Quantum Searching-Grover

PHYS/CSCI 3090

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Come see us!

- Alexandra Kolla/ Graeme Smith: Friday 3:00-4:00 pm, over zoom.
- Ariel Shlosberg: Tu/Th 2:00-4:00pm, over zoom
- Steven Kordonowy: Th 11am-12pm, over zoom.
- Matteo Wilczak: Wednesday, 1-2pm, over zoom.

Where do we stand

- We saw Shor’s algorithm, first real and very significant quantum speedup

Today

- Searching for a marked item

- Grover's algorithm.

The problem

- Suppose we know that exactly one n-bit integer satisfies some condition.
- Namely, we have a special "marked" item a, such that, for some function $f:\{0,1\}^n \to \{0,1\}$ f(x)=1 iff x=a and 0 otherwise.

The problem

- Classically? how many queries to the ~~data~~base do I need to find a?

Check everything: n bits, means $N = 2^n$ x's to check

BAD

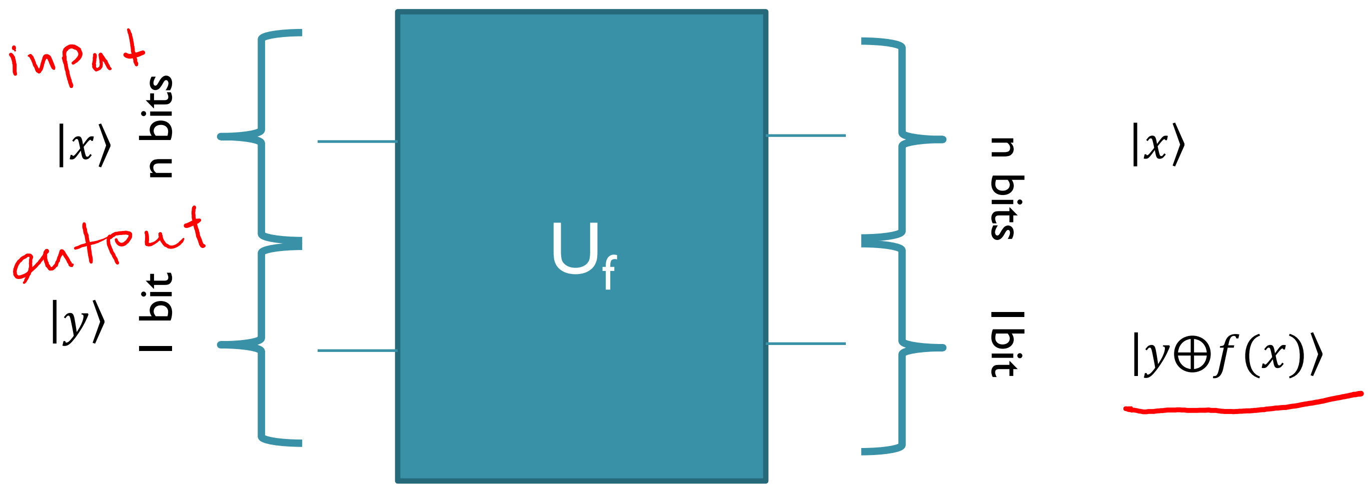
The problem

- Classically?

- Have to possibly try all $N = 2^n$ integers x to find the one such that f(x) =1.
- Randomly, need to check at least N/2 to have probability 50% of success.
- Quantumly: only $\left(\frac{\pi}{4}\right)\sqrt{N}$ queries!

