Th'm: (Row Operations + Determinant)

Let $A, B \in \mathbb{R}^{n\times n}$. Let $A[i]$ be the i^{th} row of A. Know $\det A$.

- $k \cdot A[i] + A[k] = B[j] \implies \det B = \det A$
- $A[i] = B[j] \implies \det B = -\det A$
- $k \cdot A[i] = B[j] \implies \det B = k \cdot \det A$

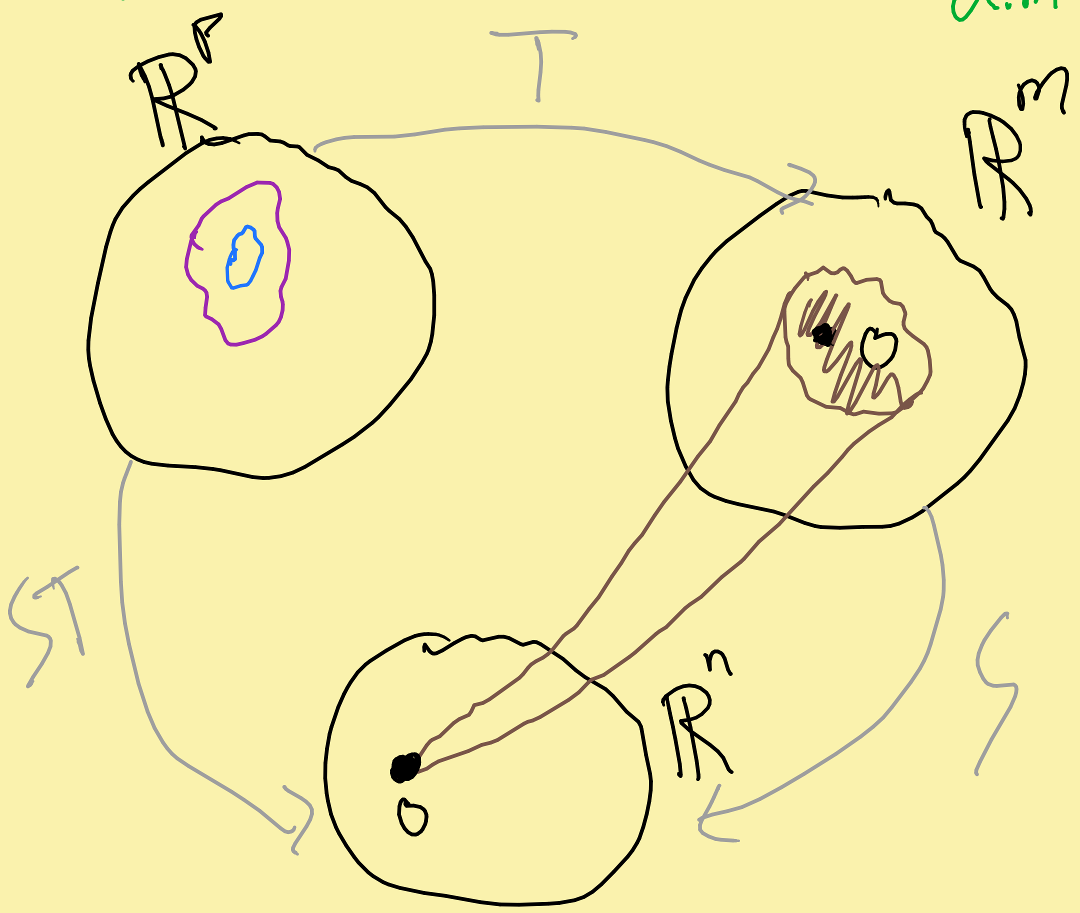
$$A = \begin{bmatrix} 3 & -1 \\ 2 & 7 \end{bmatrix}, \quad \det A = 3 \cdot 7 - -1 \cdot 2 = 21 + 2 = 23$$

$$B_1 = \begin{bmatrix} 3 & -1 \\ 8 & 5 \end{bmatrix}. \quad \det B_1 = 15 + 8 = 23 \checkmark$$

$$B_2 = \begin{bmatrix} 2 & 7 \\ 3 & -1 \end{bmatrix}. \quad \det B_2 = -2 - 21 = -23 \checkmark$$

$$B_3 = \begin{bmatrix} 12 & -4 \\ 2 & 7 \end{bmatrix}. \quad \det B_3 = 84 + 8 = 92 = \overset{k}{4} \cdot 23 \checkmark$$

Let $S \in \mathbb{R}^{n\times m}$ + $T \in \mathbb{R}^{m\times p}$. Then $\dim \operatorname{null} ST \leq \dim \operatorname{null} S + \dim \operatorname{null} T$.



$\operatorname{null} ST = \{v \in \mathbb{R}^p : STv = 0\}$

$\operatorname{null} T = \{v \in \mathbb{R}^p : Tv = 0\}$

$\operatorname{null} S = \{v \in \mathbb{R}^m : Sv = 0\}$

<u>Pf:</u> • WTS $\operatorname{null} T \subseteq \operatorname{null} ST$

Let $v \in \operatorname{null} T$. Then $Tv = 0$.

So $STv = S0 = 0$.

Then $v \in \operatorname{null} ST$.

Let $\overbrace{v_1, \dots, v_k}^{\text{basis}}$ be a basis for $\operatorname{null} T$. (Then $\dim \operatorname{null} T = k$).

Extend to $v_1, \dots, v_k, v_{k+1}, \dots, v_q$ be a basis for $\operatorname{null} ST$.

