Quantum Searching-Grover

PHYS/CSCI 3090

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Come see us!

- Alexandra Kolla/ Graeme Smith: Friday 3:00-4:00 pm, over zoom.
- Ariel Shlosberg: Tu/Th 2:00-4:00pm, over zoom
- Steven Kordonowy: Th 11am-12pm, over zoom.
- Matteo Wilczak: Wednesday, 1-2pm, over zoom.

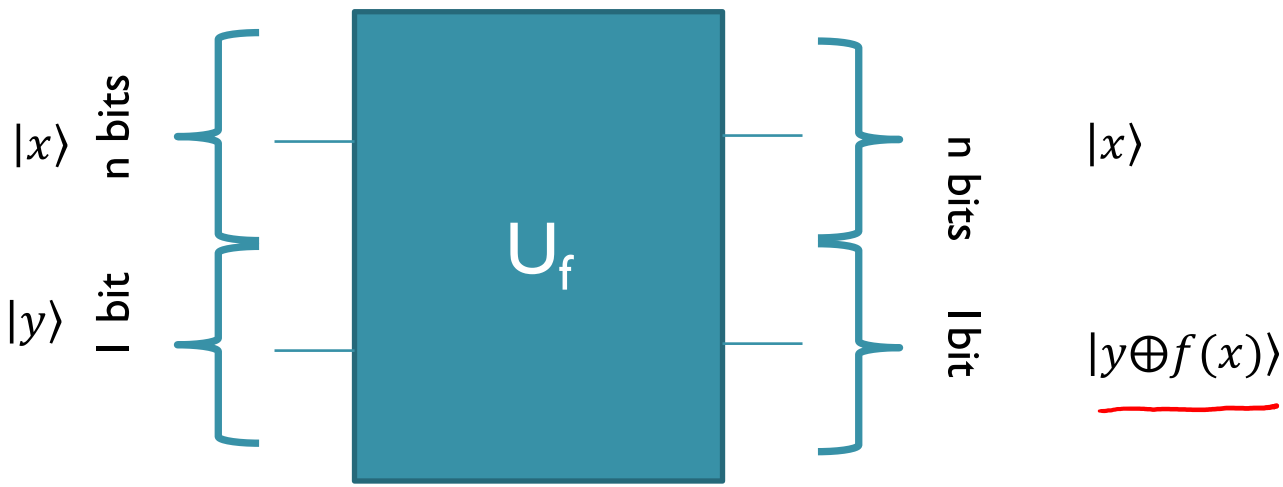
Today

- Grover's algorithm, continued.
- Optimality $\sqrt{N}$, $N = 2^n$

The problem

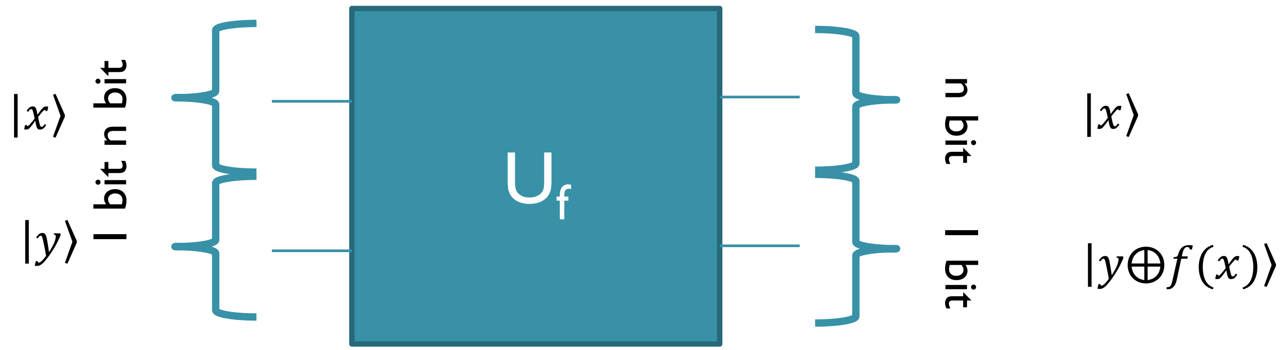
- Suppose we know that exactly one n-bit integer satisfies some condition.
- Namely, we have a special "marked" item a, such that, for some function $f: \{0,1\}^n \to \{0,1\}$ f(x)=1 iff x=a and 0 otherwise.

The setup



- We have a function $f(x) = \begin{cases} 1 \; if \; x = a \\ 0 \; o.w. \end{cases}$

The Unitary for Grover



- $V = I - 2|a\rangle\langle a|$

$$V|Y\rangle = V \sum_{0<x\leq 2^n} |x\rangle\langle x|Y\rangle = |Y\rangle - 2|a\rangle\langle a|Y\rangle$$

- See this operator as "amplifying" the amplitude of $|a\rangle$
- If we amplify the negative vectors enough then we could measure the required state with high probability.
- More of an "amplification" algorithm than search.

The Algorithm

- Start by preparing uniform superposition $|\phi\rangle = H^{\otimes n}\,|0\rangle_n \quad = \frac{1}{\sqrt{N}} \sum_{0<x\leq N} |x\rangle_n$
- $(N = 2^n)$
- Signifies maximal ignorance of special element a.
- The action of V on $|\phi\rangle$?

The Algorithm

- $V|\phi\rangle = (I - 2|a\rangle\langle a|)|\phi\rangle =$

$|\phi\rangle - 2|a\rangle\langle a||\phi\rangle = |\phi\rangle - \frac{2}{\sqrt{N}}|a\rangle =$

$\frac{1}{\sqrt{N}}(\sum_{x \neq a}|x\rangle_n) - \frac{1}{\sqrt{N}}|a\rangle$

$0.9|a\rangle + b|a'\rangle$

- **Applying the oracle to the initial state negates the amplitude of the satisfying element!**

The Algorithm

- We now introduce the Grover diffusion operator. Recall $|\phi\rangle = \frac{1}{\sqrt{N}} \sum_{0<x\leq N} |x\rangle_n$

- $\mathrm{W} = 2|\phi\rangle\langle\phi| - I$

- Question: Let x be a standard basis vector. What is $(|\phi\rangle\langle\phi|)|x\rangle$?

