimes (\beta_0|0\rangle + \beta_1|1\rangle)$$

$$= \alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle,$$

So for such a state $\frac{\alpha_{00}}{\alpha_{01}} = \frac{\alpha_{10}}{\alpha_{11}} = \frac{\beta_0}{\beta_1}$, which is not true in genearl

Arbitrary two qubit state: one CNOT suffices

Let's figure out how to make

$|\sigma\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$

with one qubit unitaries and a single CNOT

First, note that

$$|\sigma\rangle = |0\rangle \otimes |\psi\rangle + |1\rangle \otimes |\phi\rangle$$

with $|\psi\rangle = \alpha_{00}\,|0\rangle + \alpha_{01}|1\rangle$ and $|\phi\rangle = \alpha_{10}\,|0\rangle + \alpha_{11}|1\rangle$.

Now, apply $U \otimes I$ with
$U|0\rangle = a|0\rangle + b|1\rangle$ and $U|1\rangle = -b^*|0\rangle + a^*|1\rangle$ to get

$$U \otimes I|\sigma\rangle = |0\rangle \otimes |\psi\rangle + |0\rangle \otimes |\phi\rangle$$

$$|\sigma\rangle = |0\rangle \otimes |\psi\rangle + |1\rangle \otimes |\phi\rangle$$

$$U \otimes I|\sigma\rangle = (a|0\rangle + b|1\rangle) \otimes |\psi\rangle + (-b^*|0\rangle + a^*|1\rangle) \otimes |\phi\rangle$$
$$= |0\rangle \otimes |\psi'\rangle + |1\rangle \otimes |\phi'\rangle$$

Where $|\psi'\rangle = a|\psi\rangle - b^*|\phi\rangle$ and $|\phi'\rangle = b|\psi\rangle + a^*|\phi\rangle$.

We'll choose a and b so that $|\psi'\rangle$ and $|\phi'\rangle$ are orthogonal.

0 = $\langle\phi'|\psi'\rangle = a^2\langle\phi|\psi\rangle - {b^*}^2\langle\psi|\phi\rangle + ab^*(\langle\psi|\psi\rangle - \langle\phi|\phi\rangle)$

if $\langle\phi|\psi\rangle = 0$, they're already orthogonal so U= I. If not, this gives a quadratic equation for $\frac{a}{b^*}$ which we can solve for a and b

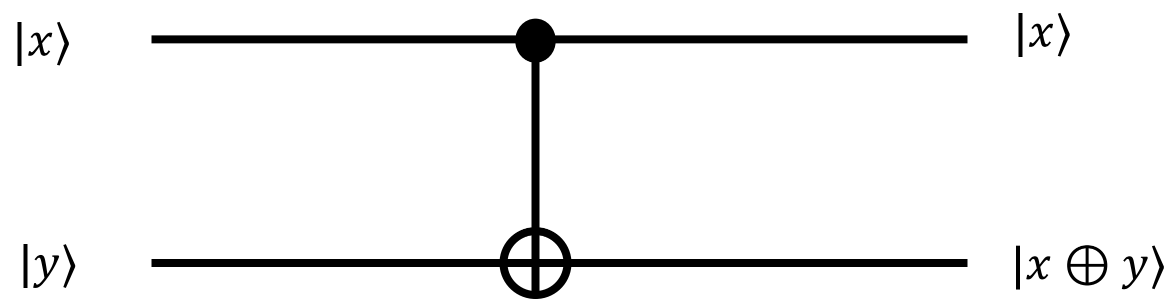
Now we have a U such that
$U \otimes I|\sigma\rangle = |0\rangle \otimes |\psi'\rangle + |1\rangle \otimes |\phi'\rangle$ and $\langle\phi'|\psi'\rangle = 0$

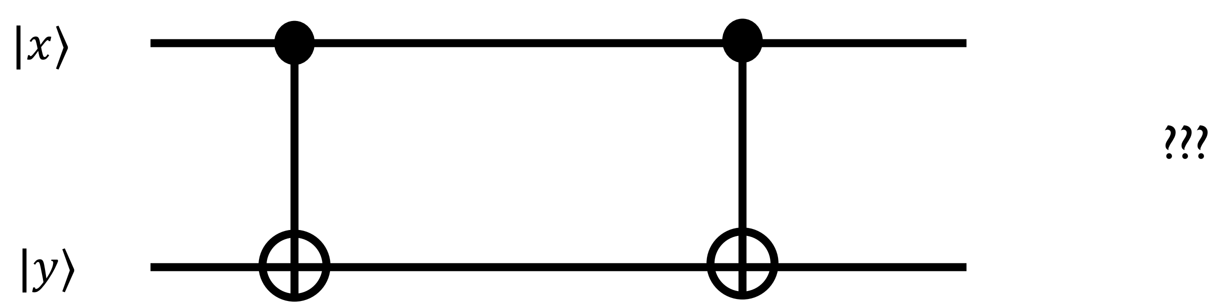
$|\psi'\rangle$ and $|\phi'\rangle$ are not properly normalized yet, but we can find real numbers μ and λ so that $|\psi''\rangle = \frac{|\psi'\rangle}{\lambda}$ and $|\phi''\rangle = \frac{|\phi'\rangle}{\mu}$ are orthogonal unit vectors. Letting V be a unitary with
$|\psi''\rangle = V|0\rangle$ and $|\phi''\rangle = V|1\rangle$ we therefore have