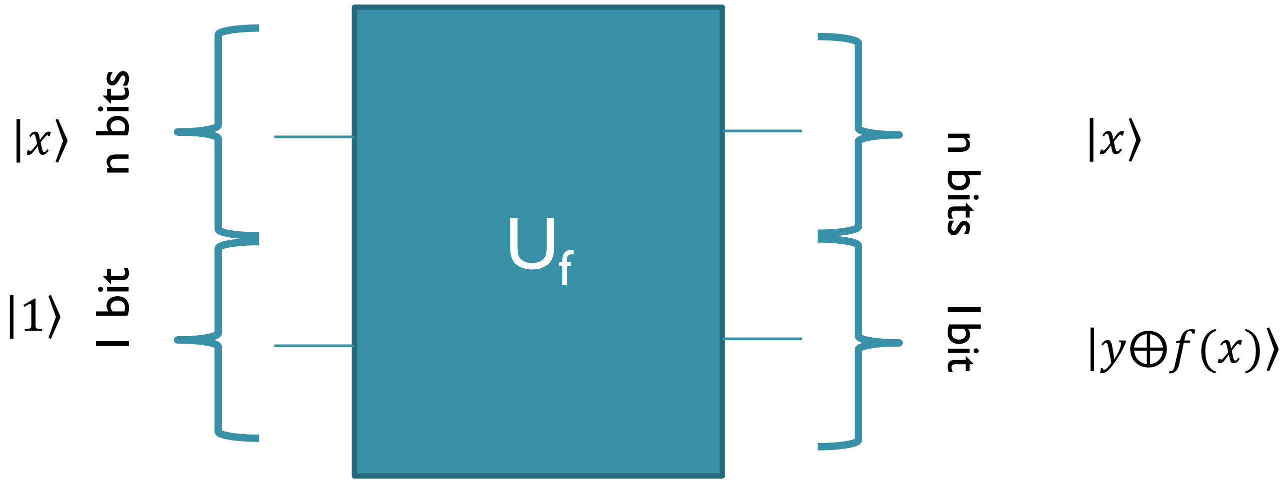
The setup

$f(x): \{0,1\}^n \to \{0,1\}$



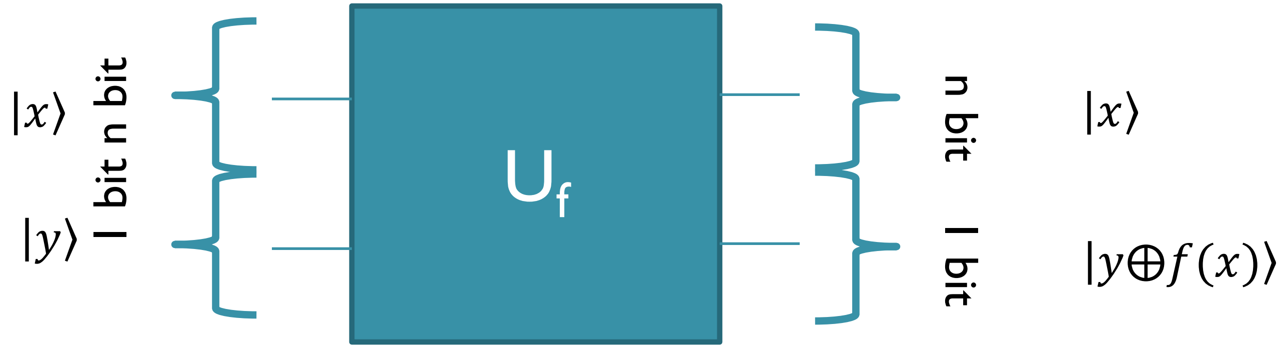
- We have a function $f(x) = \begin{cases} 1 \; if \; x = a \\ 0 \; o.w. \end{cases}$

The setup



- We have a function $f(x) = \begin{cases} 1 \; if \; x = a \\ 0 \; o.w. \end{cases}$

