Linear Algebra

CSCI 2820

Lecture 9

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ECES 122

Today

- Gram Schmidt + examples

Refresher on linear independence

Are the vectors $\overrightarrow{x_1} = (1,0,0), \overrightarrow{x_2} = (1,1,1), \overrightarrow{x_3} = (1,-1,1)$ linearly independent?

$$\underline{\beta_1 \vec{x}_1 + \beta_2 \vec{x}_2 + \beta_3 \vec{x}_3} = \vec{0} \Leftrightarrow \beta_1, \beta_2 \beta_3 = 0$$

$$\Rightarrow \beta_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \beta_3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \vec{0}$$

$$\begin{pmatrix} \beta_1 + \beta_2 + \beta_3 \\ \beta_2 - \beta_3 \\ \beta_2 + \beta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$$

$\beta_1 + \beta_2 + \beta_3 = 0 \quad \beta_1 = 0$

$$\left.\begin{array}{l} \beta_2 = \beta_3 \\ \beta_2 = -\beta_3 \end{array}\right\} \beta_2 = \beta_3 = 0$$

Gram Schmidt.

Algorithm 5.1 GRAM–SCHMIDT ALGORITHM

given n-vectors $a_1, \ldots, a_k$ (L.I)

for $i = 1, \ldots, k$,

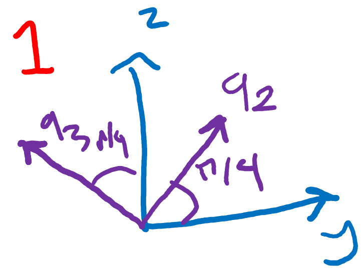
1. *Orthogonalization.* $\tilde{q}_i = a_i - (q_1^T a_i)q_1 - \cdots - (q_{i-1}^T a_i)q_{i-1}$
2. *Test for linear dependence.* if $\tilde{q}_i = 0$, quit.
3. *Normalization.* $q_i = \tilde{q}_i / \|\tilde{q}_i\|$

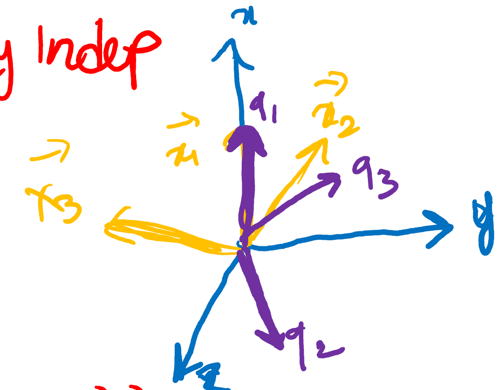
Refresher on linear independence

What happens when we run Gram Schmidt on the list ofvectors $\overrightarrow{x_1} = (1,0,0), \overrightarrow{x_2} = (1,1,1), \overrightarrow{x_3} = (1,-1,1)$? Linearly Indep

$i=1$: $\tilde{q}_1 = \vec{x}_1$, $\|\vec{x}_1\| = 1$

$\vec{q}_1 = \dfrac{\vec{x}_1}{\|\vec{x}_1\|} = \vec{x}_1$





$i=2$: $\tilde{q}_2 = \vec{x}_2 - \langle q_1, x_2 \rangle \cdot \vec{q}_1 = \vec{x}_2 - \vec{x}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\vec{q}_2 = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

$\langle q_1, x_2 \rangle = \langle x_1, x_2 \rangle = 1$

$i=3$: $\tilde{q}_3 = \vec{x}_3 - \langle q_1, x_3 \rangle \vec{q}_1 - \langle q_2, x_3 \rangle \vec{q}_2 = \vec{x}_3 - \vec{x}_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

$\langle q_1, x_3 \rangle = \langle x_1, x_3 \rangle = 1$

$\langle q_2, x_3 \rangle = -1/\sqrt{2} + 1/\sqrt{2} = 0$

$\vec{q}_3 = \begin{pmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

Gram Schmidt.

