Quantum Searching- Grover

PHYS/CSCI 3090

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Come see us!

- Alexandra Kolla/ Graeme Smith: Friday 3:00-4:00 pm, over zoom.
- Ariel Shlosberg: Tu/Th 2:00-4:00pm, over zoom
- Steven Kordonowy: Th 11am-12pm, over zoom.
- Matteo Wilczak: Wednesday, 1-2pm, over zoom.

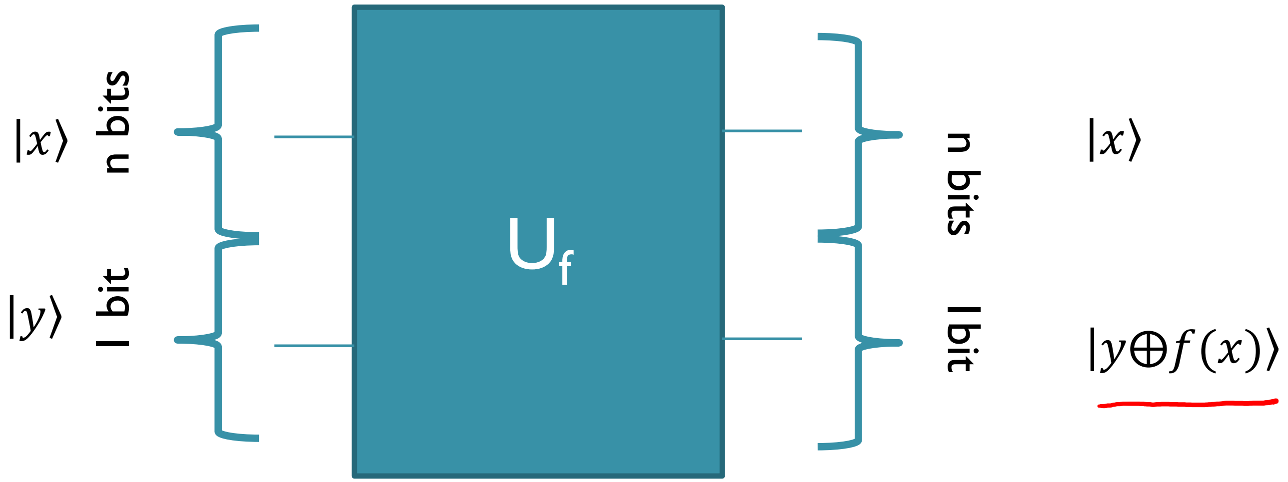
Today

- Searching for a marked item
- Grover’s algorithm.

The problem

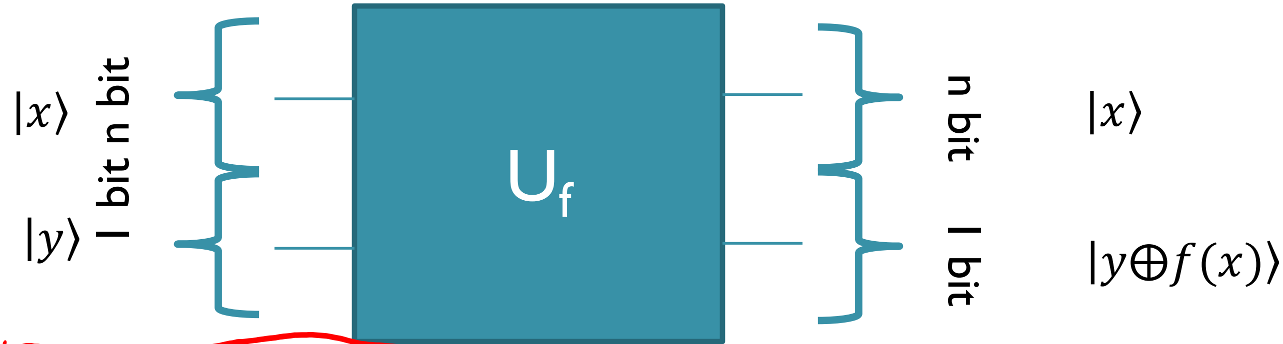
- Suppose we know that exactly one n-bit integer satisfies some condition.
- Namely, we have a special "marked" item a, such that, for some function $f: \{0,1\}^n \rightarrow \{0,1\}$ f(x)=1 iff x=a and 0 otherwise.

The setup



- We have a function $f(x) = \begin{cases} 1 \; if \; x = a \\ 0 \; o.w. \end{cases}$

The Unitary for Grover



- $V = I - 2|a\rangle\langle a|$

$$V|Y\rangle = V \sum_{0<x\leq 2^n} |x\rangle\langle x|Y\rangle = |Y\rangle - 2|a\rangle\langle a|Y\rangle$$

- See this operator as "amplifying" the amplitude of $|a\rangle$
- If we amplify the negative vectors enough then we could measure the required state with high probability.
- More of an "amplification" algorithm than search.

