# Quantum Searching- Grover , Optimality

PHYS/CSCI 3090

Prof. Alexandra Kolla

Alexandra.Kolla@Colorado.edu
ECES 122

Prof. Graeme Smith

Graeme.Smith@Colorado.edu
JILA S326

https://home.cs.colorado.edu/~alko5368/indexCSCI3090.html

# Come see us!

- Alexandra Kolla/ Graeme Smith: Friday 3:00-4:00 pm, over zoom.
- Ariel Shlosberg: Tu/Th 2:00-4:00pm, over zoom
- Steven Kordonowy: Th 11am-12pm, over zoom.
- Matteo Wilczak: Wednesday, 1-2pm, over zoom.

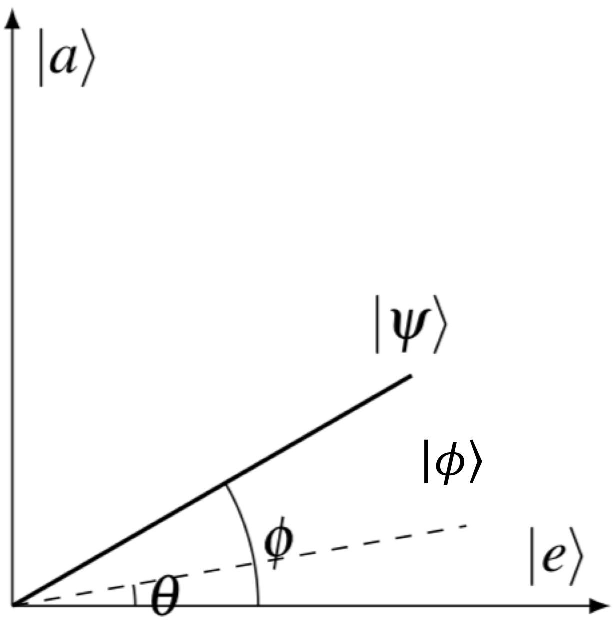
# Today

- Homework out, due next Monday at noon on Canvas

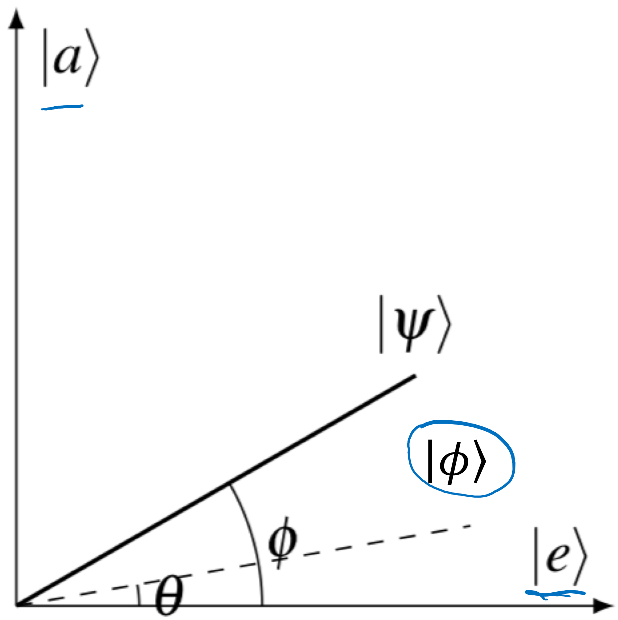
# Today

- Optimality of Grover

# The start of one iteration



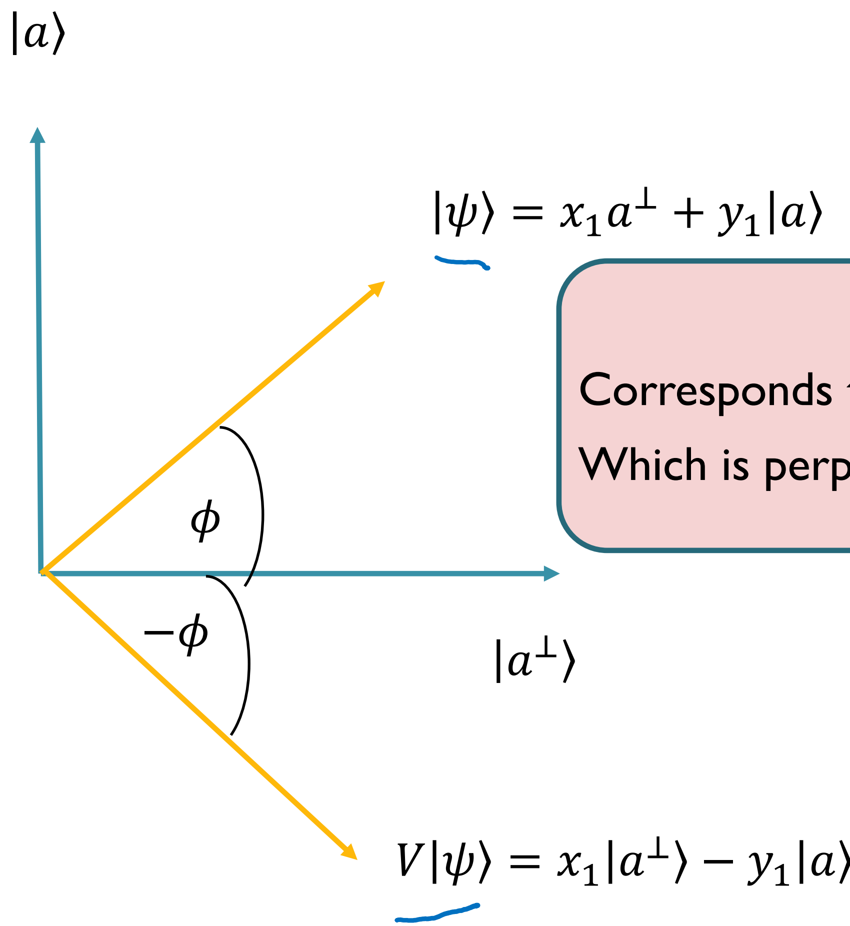
# The start of one iteration




$$|\phi\rangle = \frac{1}{\sqrt{N}} \sum_{0<x\leq N} |x\rangle_n$$

$$|e\rangle = \frac{1}{\sqrt{N-1}} \sum_{x\neq a} |x\rangle_n$$

# After V




$$V = I - 2|a\rangle\langle a|$$

Corresponds to a flip over $|e\rangle = \frac{1}{\sqrt{N-1}}\sum_{x \neq a}|x\rangle_n$