$$\begin{aligned}|\sigma\rangle &= (U^{\dagger} \otimes V)(\lambda|0\rangle \otimes |0\rangle + \mu|1\rangle \otimes |1\rangle) \\ &= (U^{\dagger} \otimes V)\mathrm{CNOT}([\lambda|0\rangle + \mu|1\rangle] \otimes |0\rangle) \\ &= (U^{\dagger} \otimes V)\mathrm{CNO}T([W|0\rangle] \otimes |0\rangle)\end{aligned}$$

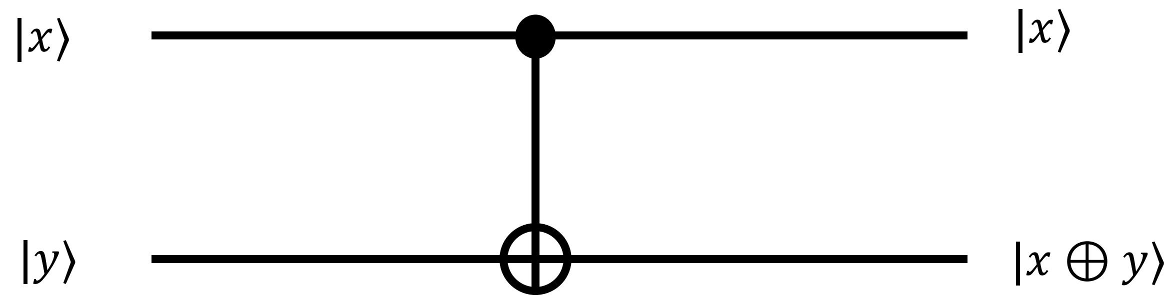
So, we can make an arbitrary two qubit state from $|0\rangle$'s, one-qubit unitaries, and a single CNOT