The Algorithm

- Start by preparing uniform superposition $|\phi\rangle \triangleq H^{\otimes n}\,|0\rangle_n \quad = \frac{1}{\sqrt{N}}\sum_{0<x\leq N}|x\rangle_n$ maximal ignorance
- $(N = 2^n)$
- Signifies maximal ignorance of special element a. knew a: start $|a\rangle \xrightarrow{U_f}$ get 1 done
- The action of V on $|\phi\rangle$? $V \triangleq$ Grover operator

$V|\phi\rangle = (I - 2|a\rangle\langle a|)|\phi\rangle = I|\phi\rangle - 2|a\rangle\langle a|\phi\rangle$

$= |\phi\rangle - \frac{2}{\sqrt{N}}|a\rangle = \frac{1}{\sqrt{N}}\sum_{x\neq a}|x\rangle_n - \frac{1}{\sqrt{N}}|a\rangle$

The Algorithm

- $V|\phi\rangle = (I - 2|a\rangle\langle a|)|\phi\rangle =$
$|\phi\rangle - 2|a\rangle\langle a||\phi\rangle = |\phi\rangle - \frac{2}{\sqrt{N}}|a\rangle =$
$\frac{1}{\sqrt{N}}(\sum_{x\neq a}|x\rangle_n) - \frac{1}{\sqrt{N}}|a\rangle$

- **Applying the oracle to the initial state negates the amplitude of the satisfying element!**

The Algorithm

- We now introduce the Grover diffusion operator. Recall $|\phi\rangle = \frac{1}{\sqrt{N}}\sum_{0<x\leq N}|x\rangle_n$

$Proj_{|\phi\rangle} = |\phi\rangle\langle\phi|$

- $W = 2|\phi\rangle\langle\phi| - I$

- Question: Let x be a standard basis vector. What is $(|\phi\rangle\langle\phi|)|x\rangle$?

$|\phi\rangle\langle\phi|x\rangle = \langle\phi|x\rangle|\phi\rangle = \frac{1}{\sqrt{N}}|\phi\rangle$

The Algorithm

- We now introduce the Grover diffusion operator.
- $W = 2|\phi\rangle\langle\phi| - I$
- One iteration of the algorithm consists of applying the operator $A = WV$ (that is, querying the oracle and then applying the diffusion operator).

$WV|\phi\rangle$ = 1 iteration.

The Algorithm

$$W = 2|\phi\rangle\langle\phi| - I, V = I - 2|a\rangle\langle a|$$

- After one iteration:

$$WV|\phi\rangle = (2|\phi\rangle\langle\phi| - I)(I - 2|a\rangle\langle a|)|\phi\rangle$$

$$= \left(2|\phi\rangle\langle\phi| + 2|a\rangle\langle a| - \frac{4}{\sqrt{N}}|\phi\rangle\langle a| - I\right)|\phi\rangle$$

$$= 2|\phi\rangle\langle\phi|\phi\rangle + 2|a\rangle\langle a|\phi\rangle - \frac{4}{\sqrt{N}}|\phi\rangle\langle a|\phi\rangle - I|\phi\rangle$$

$$= 2|\phi\rangle + \frac{2}{\sqrt{N}}|a\rangle - \frac{4}{N}|\phi\rangle - |\phi\rangle$$

$$= (1 - 4/N)|\phi\rangle + \frac{2}{\sqrt{N}}|a\rangle$$

The Algorithm

$$W = 2|\phi\rangle\langle\phi| - I, V = I - 2|a\rangle\langle a|$$

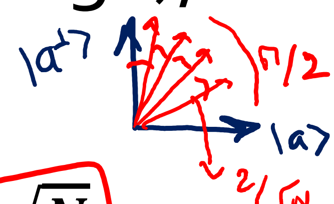
- After one iteration:

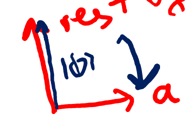
$$WV|\phi\rangle = (2|\phi\rangle\langle\phi| - I)(I - 2|a\rangle\langle a|)|\phi\rangle =$$
$$(2|\phi\rangle\langle\phi| + 2|a\rangle\langle a| - \frac{4}{\sqrt{N}}|\phi\rangle\langle a| - I)|\phi\rangle =$$
$$\left(1 - \frac{4}{N}\right)|\phi\rangle + \frac{2}{\sqrt{N}}|a\rangle$$

- We see that after one iteration, the probability of measuring *a* has increased.

The Algorithm

- It can be checked that the operation rotates the state vector by Type equation here. $\frac{2\sqrt{(N-1)}}{N} \sim \frac{2}{\sqrt{N}}$
- Since we start out almost orthogonal to $|a\rangle$ (Assuming N is large), we need to rotate by $\frac{\pi}{2}$.





- So we need about $\frac{\pi\sqrt{N}}{4}$ applications of the algorithm

The Algorithm

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- Since we start out almost orthogonal to $|a\rangle$ (Assuming N is large), we need to rotate by $\frac{\pi}{2}$.
- How many times do I need to run the algorithm to measure $|a\rangle$ w.h.p?

The Algorithm

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- Since we start out almost orthogonal to $|a\rangle$ (Assuming N is large), we need to rotate by $\frac{\pi}{2}$.
- So we need about $\frac{\pi\sqrt{N}}{4}$ applications of the algorithm.