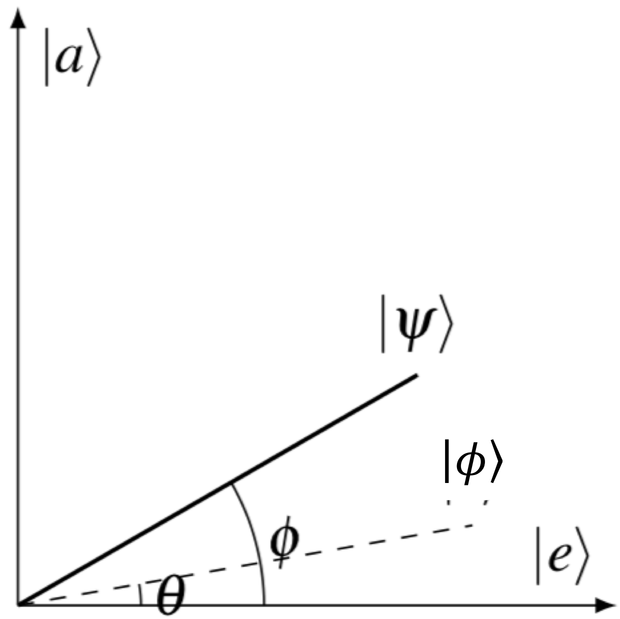
Which is perpendicular to $|a\rangle$

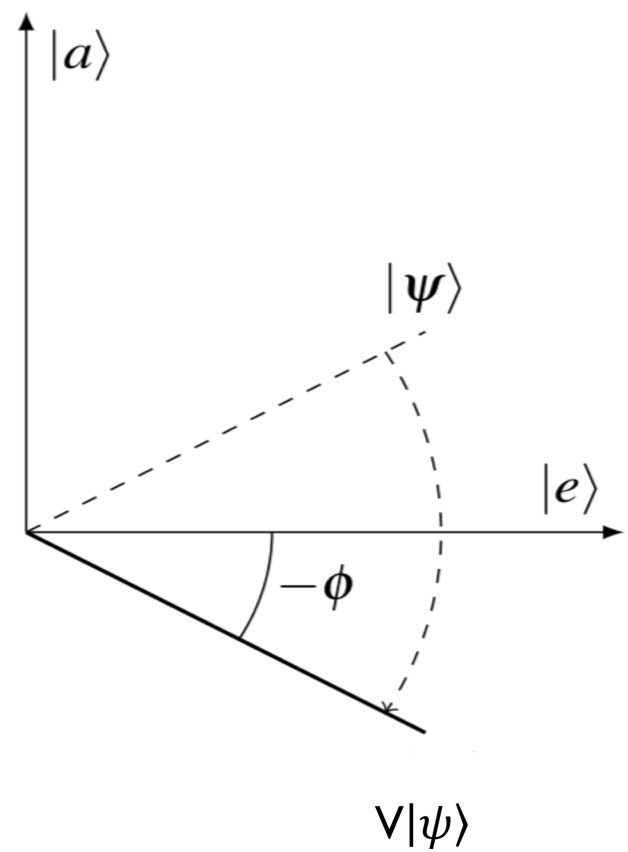
# After V

$$V = I - 2|a\rangle\langle a|$$

Corresponds to a flip over $|e\rangle$

$$|e\rangle = \frac{1}{\sqrt{N-1}}\sum_{x\neq a}|x\rangle_n$$

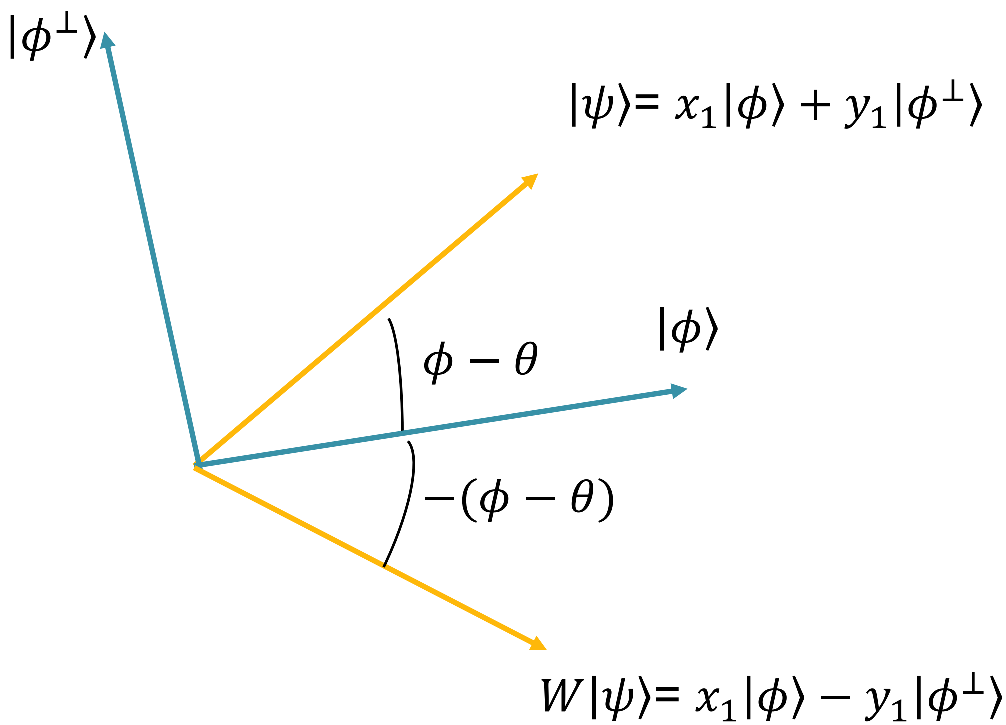