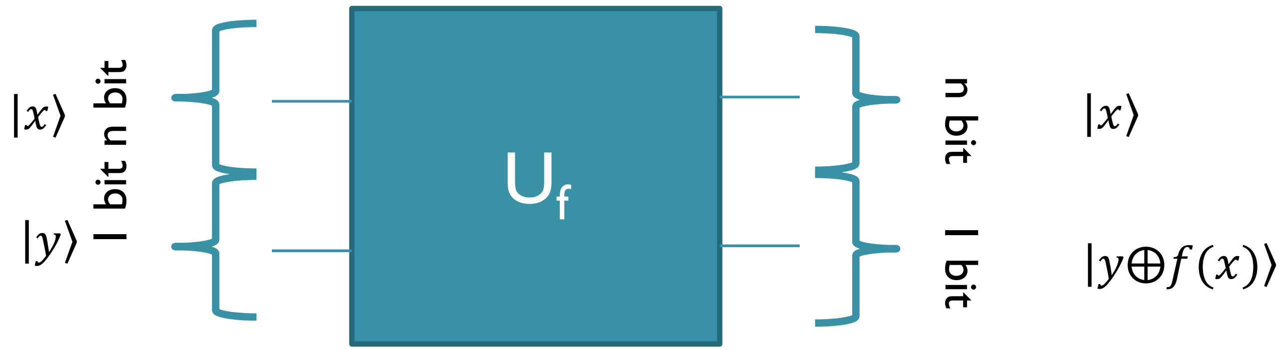
The Trick



- $U_f|x\rangle_n \otimes |0\rangle = |x\rangle_n \otimes |0 \oplus f(x)\rangle =$
  $-|x\rangle_n \otimes |0\rangle$, if f(x) =0
  $-|x\rangle_n \otimes |1\rangle$, if f(x) =1

- $U_f|x\rangle_n \otimes |1\rangle = |x\rangle_n \otimes |1 \oplus f(x)\rangle =$
  $-|x\rangle_n \otimes |1\rangle$, if f(x) =0
  $-|x\rangle_n \otimes |0\rangle$, if f(x) =1

The Trick



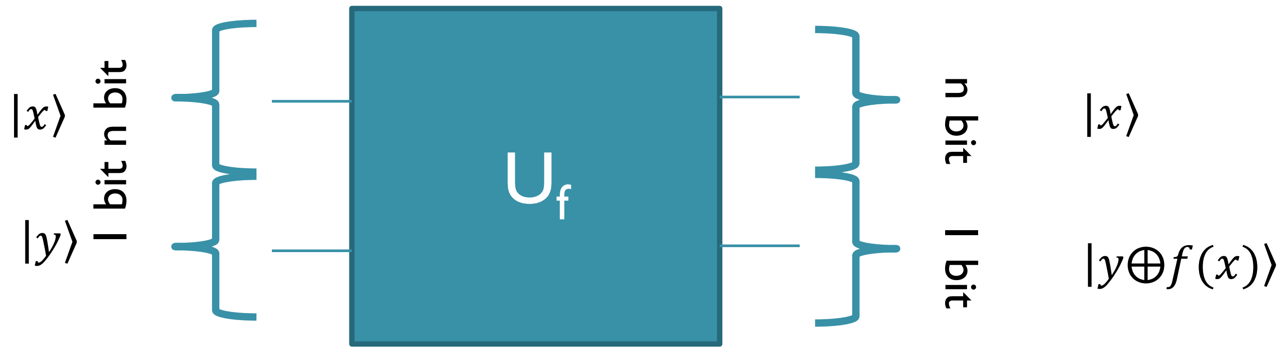
1. $U_f|x\rangle_n \otimes (|0\rangle + |1\rangle) = ??$

   $|+\rangle$ nothing

2. $U_f|x\rangle_n \otimes (|0\rangle - |1\rangle) = ??$

   $|-\rangle$ $(-1)^{f(x)} |x\rangle_n (|0\rangle - |1\rangle)$

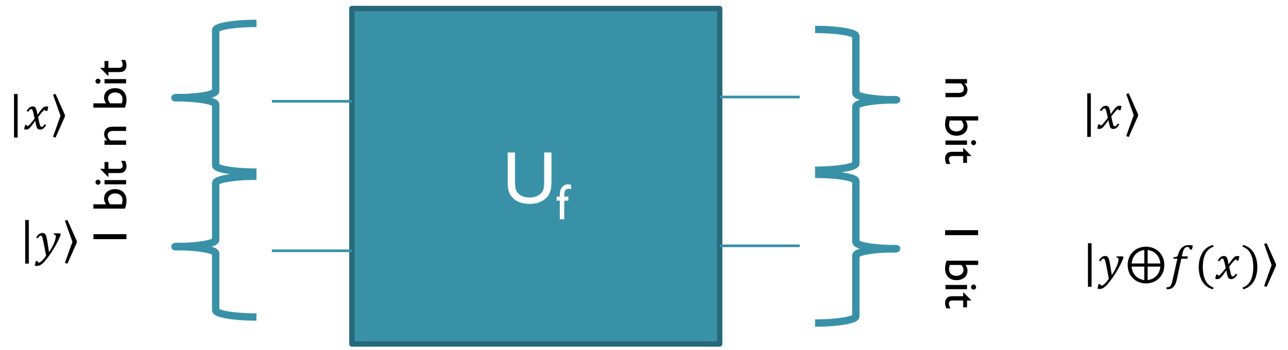
The Trick



- $U_f |x\rangle_n \otimes 1/\sqrt{2}\ (|0\rangle - |1\rangle) = (-1)^{f(x)}\ |x\rangle_n \otimes 1/\sqrt{2}\ (|0\rangle - |1\rangle)$

So taking the 1-qubit output register to be $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$, we convert a bit flip to a sign change!

