Come see us!

- Alexandra Kolla/ Graeme Smith: Friday 3:00-4:00 pm, over zoom.
- Ariel Shlosberg: Tu/Th 2:00-4:00pm, over zoom
- Steven Kordonowy: Th 11am-12pm, over zoom.
- Matteo Wilczak: Wednesday, 1-2pm, over zoom.
Today

- Grover’s algorithm, continued.
- Optimality $\sqrt{N}, N = 2^n$
The problem

• Suppose we know that exactly one n-bit integer satisfies some condition.

• Namely, we have a special “marked” item $a$, such that, for some function $f : \{0,1\}^n \rightarrow \{0,1\}$ $f(x) = 1$ iff $x = a$ and $0$ otherwise.
The setup

• We have a function $f(x) = \begin{cases} 1 & \text{if } x = a \\ 0 & \text{o.w.} \end{cases}$
The Unitary for Grover

\[
|y\rangle \quad \text{1 bit} \quad n \text{ bit} \\
\begin{array}{c}
U_f \\
|y\rangle \quad \text{1 bit} \\
\end{array} \\
\begin{array}{c}
|x\rangle \\
\text{1 bit} \quad n \text{ bit} \\
\end{array}
\]

- \[ V = I - 2|a\rangle\langle a| \]
- \[ V|Y\rangle = V \sum_{0<x\leq 2^n} |x\rangle\langle x|Y\rangle = |Y\rangle - 2|a\rangle\langle a|Y\rangle \]

- See this operator as “amplifying” the amplitude of \(|a\rangle\)
- If we amplify the negative vectors enough then we could measure the required state with high probability.
- More of an “amplification” algorithm than search.
The Algorithm

- Start by preparing uniform superposition
  \[ |\phi\rangle = H^{\otimes n} |0\rangle_n = \frac{1}{\sqrt{N}} \sum_{0 < x \leq N} |x\rangle_n \]
- \((N = 2^n)\)
- Signifies maximal ignorance of special element \(a\).
- The action of \(V\) on \(|\phi\rangle\)?
The Algorithm

\[ V |\phi\rangle = (I - 2 |a\rangle \langle a|) |\phi\rangle = \\
|\phi\rangle - 2 |a\rangle \langle a||\phi\rangle = |\phi\rangle - \frac{2}{\sqrt{N}} |a\rangle = \\
\frac{1}{\sqrt{N}} (\sum_{x\neq a} |x\rangle_n) - \frac{1}{\sqrt{N}} |a\rangle \]

- Applying the oracle to the initial state negates the amplitude of the satisfying element!
The Algorithm

- We now introduce the Grover diffusion operator. Recall $|\phi\rangle = \frac{1}{\sqrt{N}} \sum_{0<x\leq N} |x\rangle_n$

- $W = 2|\phi\rangle\langle\phi| - I$

- Question: Let $x$ be a standard basis vector. What is $(|\phi\rangle\langle\phi|)|x\rangle$?
The Algorithm

- We now introduce the Grover diffusion operator.

- \[ W = 2|\phi\rangle\langle \phi| - I \]

- One iteration of the algorithm consists of applying the operator \[ A = WV \] (that is, querying the oracle and then applying the diffusion operator).
The Algorithm

\[ W = 2|\phi\rangle\langle\phi| - I, \quad V = I - 2|a\rangle\langle a| \]

- After one iteration:
  \[ WV|\phi\rangle = (2|\phi\rangle\langle\phi| - I)(I - 2|a\rangle\langle a|)|\phi\rangle = (2|\phi\rangle\langle\phi| + 2|a\rangle\langle a| - \frac{4}{\sqrt{N}} |\phi\rangle\langle a| - I)|\phi\rangle = \left(1 - \frac{4}{N}\right)|\phi\rangle + \frac{2}{\sqrt{N}}|a\rangle \]

- We see that after one iteration, the probability of measuring \( a \) has increased.
The Algorithm

- It can be checked that the operation rotates the state vector by Type equation here. \( \frac{2\sqrt{(N-1)}}{N} \sim \frac{2}{\sqrt{N}} \)

- Since we start out almost orthogonal to \( |a\rangle \) (Assuming \( N \) is large), we need to rotate by \( \frac{\pi}{2} \).

- So we need about \( \frac{\pi \sqrt{N}}{4} \) applications of the algorithm.
The start of one iteration

- We can visualize the operators in Grover’s algorithm as reflections in state space. Consider the target vector $|a\rangle$ and the hyperplane of all other vectors $|e\rangle = \frac{1}{\sqrt{N-1}} \sum_{x\neq a} |x\rangle_n$

- Assume the algorithm is at state $|\psi\rangle$ at the current iteration
The start of one iteration

Start iteration: $|\psi\rangle$

do $WV|\psi\rangle$

what do I get?

$|a\rangle = 1e\rangle = \frac{1}{\sqrt{n-1}} \sum_{x \neq a} 1 \times 7_n$
The start of one iteration

\[ |\phi\rangle = \frac{1}{\sqrt{N}} \sum_{0<x\leq N} |x\rangle_n \]

\[ |e\rangle = \frac{1}{\sqrt{N-1}} \sum_{x\neq a} |x\rangle_n \]
After V - class exercise in 2D

\[ V = I - 2|a\rangle\langle a| \]

Question: What operator on the plane is \( V = I - 2|a\rangle\langle a| \)?