The Algorithm

- We now introduce the Grover diffusion operator.
- $W = 2|\phi\rangle\langle\phi| - I$
- One iteration of the algorithm consists of applying the operator $A = WV$ (that is, querying the oracle and then applying the diffusion operator).

how many? $|\phi\rangle \xrightarrow{\text{grover}} WV|\phi\rangle \xrightarrow{\text{grover}} WVWV|\phi\rangle \ldots$ constant $|a\rangle + b|a'\rangle$

The Algorithm

$$W = 2|\phi\rangle\langle\phi| - I, V = I - 2|a\rangle\langle a|$$

- After one iteration:

$$WV|\phi\rangle = (2|\phi\rangle\langle\phi| - I)(I - 2|a\rangle\langle a|)|\phi\rangle =$$
$$(2|\phi\rangle\langle\phi| + 2|a\rangle\langle a| - \frac{4}{\sqrt{N}}|\phi\rangle\langle a| - I)|\phi\rangle =$$
$$\left(1 - \frac{4}{N}\right)|\phi\rangle + \frac{2}{\sqrt{N}}|a\rangle$$

- We see that after one iteration, the probability of measuring *a* has increased.

The Algorithm

- It can be checked that the operation rotates the state vector by Type equation here. $\frac{2\sqrt{(N-1)}}{N} \sim \frac{2}{\sqrt{N}}$
- Since we start out almost orthogonal to $|a\rangle$ (Assuming N is large), we need to rotate by $\frac{\pi}{2}$.
- So we need about $\frac{\pi\sqrt{N}}{4}$ applications of the algorithm.

The start of one iteration

- We can visualize the operators in Grover's algorithm as reflections in state space. Consider the target vector $|a\rangle$ and the hyperplane of all other vectors $|e\rangle = \frac{1}{\sqrt{N-1}} \sum_{x \neq a} |x\rangle_n$
- Assume the algorithm is at state $|\psi\rangle$ at the current iteration

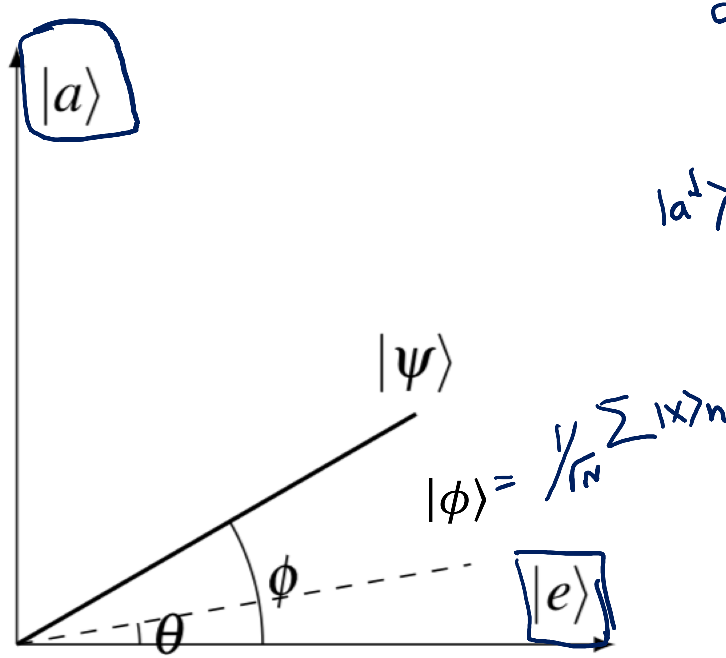
The start of one iteration

start iteration: $|\psi\rangle$

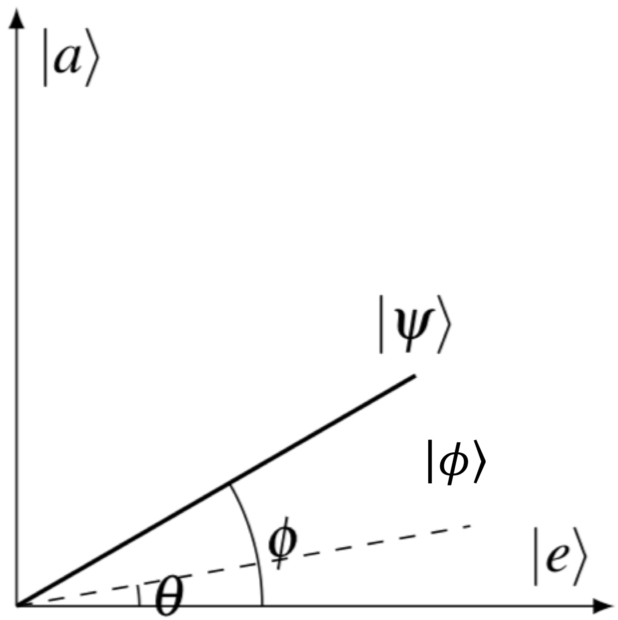
do $WV|\psi\rangle$

what do I get?

$$|a^{\perp}\rangle = |e\rangle = \frac{1}{\sqrt{N-1}} \sum_{x \neq a} |x\rangle_n$$