The Unitary for Grover

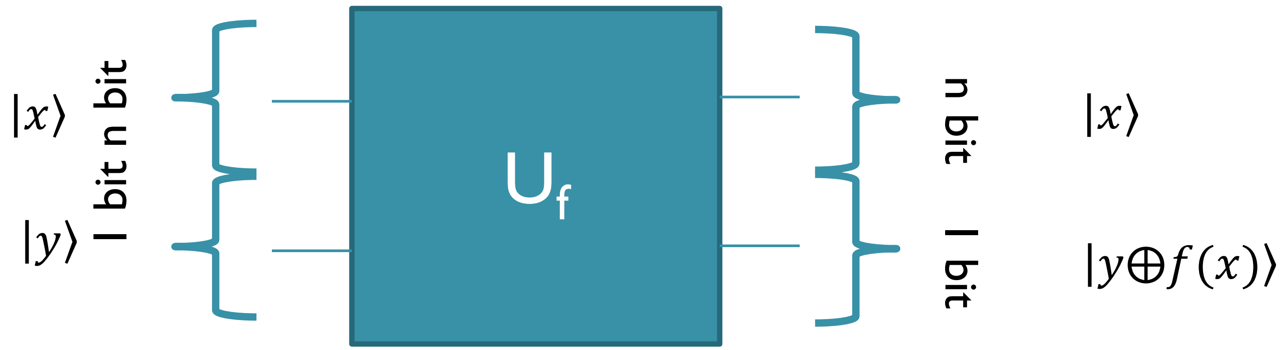


- $U_f \,|x\rangle_n \otimes 1/\sqrt{2}\;(|0\rangle - |1\rangle) = (-1)^{f(x)}\;|x\rangle_n \otimes 1/\sqrt{2}\;(|0\rangle - |1\rangle)$

Multiply the input state by -1 if and only if x=a

We can forget about the output register since it does not change

The Unitary for Grover



V = unitary for 1 query to database

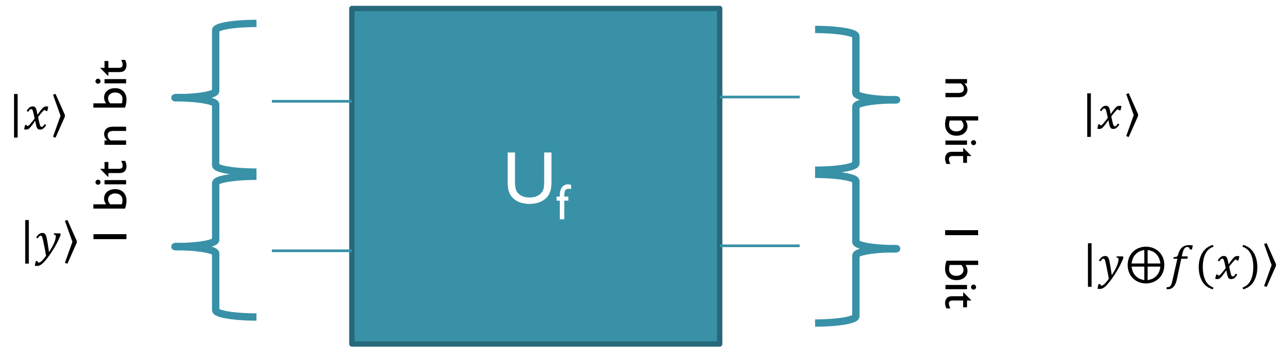
- $$V|x\rangle_n = \begin{cases} |x\rangle_n, & x \neq a \\ -|a\rangle & , x = a \end{cases}$$

$V|a\rangle = -|a\rangle$, $V|x\rangle = |x\rangle \; \forall x \neq a$

Multiply the input state by -1 if and only if x=a

We can forget about the output register since it does not change

The Unitary for Grover



- $$V|x\rangle_n = \begin{cases} |x\rangle_n, & x \neq a \\ -|a\rangle & , x = a \end{cases}$$