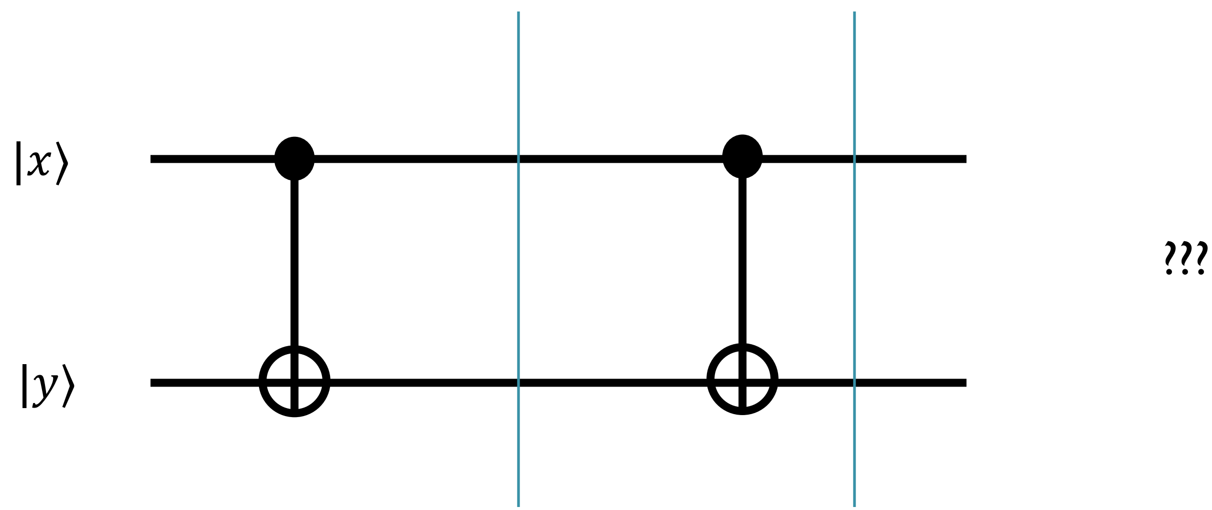
More on CNOT





More on CNOT

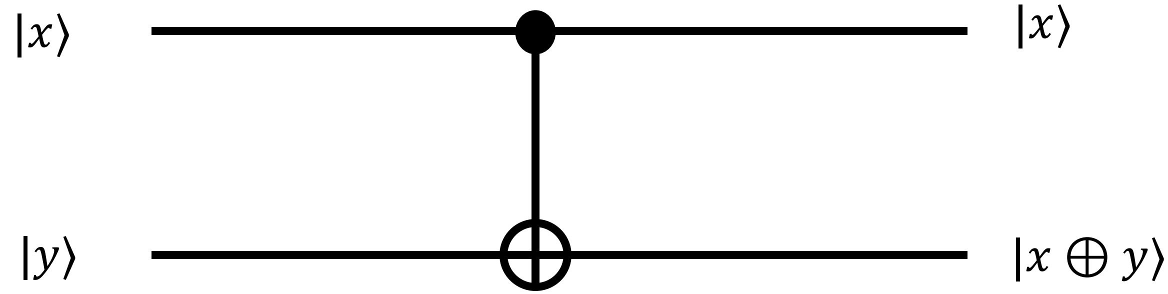


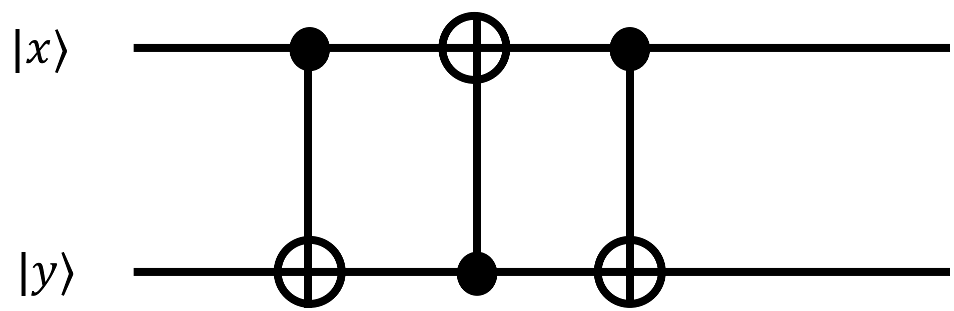


State here is
$|x\rangle|x \oplus y\,\rangle$

State here is
$|x\rangle|(x \oplus y) \oplus x\,\rangle = |x\rangle|y\rangle$

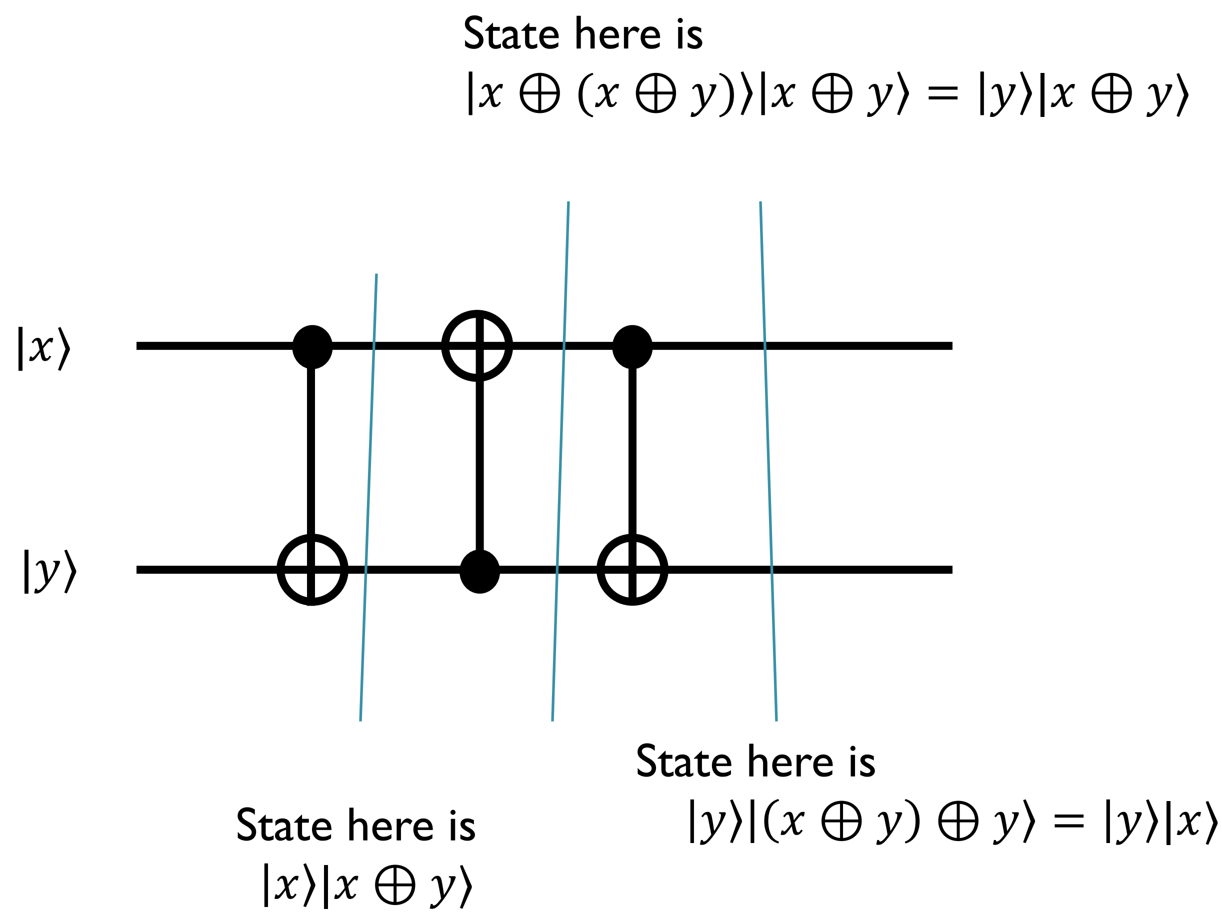
CNOTs concept test





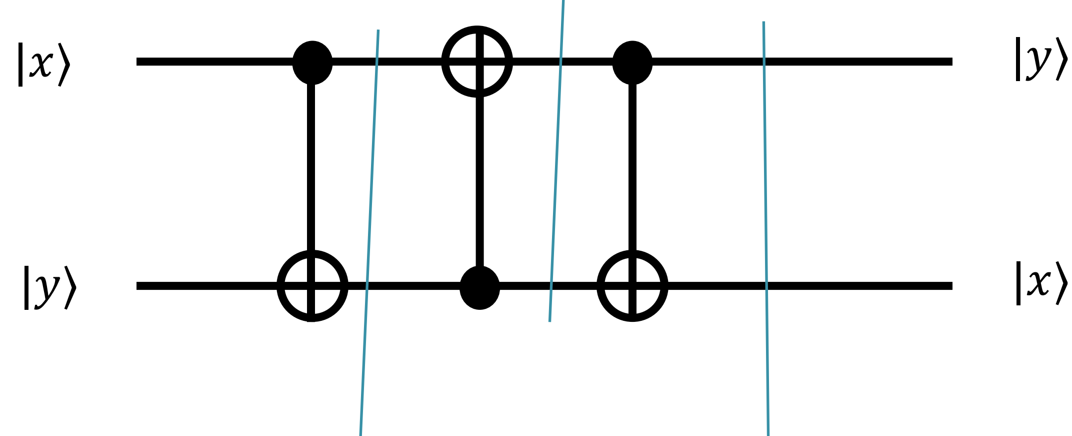
What gate is this?

CNOTs concept test



CNOTs concept test

State here is
$|x \oplus (x \oplus y)\rangle|x \oplus y\rangle = |y\rangle|x \oplus y\rangle$