The start of one iteration



$$|\phi\rangle = \frac{1}{\sqrt{N}} \sum_{0<x\leq N} |x\rangle_n$$

$$|e\rangle = \frac{1}{\sqrt{N-1}} \sum_{x\neq a} |x\rangle_n$$

$\mathbb{C}^2$

After V - class exercise in 2D

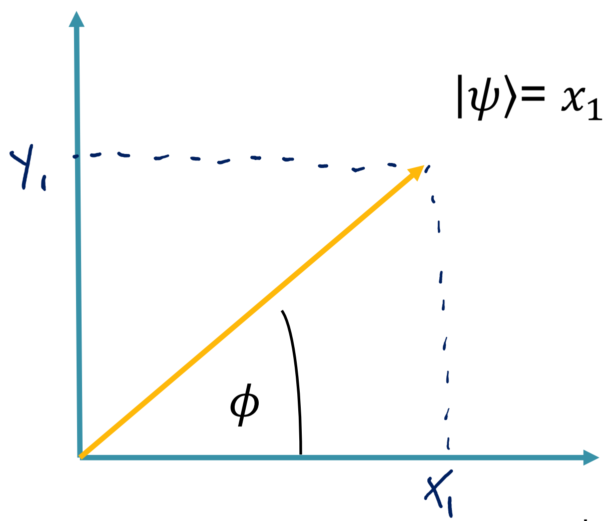
$|a\rangle$

$V = I - 2|a\rangle\langle a|$

$|\psi\rangle = (x_1, y_1)$

$V|\psi\rangle = (?, ?)$

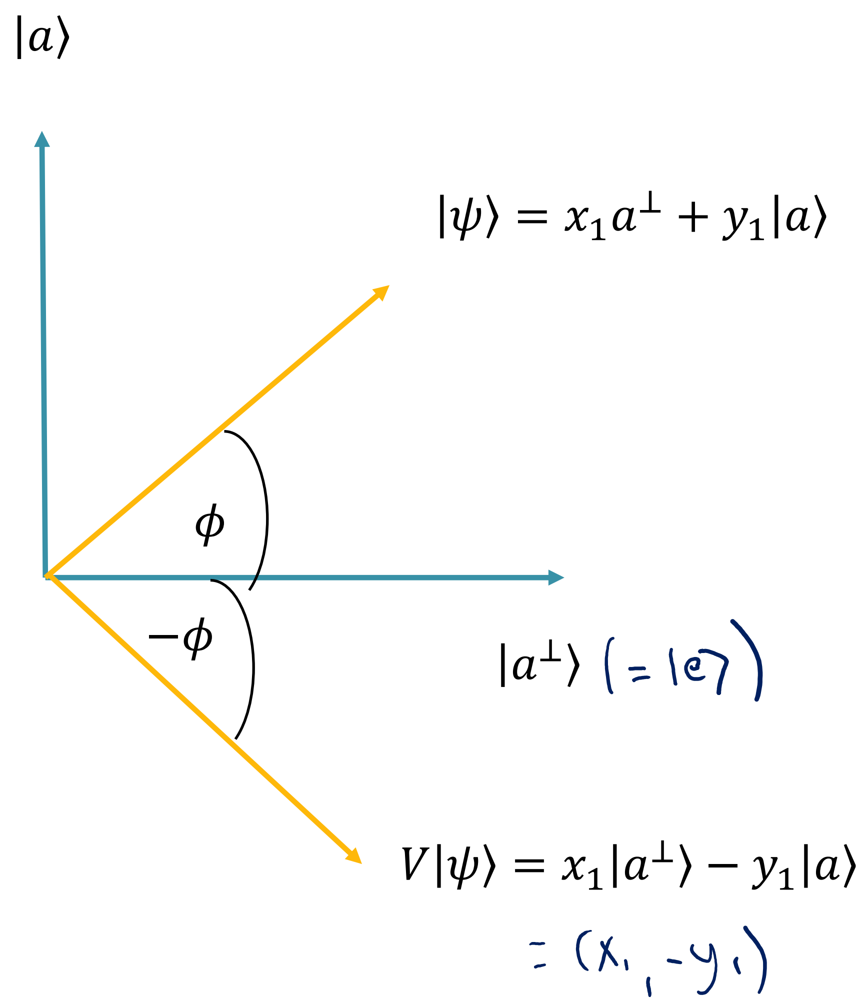
$|\psi\rangle = x_1|a^{\perp}\rangle + y_1|a\rangle$



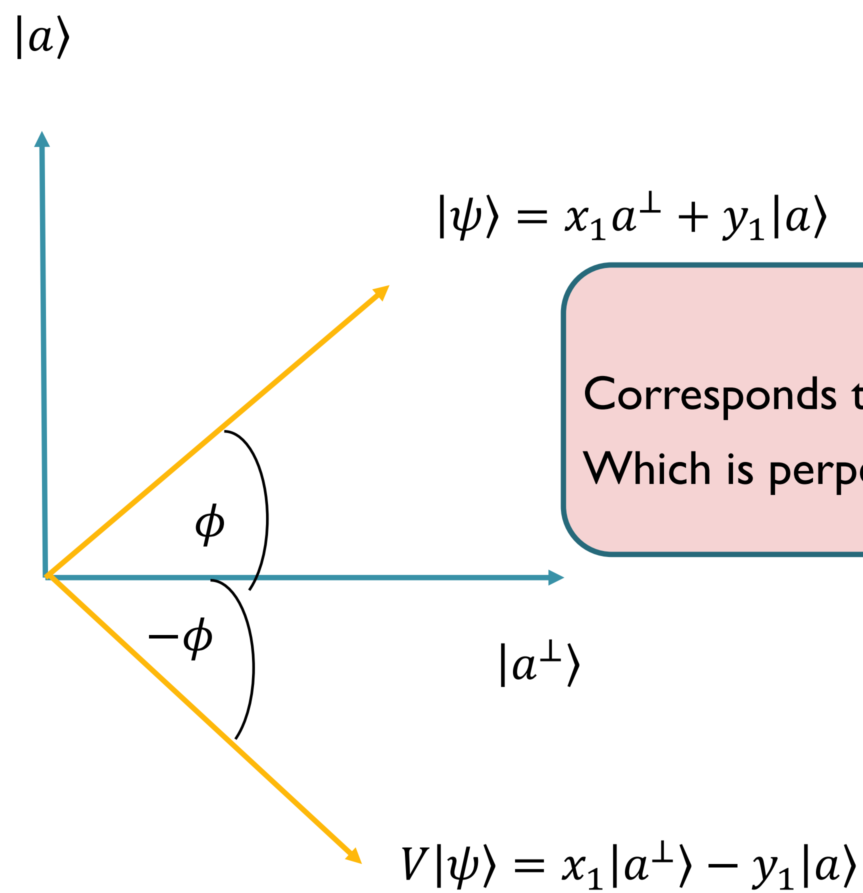
$|a^{\perp}\rangle = |e\rangle$

Question: What operator on the plane is $V = I - 2|a\rangle\langle a|$?
(Draw the vector $V|\psi\rangle$ on the plane)