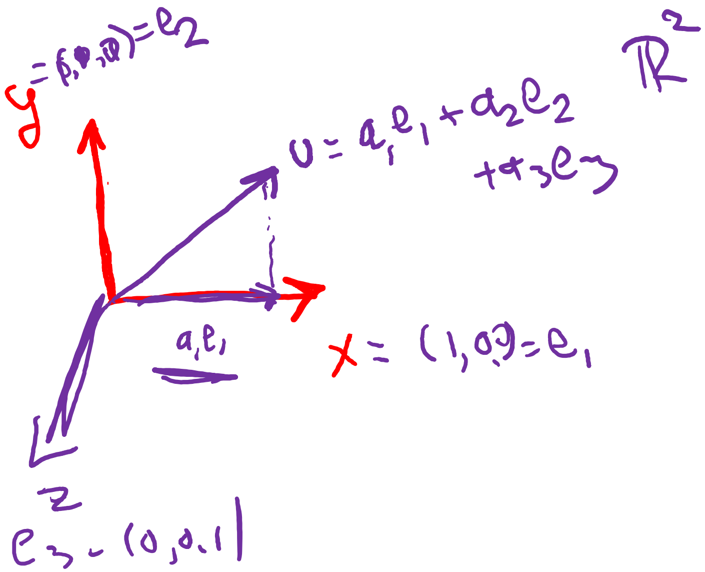
How does V act on an arbitrary superposition?

Parenthesis-Projections

- Question:

Let Π_a be the projection operator on the basis vector $|a\rangle_n$. What is $\Pi_a(\sum_{0<x\leq 2^n}|x\rangle_n)$?

- Definition: we denote Π_a with $|a\rangle\langle a|$



$\Pi_{e_1}(v) = a_1 e_1$

$\Pi_a(\sum|x\rangle) = ?\ |a\rangle$

$\Pi_a \triangleq |a\rangle\langle a|$

Parenthesis-Projections,II

- Question:

Let $|Y\rangle = \sum_{0<x\leq 2^n} y_x |x\rangle_n$ be an arbitrary superposition. How can we express the coefficients y_x as inner products?

$y_x = \langle x|Y\rangle$

$$|Y\rangle = \sum_{0<x\leq 2^n} |x\rangle\langle x|Y\rangle$$

Parenthesis-Projections,II

- Question:

Let $|Y\rangle = \sum_{0<x\leq 2^n} y_x |x\rangle_n$be an arbitrary superposition. How can we express the coefficients y_x as inner products?

$$y_x = \langle x|Y\rangle$$

So $|Y\rangle = \sum_{0<x\leq 2^n}|x\rangle\langle x|Y\rangle$

Parenthesis-Projections, II

- Question:

What is $\Pi_a \, |Y\rangle$? (or, equivalently, $|a\rangle\langle a| \; |Y\rangle$?

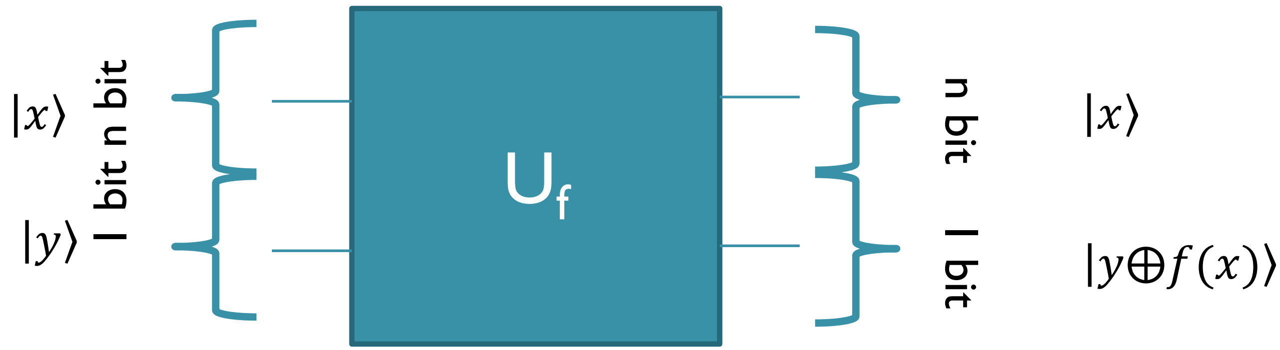
$\Pi_a|Y\rangle = |a\rangle\langle a| \sum |x\rangle \langle x|Y\rangle$