# After W

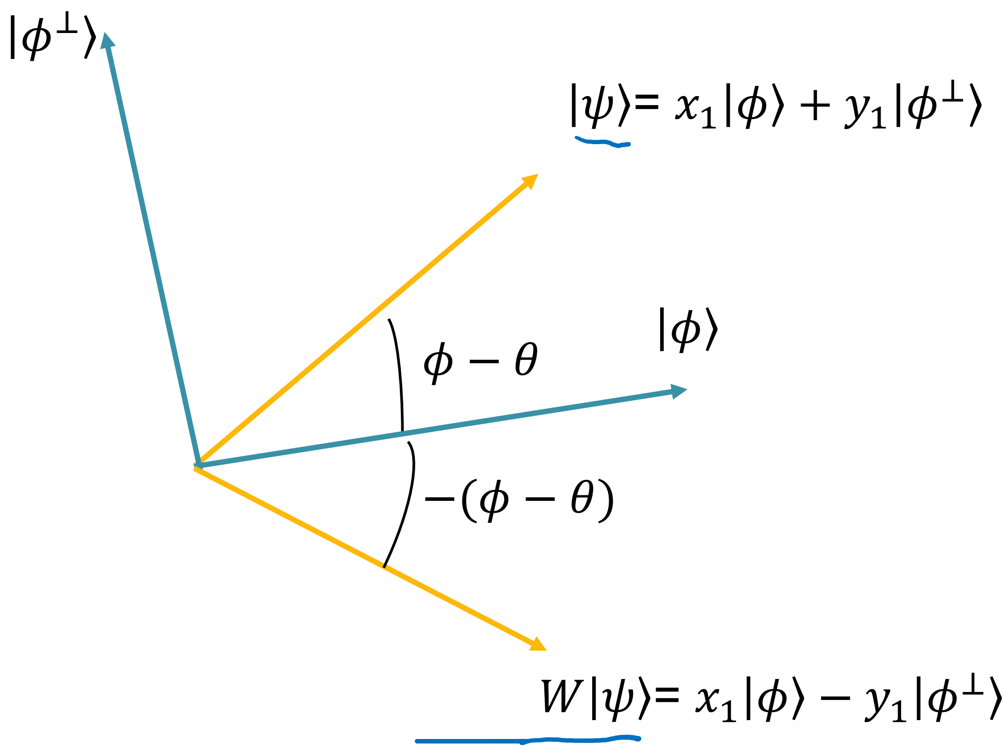
W= $2|\phi\rangle\langle\phi| - I$

Corresponds to a flip over $|\phi\rangle$



# After W

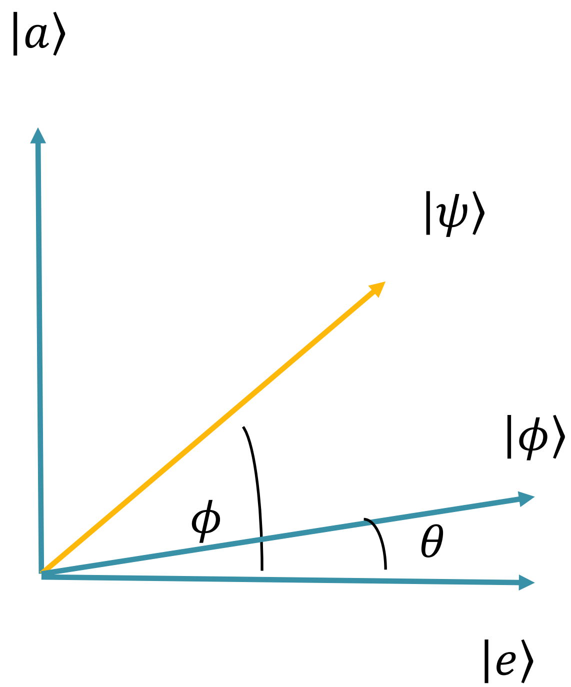
W= $2|\phi\rangle\langle\phi| - I$
Corresponds to a flip over $|\phi\rangle$



# Putting it together

$$|\phi\rangle = \frac{1}{\sqrt{N}} \sum_{0<x\leq N} |x\rangle_n$$