After V - class exercise in 2D



After V - class exercise in 2D



$$V = I - 2|a\rangle\langle a|$$

Corresponds to a flip over $|e\rangle = \frac{1}{\sqrt{N-1}} \sum_{x \neq a} |x\rangle_n$

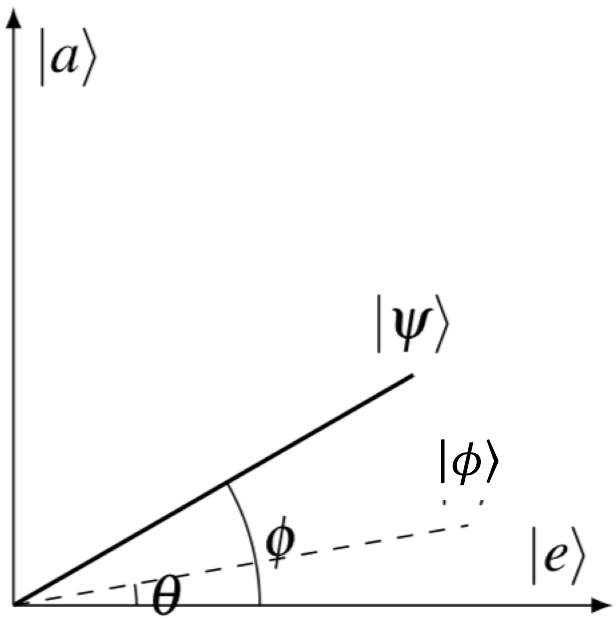
Which is perpendicular to $|a\rangle$

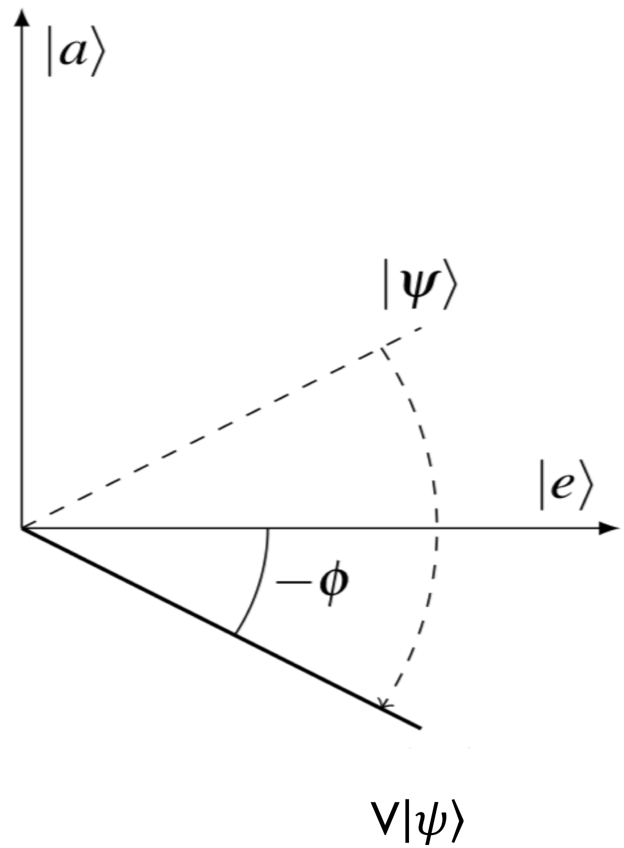
After V

$$V = I - 2|a\rangle\langle a|$$

Corresponds to a flip over $|e\rangle$

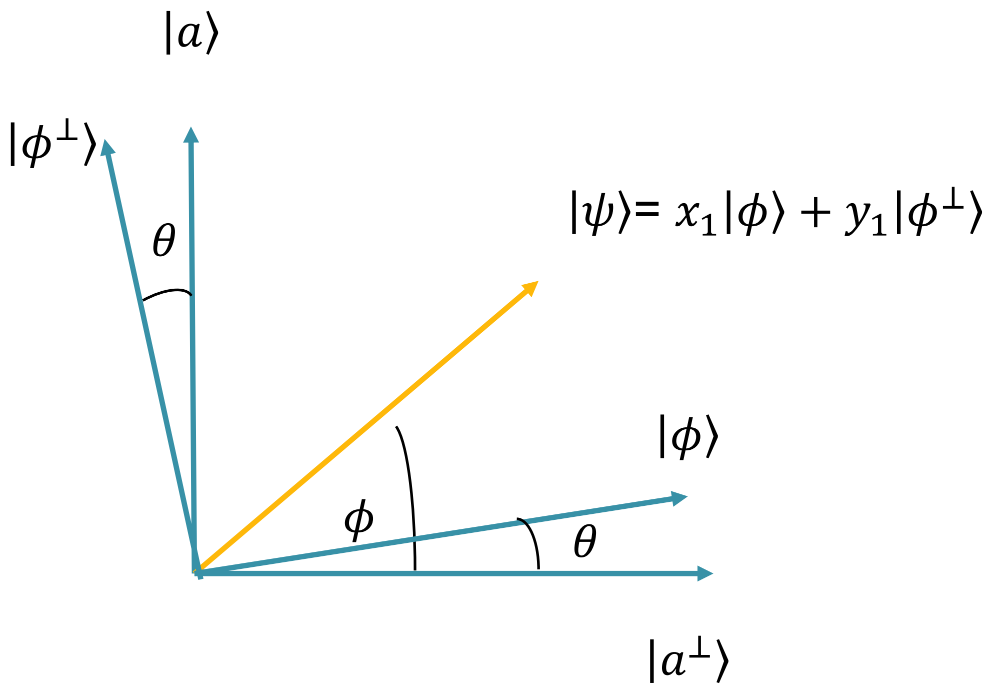
$$|e\rangle = \frac{1}{\sqrt{N-1}} \sum_{x \neq a} |x\rangle_n$$