$= \sum_x |a\rangle\langle a|x\rangle\langle x|Y\rangle$

$$|Y\rangle = \sum_{0<x\leq 2^n} |x\rangle\langle x|Y\rangle$$

$y_a|a\rangle$

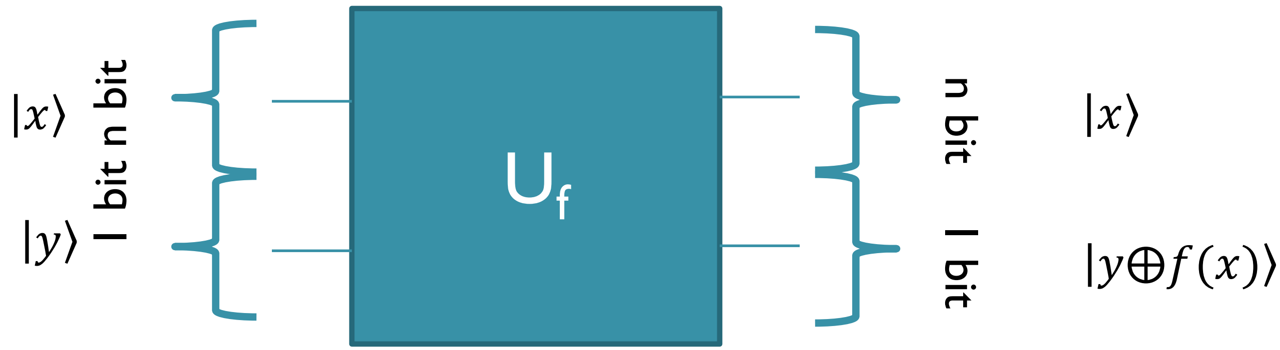
The Unitary for Grover



- $$V|x\rangle_n = \begin{cases} |x\rangle_n, & x \neq a \\ -|a\rangle & , x = a \end{cases}$$

How does V act on an arbitrary superposition $|Y\rangle = \sum_{0<x\leq 2^n} |x\rangle\langle x|Y\rangle$?

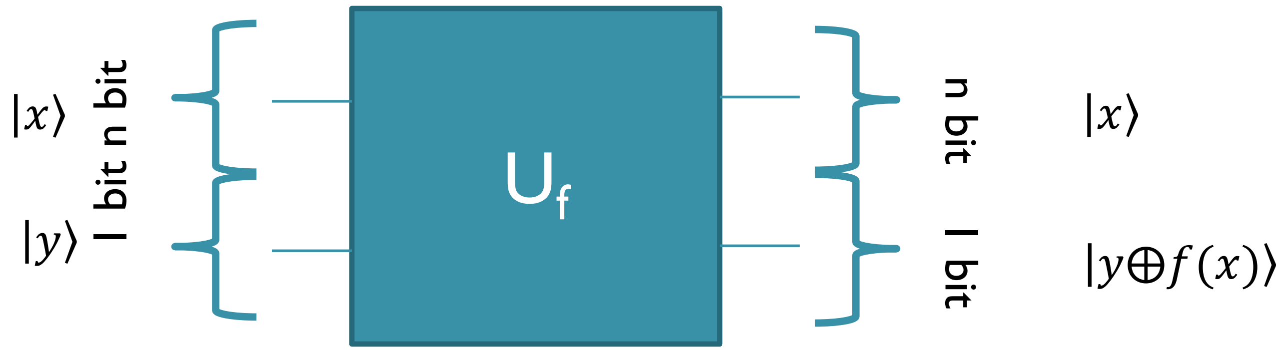
The Unitary for Grover



- $V|x\rangle_n = \begin{cases} |x\rangle_n, & x \neq a \\ -|a\rangle & , x = a \end{cases}$

How does V act on an arbitrary superposition
$|Y\rangle = \sum_{0<x\leq 2^n} |x\rangle\langle x|Y\rangle$?
Changes the sign of the component along $|a\rangle$, while
Leaving unchanged the component orthogonal to $|a\rangle$