Algorithm 5.1 GRAM–SCHMIDT ALGORITHM

given n-vectors $a_1, \ldots, a_k$

for $i = 1, \ldots, k$,

1. *Orthogonalization.* $\tilde{q}_i = a_i - (q_1^T a_i)q_1 - \cdots - (q_{i-1}^T a_i)q_{i-1}$
2. *Test for linear dependence.* if $\tilde{q}_i = 0$, quit.
3. *Normalization.* $q_i = \tilde{q}_i / \|\tilde{q}_i\|$

T F hold for this algo: If $\vec{a}_1, \ldots, \vec{a}_k$ is linearly independent for every iteration i

1. $\tilde{q}_i \neq 0$ (also means linear dependence test in step 2 not sat.)

2. $\vec{q}_1, \ldots, \vec{q}_i$ are orthonormal

3. $\vec{a}_i$ is linear comb. of $\vec{q}_1, \ldots, \vec{q}_i$

4. $\vec{q}_i$ is linear comb. of $\vec{a}_1 \ldots \vec{a}_i$

Algorithm 5.1 GRAM–SCHMIDT ALGORITHM

given n-vectors $a_1, \ldots, a_k$

for $i = 1, \ldots, k$,

1. *Orthogonalization.* $\boxed{\tilde{q}_i = a_i - (q_1^T a_i)q_1 - \cdots - (q_{i-1}^T a_i)q_{i-1}}$
2. *Test for linear dependence.* if $\tilde{q}_i = 0$, quit.
3. *Normalization.* $q_i = \tilde{q}_i / \|\tilde{q}_i\|$

Proof (induction)

Base case $i=1$, $\tilde{q}_1 = \vec{a}_1 \neq \vec{0}$, so $\boxed{\tilde{q}_1 \neq \vec{0}}$ point 1 ✓

point 2: $\vec{q}_1 = \tilde{q}_1 / \|\tilde{q}_1\|$, $\|\vec{q}_1\| = 1$ ✓

point 3,4: $\left.\begin{aligned} \vec{a}_1 &= \tilde{q}_1 = \|\tilde{q}_1\| \cdot \vec{q}_1 \\ \vec{q}_1 &= 1/\|\tilde{q}_1\| \cdot \vec{a}_1 \end{aligned}\right\}$

I.H: suppose that points 1-4 hold for some $i-1$, $i<k$

I.Proof: we will show they hold for iteration i.

✓ point 1: Assume, towards contradiction, that $\tilde{q}_i = \vec{0}$

then $\vec{0} = \vec{a}_i - \underbrace{\langle q_1, a_i \rangle \vec{q}_1 - \cdots}$ $\Rightarrow$

$* \; \vec{a}_i = c_1 \vec{q}_1 + \ldots + c_{i-1} \vec{q}_{i-1}$. from I.H each $\vec{q}_1 \ldots \vec{q}_{i-1}$ is linear comb. of $\vec{a}_1 \ldots \vec{a}_{i-1}$ $*$

$\vec{a}_i$ is linear comb. of $\vec{a}_1 \ldots \vec{a}_{i-1}$ $\Rightarrow$ means $\tilde{q}_i \neq \vec{0}$

Gram Schmidt.

point 2 : Show orthogonality

we will show that $\vec{q}_i \perp \vec{q}_j$ for $j = 1 \ldots i-1$

(I.H we have assumed $\boxed{\vec{q}_r \perp \vec{q}_s}$ for $r, s < i$)

↳ assuming for $i-1$

proof for i : $\tilde{q}_i = \vec{a}_i - \langle q_1, a_i\rangle \vec{q}_1 - \cdots \langle q_{i-1}, a_i\rangle \vec{q}_{i-1}$ *

for any $j = 1, \ldots, i-1$:

$\langle q_j, q_i \rangle = 0 \iff$ instead: $\langle q_j, \tilde{q}_i \rangle = 0$ enough? $\quad q_i = \dfrac{\tilde{q}_i}{\|\tilde{q}_i\|}$

should show

* $\Rightarrow$ $\langle q_j, \tilde{q}_i \rangle = \langle q_j, a_i \rangle - \langle q_1, a_1 \rangle \cancel{\langle q_1, q_j \rangle}^{0} \cdots - \langle q_{i-1}, a_i \rangle \cancel{\langle q_{i-1}, q_j \rangle}^{0}$