After W - class exercise in 2D

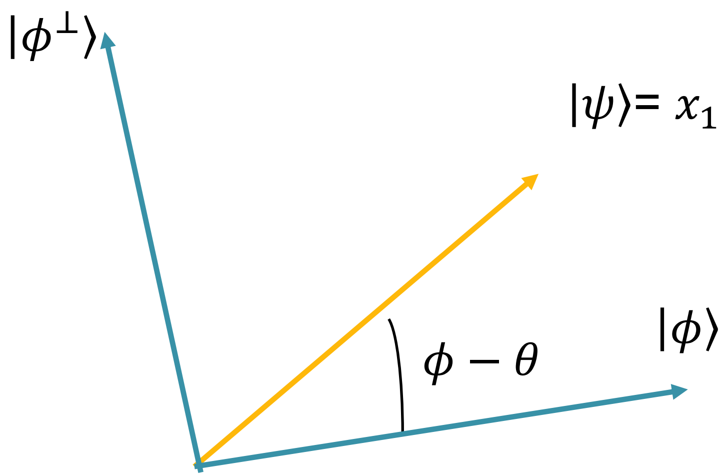
$W = 2|\phi\rangle\langle\phi| - I$



Question: What operator on the plane is $W = 2|\phi\rangle\langle\phi| - I$?
(Draw the vector $W|\psi\rangle$ on the plane)

After W - class exercise in 2D

$W = 2|\phi\rangle\langle\phi| - I$



$|\psi\rangle = x_1|\phi\rangle + y_1|\phi^{\perp}\rangle$

$|\psi\rangle = (x_1, y_1)$

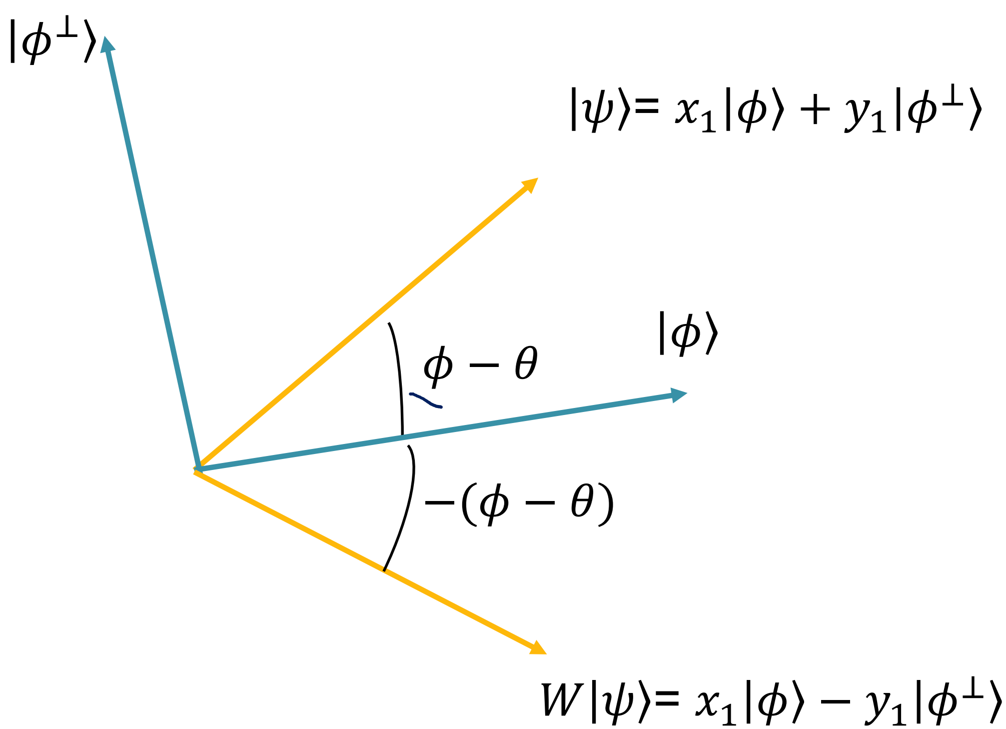
$W|\psi\rangle = (?, ?)$

Question: What operator on the plane is $W = 2|\phi\rangle\langle\phi| - I$?
(Draw the vector $W|\psi\rangle$ on the plane)