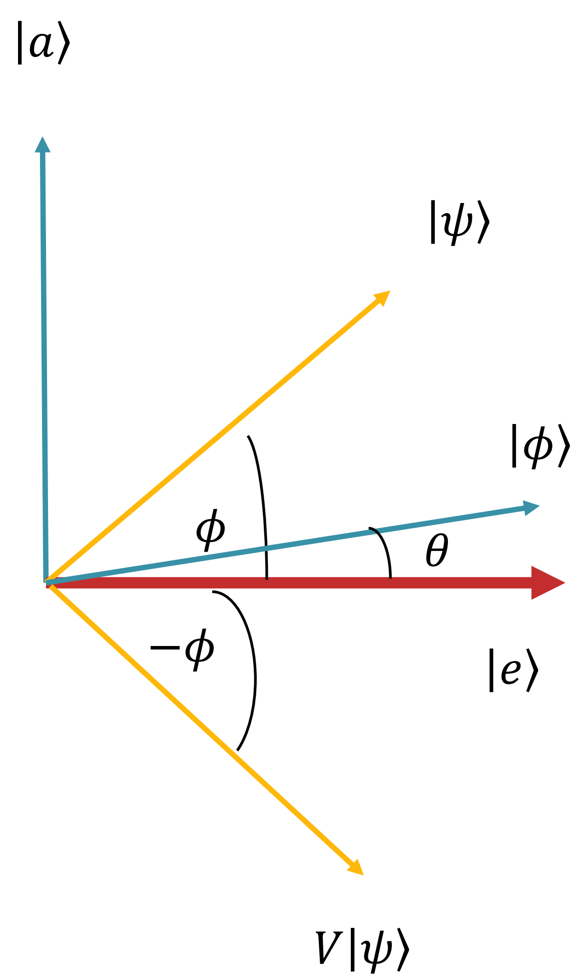
$$|e\rangle = \frac{1}{\sqrt{N-1}} \sum_{x\neq a} |x\rangle_n$$



# Putting it together

$$|\phi\rangle = \frac{1}{\sqrt{N}} \sum_{0<x\leq N} |x\rangle_n$$

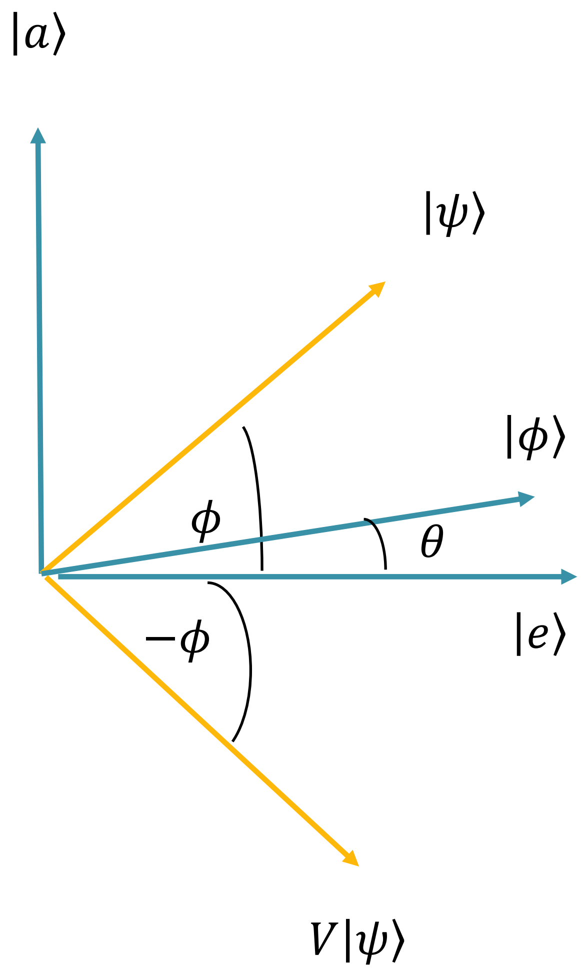
$$|e\rangle = \frac{1}{\sqrt{N-1}} \sum_{x\neq a} |x\rangle_n$$