(Draw the vector \( V|\psi\rangle \) on the plane)
After V - class exercise in 2D

$$|\psi\rangle = x_1 a^\perp + y_1 |a\rangle$$

$$|a^\perp\rangle \left( = |e_\gamma\rangle \right)$$

$$V|\psi\rangle = x_1 |a^\perp\rangle - y_1 |a\rangle$$

$$= (x_1, -y_1)$$
After V - class exercise in 2D

\[ |\psi\rangle = x_1 a^\perp + y_1 |a\rangle \]

Corresponds to a flip over \(|e\rangle = \frac{1}{\sqrt{N-1}} \sum_{x \neq a} |x\rangle_n\)

Which is perpendicular to \(|a\rangle\)

\[ V = I - 2|a\rangle\langle a| \]

\[ V |\psi\rangle = x_1 |a^\perp\rangle - y_1 |a\rangle \]
After $V$

$$V = I - 2|a⟩⟨a|$$

Corresponds to a flip over $|e⟩$

$$|e⟩ = \frac{1}{\sqrt{N-1}} \sum_{x\neq a} |x⟩_n$$

$$\psi$$
After W - class exercise in 2D

\[ W = 2|\phi\rangle\langle\phi| - I \]

\[ |\psi\rangle = x_1|\phi\rangle + y_1|\phi^\perp\rangle \]

**Question:** What operator on the plane is \( W = 2|\phi\rangle\langle\phi| - I \)?

*(Draw the vector \( W|\psi\rangle \) on the plane)*
After W - class exercise in 2D

$$W = 2 |\phi\rangle\langle\phi| - I$$

$$|\psi\rangle = x_1 |\phi\rangle + y_1 |\phi^\perp\rangle$$

$$w |\psi\rangle = (?, ?)$$

**Question:** What operator on the plane is $W = 2 |\phi\rangle\langle\phi| - I$? (Draw the vector $W|\psi\rangle$ on the plane)
After \( W \) - class exercise in 2D

\[
W = 2|\phi\rangle\langle\phi| - I
\]

Corresponds to a flip over \(|\phi\rangle\)

\[
|\psi\rangle = x_1|\phi\rangle + y_1|\phi^\perp\rangle
\]

\[
W|\psi\rangle = x_1|\phi\rangle - y_1|\phi^\perp\rangle
\]
After W - class exercise in 2D

\[ W = 2|\phi\rangle\langle\phi| - I \]

Corresponds to a flip over \(|\phi\rangle\)

\[ |\psi\rangle = x_1|\phi\rangle + y_1|\phi^{\perp}\rangle \]

\[ W|\psi\rangle = x_1|\phi\rangle - y_1|\phi^{\perp}\rangle \]
Putting it together

\[ |\phi\rangle = \frac{1}{\sqrt{N}} \sum_{0 < x \leq N} |x\rangle_n \]

\[ |e\rangle = \frac{1}{\sqrt{N - 1}} \sum_{x \neq a} |x\rangle_n \]
Putting it together

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Putting it together

\[ |\phi\rangle = \frac{1}{\sqrt{N}} \sum_{0<x\leq N} |x\rangle_n \]

\[ |e\rangle = \frac{1}{\sqrt{N-1}} \sum_{x \neq a} |x\rangle_n \]
In the end

\[ |e\rangle = \frac{1}{\sqrt{N-1}} \sum_{x \neq a} |x\rangle_n \]
The Algorithm

- We concluded that V corresponds to a reflection over $|e\rangle$ (perp to $|a\rangle$) and W corresponds to a reflection over $|\phi\rangle$ (uniform superposition, and also starting state).
- If we define $\theta$ as the angle between $|\phi\rangle$ and $|e\rangle$, and $\phi$ as the angle between $|\psi\rangle$ and $|e\rangle$ (where $|\psi\rangle$ is the state at the current iteration), we see that the transformations perform the following rotations:
The Algorithm

- We concluded that V corresponds to a reflection over |e⟩ (perp to |a⟩) and W corresponds to a reflection over |ϕ⟩ (uniform superposition, and also starting state).
- If we define θ as the angle between |ϕ⟩ and |e⟩, and φ as the angle between |ψ⟩ and |e⟩ (where |ψ⟩ is the state at the current iteration), we see that the transformations perform the following rotations:
  \[
  VV \rightarrow -ϕ \rightarrow ϕ + 2θ
  \]
The Algorithm

- Question: what is $\theta$ (the angle between uniform superposition and uniform superposition minus $a$)?
The Algorithm

• After one iteration, we rotate the state vector by $2\theta = 2 \arcsin \left( \frac{1}{\sqrt{N}} \right) \sim \frac{2}{\sqrt{N}}$.

• Since we start out at state $|\phi\rangle$ (uniform) almost orthogonal to $|a\rangle$, (Assuming $N$ is large), we need to rotate by $\frac{\pi}{2}$.

• So we need about $\frac{\pi}{2\theta} \sim O(\sqrt{N})$ applications of the algorithm.

Algo: $\overbrace{\text{WVWV... WV} |\phi\rangle}^{O(\sqrt{N})\text{times}}, \text{measure, get a whp.}$