Define $U \equiv \operatorname{span}\{v_{k+1}, \dots, v_q\}$. WTP: T 1-to-1 on U.

Let $u = \overset{\text{Asm}}{c_{k+1}} v_{k+1} + \dots + c_q v_q$ s.t. $Tu = 0$.

Then $u = \overset{u \in \operatorname{null} T}{c_1 v_1 + \dots + c_k v_k}$. $\overbrace{u = c_1 v_1 + \dots + c_k u_k = c_{k+1} v_{k+1} + \dots + c_q v_q}^{\text{subtract}}$

$\Leftrightarrow c_1 v_1 + \dots + c_k v_k - c_{k+1} v_{k+1} - \dots - c_q v_q = 0. \Rightarrow c_i = 0 \ \forall i.$

$$\underbrace{\{Tv_{k+1}, Tv_{k+2}, \dots, Tv_q\}}_{\cdot\, \text{LI}} \,\dot{\subseteq}\, \operatorname{null} S$$

$\Rightarrow q - k \leq \dim \operatorname{null} S$

$\Leftrightarrow \dim \operatorname{null} ST - \dim \operatorname{null} T \leq \dim \operatorname{null} S$.

$\Leftrightarrow \dim \operatorname{null} ST \leq \dim \operatorname{null} S + \dim \operatorname{null} T$

If U is diagonalizable, then $U^n = A^{-1} D^n A = A^{-1} \begin{bmatrix} d_{11}^n & & \\ & \ddots & \\ & & d_{mm}^n \end{bmatrix} A$.

Pf: $U = ADA^{-1}$, $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_m \end{bmatrix}$, $A = [\vec{v}_1 \cdots \vec{v}_m]$

Base Case: $U^1 = AD^1A^{-1}$ ✓

I.H.: Asm that $U^k = AD^kA^{-1}$ for some int $k \geq 1$.

Then $U^{k+1} = UU^k = (ADA^{-1})(AD^kA^{-1})$

$= AD\overbrace{\boxed{A^{-1}A}}^{I}D^kA^{-1} = ADD^kA^{-1} = AD^{k+1}A^{-1}$. ■

Sps $\lambda_1, \dots, \lambda_n$ are eigenvalues of A (with possible repeats)

then $\det A = \prod \lambda_i = \lambda_1 \cdots \lambda_n$.

<u>Pf</u>: $\det(A - \lambda I) = 0 \iff \lambda = \lambda_i$ for some $i \in [n]$.

$$p(\lambda) \equiv \det(A - \lambda I) = (-1)^n (\lambda - \lambda_1) \cdots (\lambda - \lambda_n) = (\lambda_1 - \lambda) \cdots (\lambda_n - \lambda).$$

$$p(0) = \det(A) = \lambda_1 \cdots \lambda_n. \blacksquare$$

$$\begin{vmatrix} 2 & -4 \\ 5 & 1 \end{vmatrix} = 2+20 = 22$$

$$\begin{vmatrix} 3 & -1 & 2 \\ -7 & 5 & 1 \\ 9 & -8 & -1 \end{vmatrix} = 3\begin{vmatrix} 5 & 1 \\ -8 & -1 \end{vmatrix} + 1\begin{vmatrix} -7 & 1 \\ 9 & -1 \end{vmatrix} + 2\begin{vmatrix} -7 & 5 \\ 9 & -8 \end{vmatrix}$$

$$= 3(3) + (-2) + 2(11) = 29 . (?)$$

$$\begin{bmatrix} 3 & -1 & 2 \\ -7 & 5 & 1 \\ 9 & -8 & -1 \end{bmatrix} = A . \quad \det(A - \lambda I) = 0$$

$$\Longleftrightarrow \begin{vmatrix} 3-\lambda & -1 & 2 \\ -7 & 5-\lambda & 1 \\ 9 & -8 & -1-\lambda \end{vmatrix}$$

$$= (3-\lambda)\left[\overbrace{(5-\lambda)(-1-\lambda)+8}\right] - (-1)\left[-7(-1-\lambda) - 9\right] + 2\left[56 - 9(5-\lambda)\right]$$

$$= (3-\lambda)(\lambda^2 - 4\lambda + 3) + ($$

$$A = \begin{pmatrix} 2 & -1 \\ 4 & -3 \end{pmatrix} . \quad \begin{vmatrix} 2-\lambda & -1 \\ 0 & -3-\lambda \end{vmatrix} = -(2-\lambda)(3+\lambda) \quad = \lambda^2 + \lambda - 6 = (\lambda-2)(\lambda+3)$$

$$\lambda_1 = -3 : \text{null}(A - \lambda_1 I) = \text{null}\begin{pmatrix} 5 & -1 \\ 0 & 0 \end{pmatrix} .$$

$$5x_1 - x_2 = 0 , \quad 5x_1 = x_2 \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

$$\lambda_2 = 2 : \text{null}(A - \lambda_2 I) = \text{null}\begin{pmatrix} 0 & -3 \\ 0 & -1 \end{pmatrix}$$

$$-x_2 = 0 \Rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$A = \underbrace{\begin{pmatrix} 1 & 1 \\ 5 & 0 \end{pmatrix}}_{V} \underbrace{\begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix}}_{D_\lambda} \begin{pmatrix} 1 & 1 \\ 5 & 0 \end{pmatrix}^{-1}$$

m

n

$m \gg n$

m

n

$n \gg m$