$j \leq i-1$

IH $\rightarrow \langle q_j, q_k \rangle = 0 \quad k \neq j$, $= \langle q_j, a_i \rangle - \langle q_j, a_i \rangle \cancel{\langle q_j, q_j \rangle}^{1}$

$\langle q_j, q_j \rangle = 1$

$= \langle q_j, a_i \rangle - \langle q_j, a_i \rangle = 0$ ✓

Gram Schmidt.

point 3 : Immediate from step 1

$$\vec{a}_i = \tilde{q}_i + \underbrace{\langle q_1, a_i\rangle \vec{q}_1 + \cdots + \langle q_{i-1}, a_i\rangle q_{i-1}}$$

$$= \|\tilde{q}_i\| \cdot q_i + \quad \checkmark$$

point 4 : $\tilde{q}_i \Rightarrow$ linear comb. of $\vec{a}_i, \vec{q}_1 \cdots \vec{q}_{i-1}$

I.H tells us that each of the $\vec{q}_1, \cdots \vec{q}_{i-1}$ is linear comb of $\vec{a}_1, \ldots, \vec{a}_{i-1}$, $\Rightarrow \tilde{q}_i$ (thus also $\vec{q}_i$) is linear comb. of $\vec{a}_1, \ldots, \vec{a}_i$. ✓

Gram Schmidt.

if Gram-Schmidt gets completed then $\vec{a}_1, \ldots, \vec{a}_k$ are linearly indep

Proof: suppose

$$\beta_1 \vec{a}_1 + \cdots + \beta_k \vec{a}_k = \vec{0}$$

for some β_i. Will show that $\beta_1 = \beta_2 = \cdots = 0$

- Focus on some i, we will show $\beta_i = 0$. Then will repeat same reasoning for all $i = 1 \ldots k$.

First show for $i = k$.

$$* \quad \beta_1 \langle q_k, a_1 \rangle + \beta_2 \langle q_k, a_2 \rangle + \cdots + \boxed{\beta_k \langle q_k, a_k \rangle} = 0$$

note1: any linear comb. of $\vec{q}_1 \ldots \vec{q}_{k-1}$ is orthogonal to $\vec{q}_k$

note2: each $\vec{a}_1, \ldots, \vec{a}_{k-1}$ is linear comb. of $\vec{q}_1 \ldots \vec{q}_{k-1}$

$\Rightarrow \langle q_k, a_1 \rangle = \langle q_k, a_2 \rangle = \cdots = \langle q_k, a_{k-1} \rangle = 0$

$\beta_k \langle q_k, a_k \rangle = 0$, $\beta_k \|\tilde{q}_k\| = 0$

$\langle q_k, a_k \rangle = \|\tilde{q}_k\| \Rightarrow \beta_k = 0$

- what happens if G-S terminates early?

suppose it terminates at iteration j,

$\tilde{q}_j = \vec{0}$.

- points 1-4 above hold for $i=1,\dots,j-1$

since $\tilde{q}_j = 0$:

$$\vec{a}_j = \langle q_1, a_j\rangle \vec{q}_1 + \dots + \langle q_{j-1}, a_j\rangle \vec{q}_{j-1}$$

each of $\vec{q}_1, \dots, \vec{q}_{j-1}$ is linear combo of $\vec{a}_1, \dots, \vec{a}_{j-1}$

$\Rightarrow$ $\vec{a}_j$ is linear combo of $\vec{a}_1, \dots, \vec{a}_{j-1}$

$\Rightarrow$ $\vec{a}_1, \dots, \vec{a}_j$ is <u>linearly dependent</u>

$\Rightarrow$ $\vec{a}_1, \dots, \vec{a}_k$ is <u>linearly dependent</u>

$(\vec{b}, \vec{a}_1, \dots, \vec{a}_k)$ see if $\tilde{q}_{k+1} = \vec{0}$ or not