# Putting it together

$$|\phi\rangle = \frac{1}{\sqrt{N}} \sum_{0<x\leq N} |x\rangle_n$$

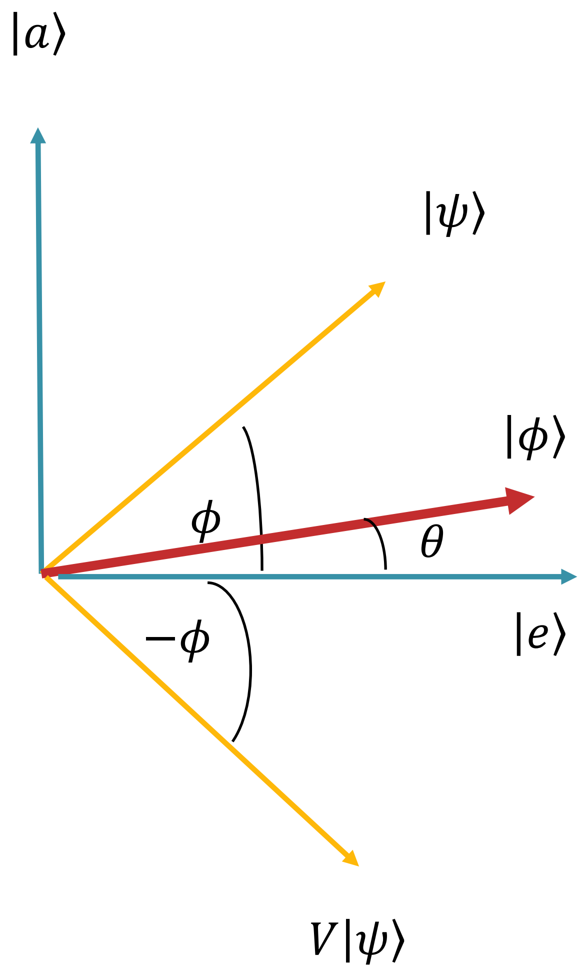
$$|e\rangle = \frac{1}{\sqrt{N-1}} \sum_{x\neq a} |x\rangle_n$$



# Putting it together

$$|\phi\rangle = \frac{1}{\sqrt{N}} \sum_{0<x\leq N} |x\rangle_n$$

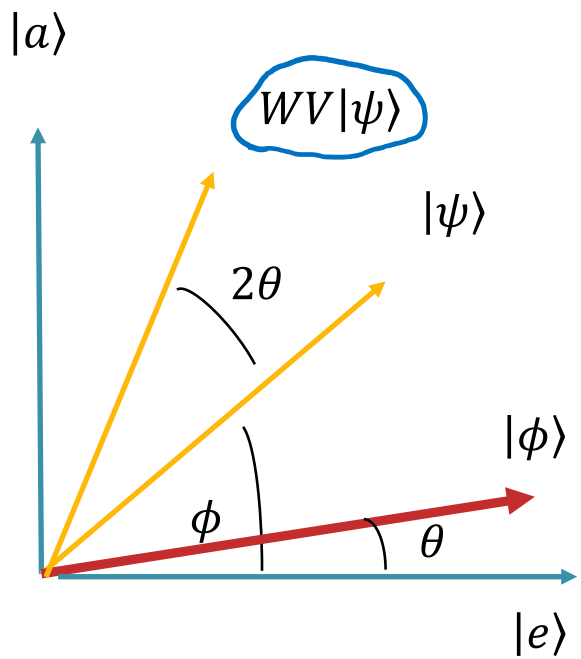
$$|e\rangle = \frac{1}{\sqrt{N-1}} \sum_{x\neq a} |x\rangle_n$$