After W - class exercise in 2D

W= $2|\phi\rangle\langle\phi| - I$

Corresponds to a flip over $|\phi\rangle$



After W - class exercise in 2D

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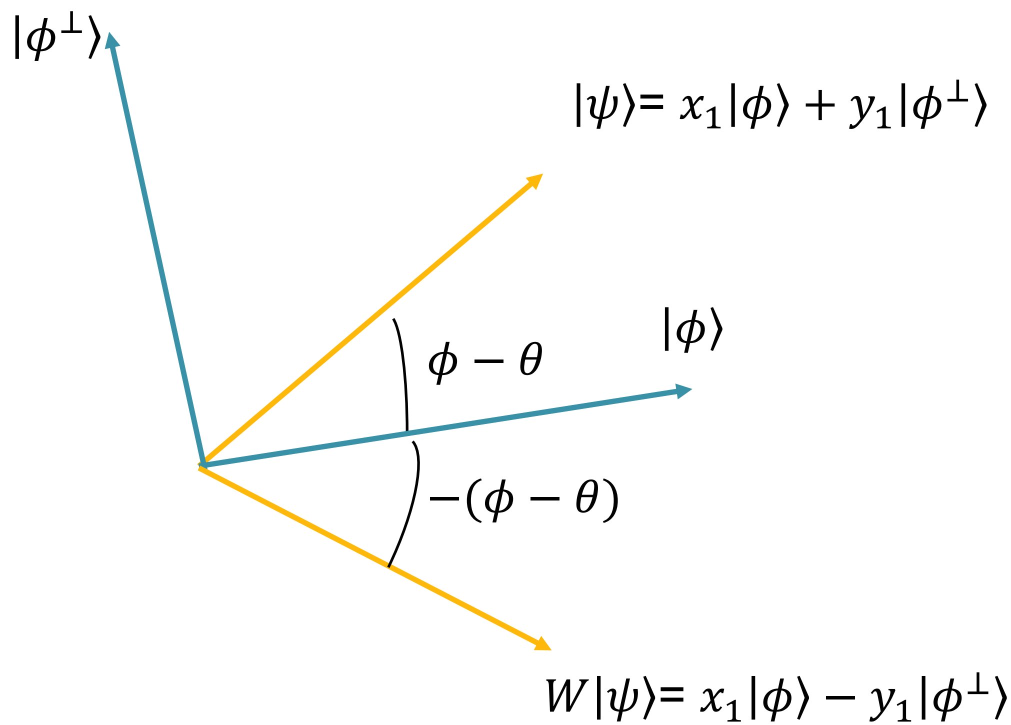
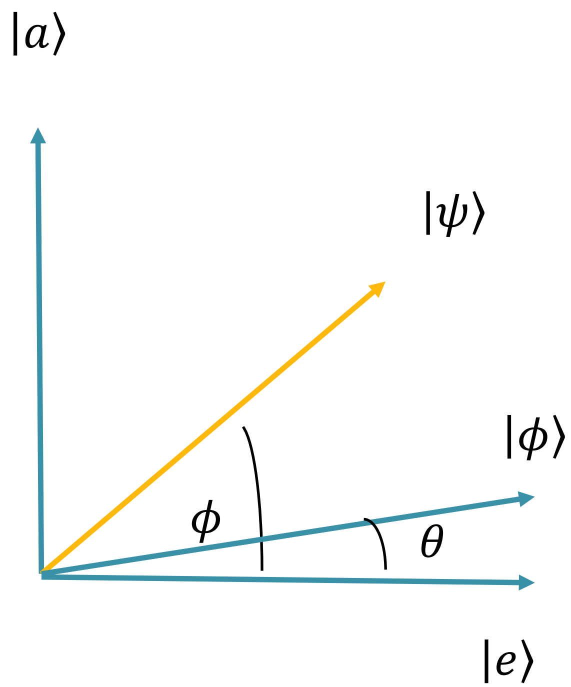


Putting it together

$$|\phi\rangle = \frac{1}{\sqrt{N}} \sum_{0<x\leq N} |x\rangle_n$$

$$|e\rangle = \frac{1}{\sqrt{N-1}} \sum_{x\neq a} |x\rangle_n$$



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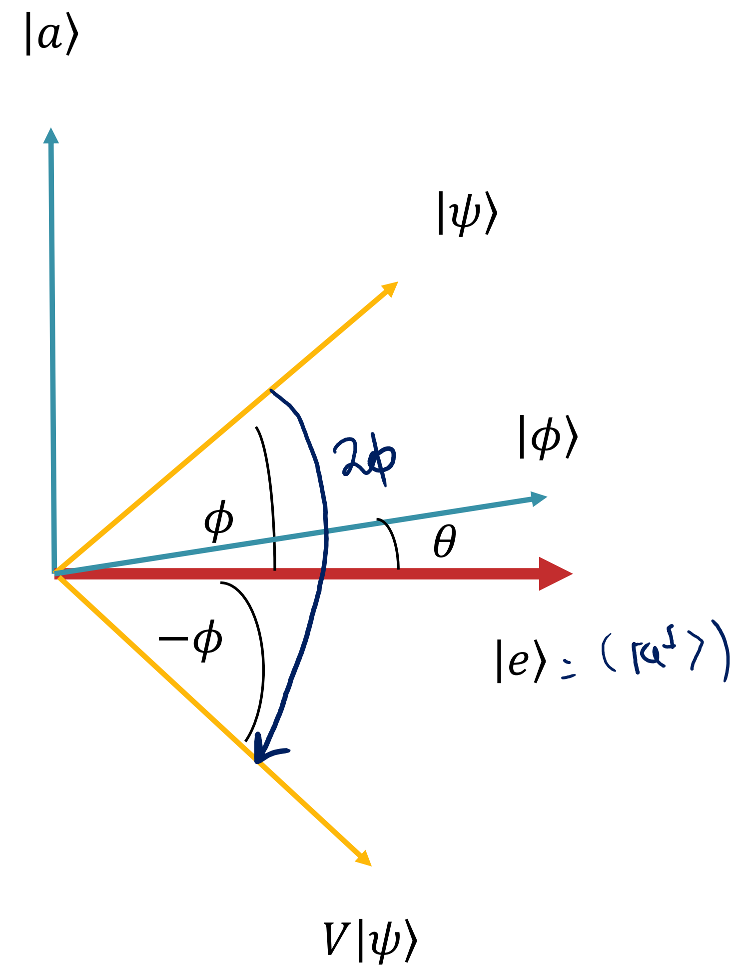