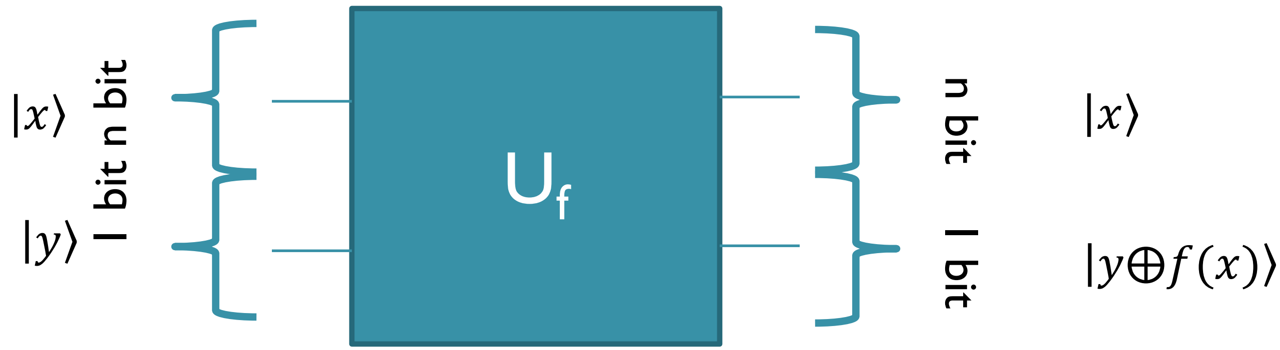
The Unitary for Grover



- $V|x\rangle_n = \begin{cases} |x\rangle_n, & x \neq a \\ -|a\rangle & , x = a \end{cases}$

So, at the end, $V|Y\rangle = \sum_{0<x\leq 2^n, x\neq a} |x\rangle\langle x|Y\rangle - |a\rangle\langle a|Y\rangle$

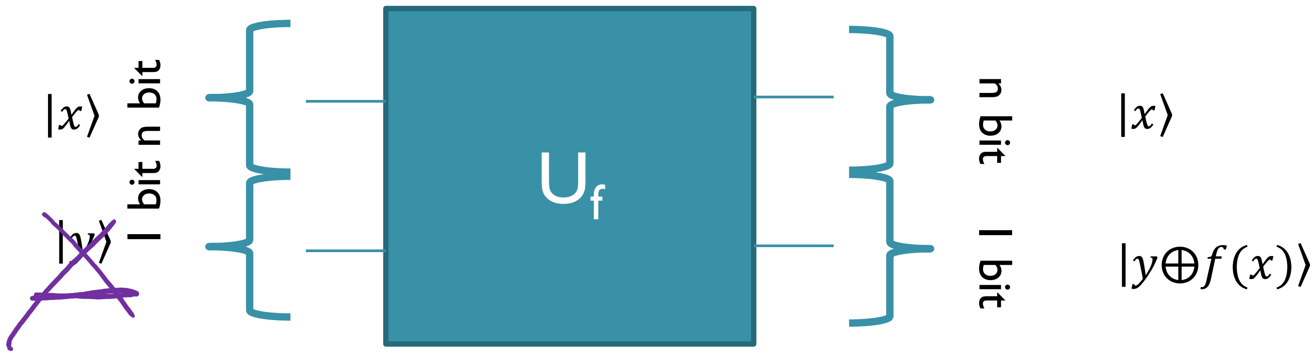
The Unitary for Grover



- $V|x\rangle_n = \begin{cases} |x\rangle_n, & x \neq a \\ -|a\rangle & , x = a \end{cases}$

$$V|Y\rangle = V \sum_{0<x\leq 2^n} |x\rangle\langle x|Y\rangle = |Y\rangle - 2|a\rangle\langle a|Y\rangle$$

The Unitary for Grover



- $V = I - 2|a\rangle\langle a|$

$$V|Y\rangle = V \sum_{0<x\leq 2^n} |x\rangle\langle x|Y\rangle = |Y\rangle - 2|a\rangle\langle a|Y\rangle$$

The Uniform Input Superposition

- Recall: $H^{\otimes n}\ |0\rangle_n \quad = \frac{1}{2^{n/2}} \sum_{0<x\leq 2^n} |x\rangle_n$