# Putting it together

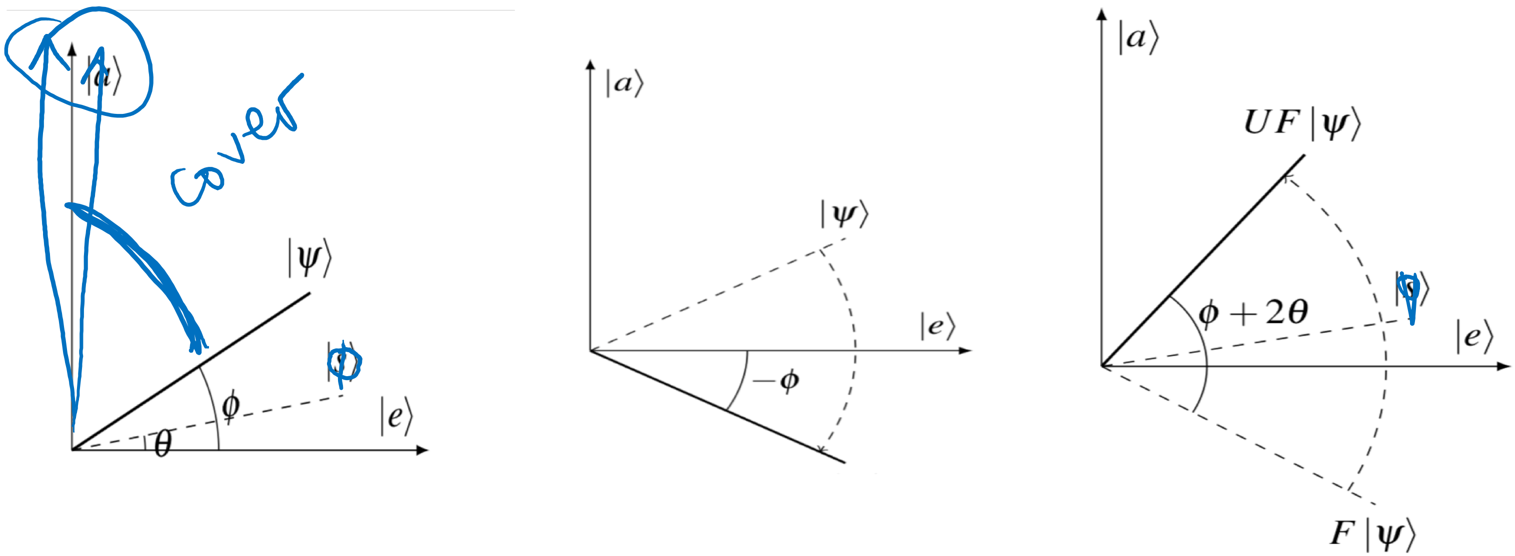
$$|\phi\rangle = \frac{1}{\sqrt{N}} \sum_{0<x\leq N} |x\rangle_n$$

$$|e\rangle = \frac{1}{\sqrt{N-1}} \sum_{x\neq a} |x\rangle_n$$



# In the end

$$|e\rangle = \frac{1}{\sqrt{N-1}} \sum_{x \neq a} |x\rangle_n$$



# The Algorithm

- We concluded that V corresponds to a reflection over $|e\rangle$ (perp to $|a\rangle$) and W corresponds to a reflection over $|\phi\rangle$ (uniform superposition, and also starting state).
- If we define θ as the angle between $|\phi\rangle$ and $|e\rangle$, and ϕ as the angle between $|\psi\rangle$ and $|e\rangle$ (where $|\psi\rangle$ is the state at the current iteration), we see that the transformations perform the following rotations:
- $\phi \overset{V}{\rightarrow} -\phi \overset{W}{\rightarrow} \phi + 2\theta$

# The Algorithm

- After one iteration, we rotate the state vector by $2\theta = 2\arcsin\left(\frac{1}{\sqrt{N}}\right) \sim 2/\sqrt{N}$
- Since we start out at state $|\phi\rangle$(uniform) almost orthogonal to $|a\rangle$ ,(Assuming N is large), we need to rotate by $\frac{\pi}{2}$.
- So we need about $\frac{\frac{\pi}{2}-\theta}{2\theta} \sim \Theta(\sqrt{N})$ applications of the algorithm.

# The Algorithm-Is it optimal?

- Why can't we design another quantum algorithm with less queries?
- How do we even prove optimality?
- To prove optimality, we show that any sequence of unitary operators (combined with calls to the oracle) that distinguish between the function that has 0 everywhere and the function which is 1 at the $a$'th position requires at least $Q(\sqrt{N})$ calls of the oracle.

$f_0(x) = 0 \quad \forall x$

$f_a(x) = \begin{cases} 0 & \forall x \neq a \\ 1 & x = a \end{cases}$

$2^n$ inputs. $f_0, f_a$ differ in only 1

# The Algorithm-Is it optimal?

- Let $U_1, U_2, \dots$ be some unitaries and $U_f$ the oracle corresponding to a function f.
- Let $|v_{f,k}\rangle = U_k U_f U_{k-1} U_f \dots U_1 \, |\phi\rangle$ be the state of the input register after k iterations of this new algorithm.
- Let $|\phi_k\rangle = U_k U_{k-1} \dots U_1 \, |\phi\rangle$

1 iteration: $U_i U_f$

$f_0$ , $f_a$

# The Algorithm-Is it optimal?

- Let $U_1, U_2, \ldots$ be some unitaries and $U_f$ the oracle corresponding to a function f.
- ① Let $|v_{f_a,k}\rangle = U_k U_{f_a} U_{k-1} U_{f_a} \ldots U_1\ |\phi\rangle$ be the state of the input register after k iterations of this new algorithm.
- ② Let $|\phi_k\rangle = U_k U_{k-1} \ldots U_1\ |\phi\rangle$
- Question: For what f is $|v_{f,k}\rangle = |\phi_k\rangle$?