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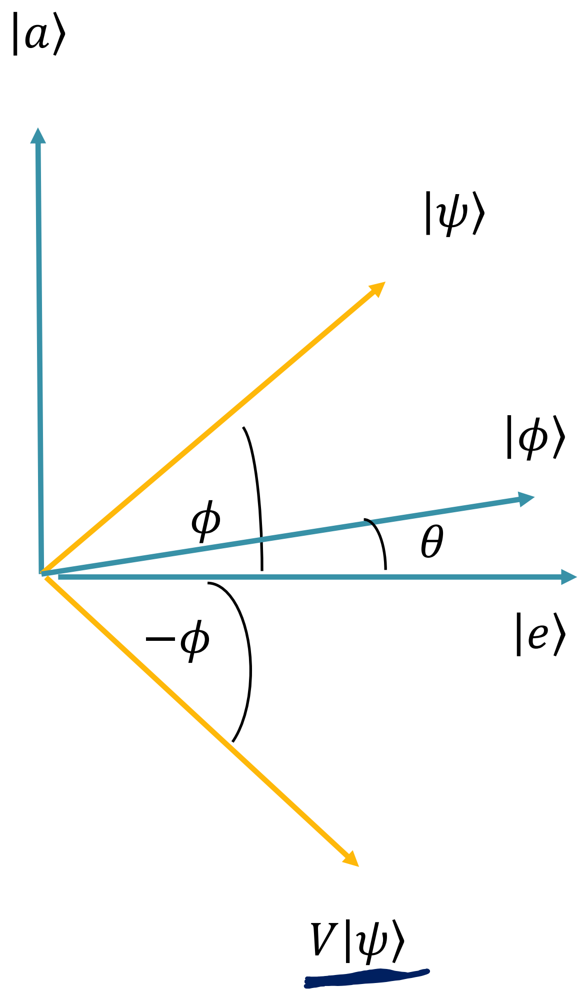

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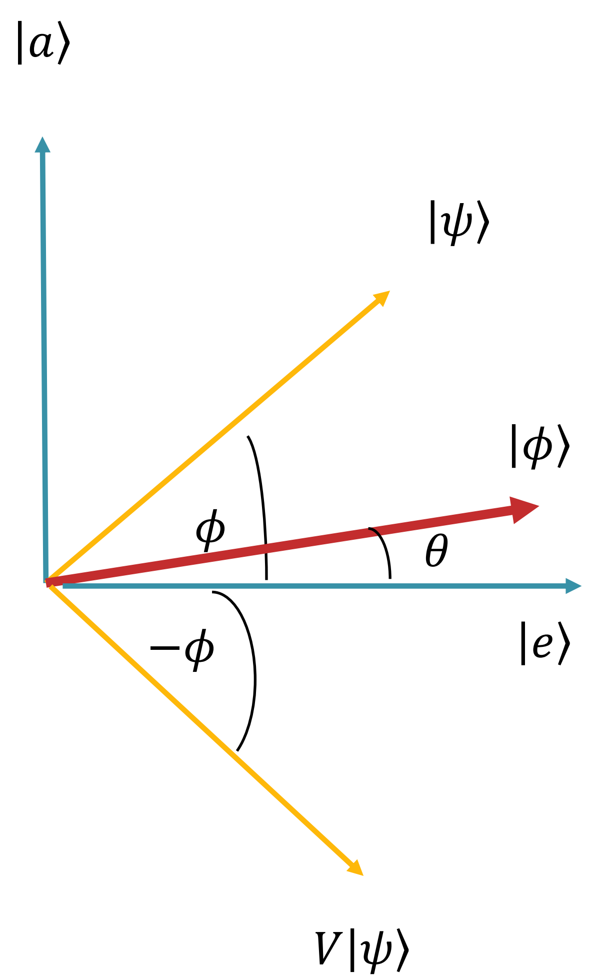

Putting it together

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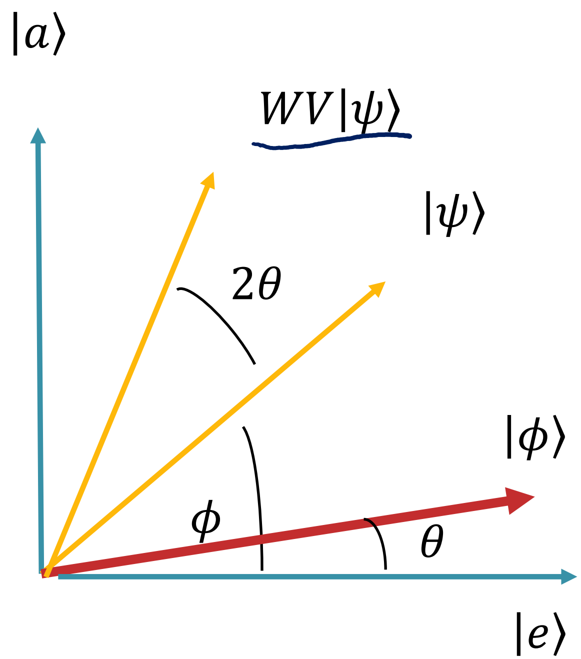
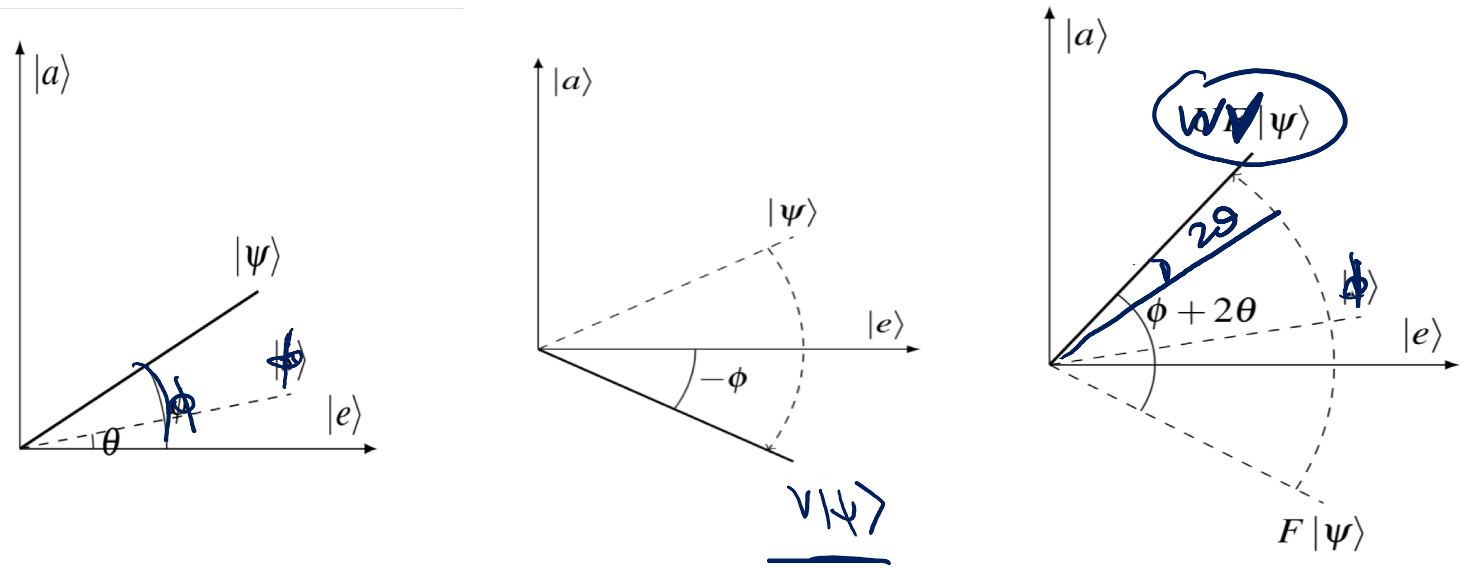