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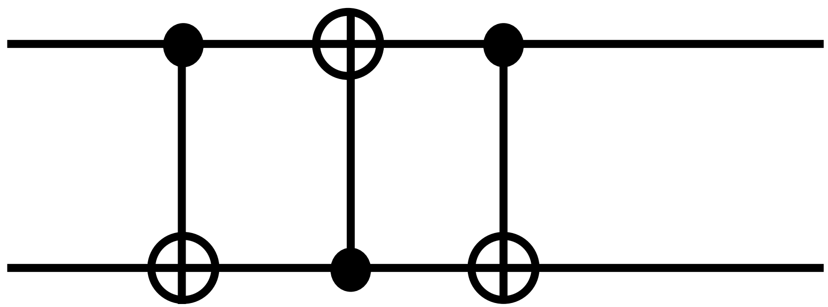
CNOTs concept test



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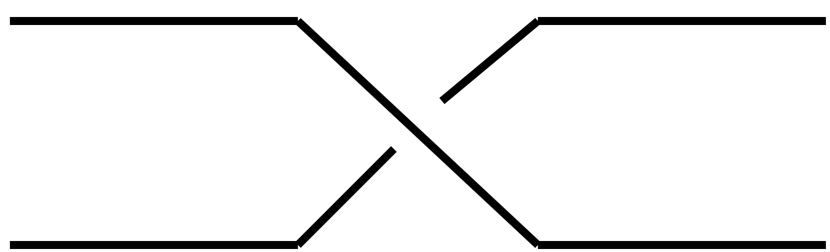

Summary: Axioms

States

Dynamics

Measurements

Composite Systems

Summary:

- Tensor products
- 1 qubit unitaries plus 1 CNOT can prepare arbitrary 2 qubit state
- You NEED the CNOT

- CNOTS can combine in interesting ways.

Reading

- We have finished Chapter 1
- Please read 2.1-2.4 for Wednesday (Monday is MLK day)