Answer: $U_f = I \Leftrightarrow f = f_0$ ✓

$f_0 \rightarrow U_{f_0} = I_n$

$f_a \rightarrow U_{f_a} = I - 2|a\rangle\langle a|$

$f: \{0,1\}^n \rightarrow \{0,1\}$

# The Algorithm-Is it optimal?

- Let $U_1, U_2, \ldots$ be some unitaries and $U_f$ the oracle corresponding to a function f.
- Let $|v_{f,k}\rangle = U_k U_f U_{k-1} U_f \ldots U_1\ |\phi\rangle$ be the state of the input register after k iterations of this new algorithm.
- Let $|\phi_k\rangle = U_k U_{k-1} \ldots U_1\ |\phi\rangle$
- Question: For what f is $|v_{f,k}\rangle = |\phi_k\rangle$?
- $|\phi_k\rangle$ corresponds to the function which is zero everywhere (no marked element)

# The Algorithm-Is it optimal?

- Let $U_1, U_2, \ldots$ be some unitaries and $U_f$ the oracle corresponding to our "marked element" function f, f(a)=1.
- Let $|v_{f,k}\rangle = U_k U_f U_{k-1} U_f \ldots U_1 \; |\phi\rangle$ be the state of the input register after k iterations of this new algorithm.
- Let $|\phi_k\rangle = U_k U_{k-1} \ldots U_1 \; |\phi\rangle$
- Question: is $|\phi_k\rangle$ independent of a?

# The Algorithm-Is it optimal?

- Let $U_1, U_2, \ldots$ be some unitaries and $U_f$ the oracle corresponding to our "marked element" function f, f(a)=1.
- Let $\left|v_{f,k}\right\rangle = U_k U_f U_{k-1} U_f \ldots U_1\, |\phi\rangle$ be the state of the input register after k iterations of this new algorithm.
- Let $|\phi_k\rangle = U_k U_{k-1} \ldots U_1\, |\phi\rangle$
- Since the oracle depends on a, $U_f$ changes with a, and so does $\left|v_{f,k}\right\rangle$.
- Since no measurement is done till the end, we can get no information about a before measuring, so the $U_i$ are independent of a, and so is $|\phi_k\rangle$.

# The Algorithm-Is it optimal?

- $|v_{f,k}\rangle = U_k U_f U_{k-1} U_f \ldots U_1\ |\phi\rangle$, $|\phi_k\rangle = U_k U_{k-1} \ldots U_1\ |\phi\rangle$
- Define $t_{a,k} = ||v_{f,k}\rangle$- $|\phi_k\rangle|$.
- Measures the error between a run of the algorithm where f is zero everywhere, or 1 at a marked element a.

# Optimality Proof- Big picture

We will show:

Claim 1: $t_{a,k} = \left|\left|v_{f,k}\right\rangle - \left|\phi_k\right\rangle\right| \leq \sum_{i=1\ to\ k} 2\langle a, \phi_i\rangle$

- Assume our new algorithm runs for T steps.
- Since our algorithm needs to distinguish between the function which is zero everywhere and the one which has 1 at $a$, $t_{a,T}$ must be large.
- Recall $t_{a,T}$ is the difference between the output vectors when these two functions (zero everywhere, or 1 at a) are used as inputs.
- So for the sake or argument say that we need $t_{a,T}$ > ½ for the algorithm to be successful.

# Optimality Proof

- Assume Claim 1:

$$t_{a,k} = \left\| |v_{f,k}\rangle - |\phi_k\rangle \right\| \leq \sum_{i=1 \text{ to } k} 2\langle a, \phi_i \rangle$$

k-steps algo , $|\phi_i\rangle = U_i U_{i-1} \ldots U_1 |0\rangle$

- Show that we need at least $\mathrm{T} = \Omega(\sqrt{N})$ iterations (invocations of $U_f$) to achieve $t_{a,T} > 1/2$.

# Optimality Proof-Main Tool

- Cauchy-Schwartz Inequality

Version 1: For any two vectors $|v\rangle$, $|u\rangle$ the following is true:

$$|\langle v,u\rangle| \le ||v||\,||u||$$

$\sqrt{\langle v,v\rangle}$ $\sqrt{\langle u,u\rangle}$

$= \sqrt{\sum v_i^2}$ $\sqrt{\sum u_i^2}$

Version 2 (for reals):

$$\sum_{i=1}^{n} v_i u_i \le \sqrt{\sum_{i=1}^{n} v_i^2}\ \sqrt{\sum_{i=1}^{n} u_i^2}$$

Question: derive Version 2 from Version 1.