In the end

$$|e\rangle = \frac{1}{\sqrt{N-1}} \sum_{x \neq a} |x\rangle_n$$



The Algorithm

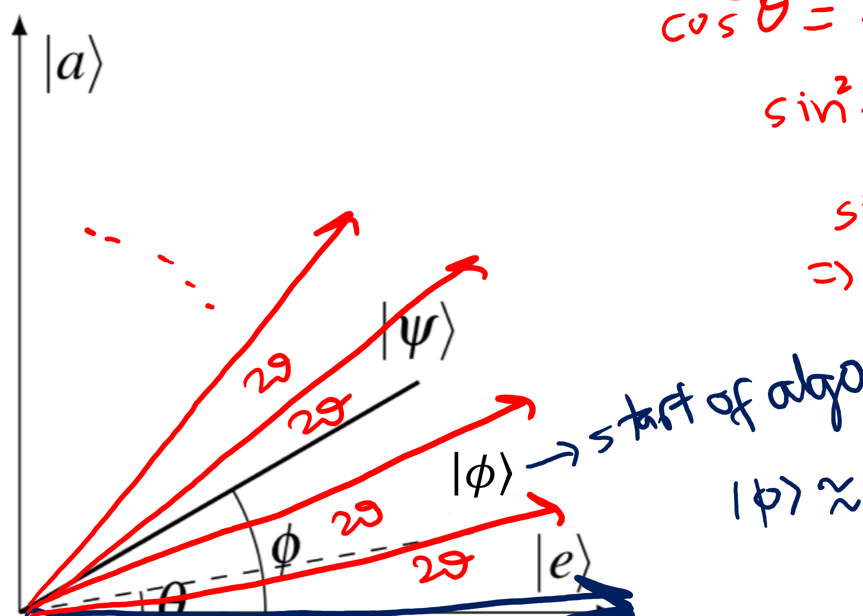
- We concluded that V corresponds to a reflection over $|e\rangle$ (perp to $|a\rangle$) and W corresponds to a reflection over $|\phi\rangle$ (uniform superposition, and also starting state).
- If we define θ as the angle between $|\phi\rangle$ and $|e\rangle$, and ϕ as the angle between $|\psi\rangle$ and $|e\rangle$ (where $|\psi\rangle$ is the state at the current iteration), we see that the transformations perform the following rotations:

The Algorithm

- We concluded that V corresponds to a reflection over $|e\rangle$ (perp to $|a\rangle$) and W corresponds to a reflection over $|\phi\rangle$ (uniform superposition, and also starting state).
- If we define θ as the angle between $|\phi\rangle$ and $|e\rangle$, and ϕ as the angle between $|\psi\rangle$ and $|e\rangle$ (where $|\psi\rangle$ is the state at the current iteration), we see that the transformations perform the following rotations:
- $\phi \xrightarrow{V} -\phi \xrightarrow{W} \phi + 2\theta$

The Algorithm

- Question: what is θ (the angle between uniform superposition and uniform superposition minus a)?



$|\phi\rangle \approx |e\rangle$

$|e\rangle = \frac{1}{\sqrt{N-1}} \sum_{x \neq a} |x\rangle$

$\cos^2\theta = \langle \phi, e\rangle^2$

$\sin^2\theta = 1 - \cos^2\theta = 1 - \langle \phi, e\rangle^2$

$\sin\theta = \sqrt{1 - \langle \phi, e\rangle^2}$

$\Rightarrow \theta = \arcsin$ ↗

$\langle \phi, e\rangle = ?$

$\sqrt{\frac{N-1}{N}}$ (ex.)

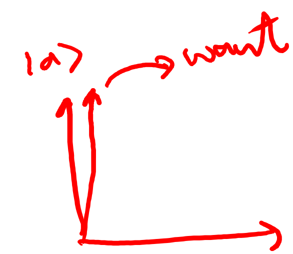
$\sqrt{1 - \langle \phi, e\rangle^2} = \sqrt{1/N}$

$\theta = \arcsin 1/\sqrt{N} \approx 1/\sqrt{N}$

$\theta \sim 1/\sqrt{N}$

The Algorithm

- After one iteration, we rotate the state vector by $2\theta = 2\arcsin\left(\frac{1}{\sqrt{N}}\right) \sim 2/\sqrt{N}$
- Since we start out at state $|\phi\rangle$(uniform) almost orthogonal to $|a\rangle$,(Assuming N is large), we need to rotate by $\frac{\pi}{2}$.
- So we need about $\frac{\frac{\pi}{2}-\theta}{2\theta} \sim O(\sqrt{N})$ applications of the algorithm.



Algo : WVWV.... WV $|\phi\rangle$, measure, get a whp.

O(√N) times