# Optimality Proof

- Assume Claim 1:

$$t_{a,k} = \left\| |v_{f,k}\rangle - |\phi_k\rangle \right\| \leq \sum_{i=1 \; to \; k} 2\langle a, \phi_i \rangle$$

- Show that we need at least $\mathrm{T} = \Omega(\sqrt{N})$ iterations (invocations of $U_f$) to achieve $t_{a,T} > 1/2$.

Proof:

$$\frac{1}{2} < t_{a,T} \overset{\text{claim 1}}{\leq} \sum_{i=1}^{T-1} \underbrace{2}_{v_i} \underbrace{\langle a, \phi_i \rangle}_{u_i} \overset{\text{Cauchy-Schwartz}}{\leq}$$

$$\left( \sum v_i u_i \leq \sqrt{\sum v_i^2} \cdot \sqrt{\sum u_i^2} \right)$$

$$\leq \sqrt{\sum_{i=1}^{T-1} 4} \cdot \sqrt{\sum_{i=1}^{T-1} \langle a, \phi_i \rangle^2} = 2\sqrt{T-1} \cdot \sqrt{\sum_{i=1}^{T-1} \langle a, \phi_i \rangle^2}$$

# Optimality Proof

- Assume Claim 1:

$t_{a,k} = \big|\,|v_{f,k}\rangle - |\phi_k\rangle\big| \leq \sum_{i=1\ to\ k} 2\langle a, \phi_i \rangle$

- Show that we need at least $\mathrm{T} = \Omega(\sqrt{N})$ iterations (invocations of $U_f$) to achieve $t_{a,T} > 1/2$.
- Proof:

$$\frac{1}{2} < t_{a,T} \leq \sum_{i=1\ to\ T-1} 2\langle a, \phi_i \rangle \leq \sqrt{\sum_{i=1}^{T-1} 2^2} \sqrt{\sum_{i=1}^{T-1} \langle a, \phi_i \rangle^2} \leq$$

$$2\sqrt{T-1} \sqrt{\sum_{i=1}^{T-1} \langle a, \phi_i \rangle^2}$$

# Optimality Proof

- Assume Claim 1:

$t_{a,k} = ||v_{f,k}\rangle - |\phi_k\rangle| \leq \sum_{i=1 \ to \ k} 2\langle a, \phi_i \rangle$

- Show that we need at least $T = \Omega(\sqrt{N})$ iterations (invocations of $U_f$) to achieve $t_{a,T} > 1/2$.
- Proof:

$$\frac{1}{2} < t_{a,T} \leq 2\sqrt{T-1}\sqrt{\sum_{i=1}^{T-1} \langle a, \phi_i \rangle^2}$$

Question: how to bound $\sqrt{\sum_{i=1}^{T-1} \langle a, \phi_i \rangle^2}$?

# Optimality Proof

- Question:

Assume we are given that $\{|a_1\rangle, \ldots, |a_N\rangle\}$ is an orthonormal basis for the N-dimensional Hilbert space. Let $|\phi\rangle$ be an arbitrary unit vector on this space. What is $\sum_{i=1}^{N} \langle a_i, \phi\rangle^2$ ?

$= 1$

$\Longleftrightarrow \|\phi\| = 1$

# Optimality Proof

$$\frac{1}{2} < t_{a,T} \leq 2\sqrt{T-1}\sqrt{\sum_{i=1}^{T-1} \langle a, \phi_i \rangle^2}$$

Question: how to bound $\sqrt{\sum_{i=1}^{T-1} \langle a, \phi_i \rangle^2}$?

# Optimality Proof

$\sum_a \langle a, \phi_i \rangle^2 = 1$ for all $i$

↳ sum over all n-bit vectors, {a}

So $\sum_{i=1\ to\ T-1} \sum_a \langle a, \phi_i \rangle^2 = T - 1$

$\sum_{i=1\ to\ T-1} \sum_a \langle a, \phi_i \rangle^2 = \sum_a \sum_i \langle a, \phi_i \rangle^2 = T - 1$

Thus there exists an a such that

$\sum_i \langle a, \phi_i \rangle^2 < \frac{T-1}{N}$

- Chose that a as our worst case for the algorithm.

# Optimality Proof

$$\frac{1}{2} < t_{a,T} \leq 2\sqrt{T-1}\sqrt{\sum_{i=1}^{T-1}\langle a, \phi_i\rangle^2}$$

For the worst case a we chose:

$$\frac{1}{2} < t_{a,T} \leq 2\sqrt{T-1}\sqrt{\frac{T-1}{N}} = \frac{T-1}{\sqrt{N}}$$

$T-1 > \sqrt{N}/2$

$T > \frac{\sqrt{N}}{2}+1$

We need to take $T = \Omega(\sqrt{N})$ as desired.