Linear Algebra

CSCI 2820

Lecture 8

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ECES 122
Today

- Problem solving on chapters 3 and 5
- Vector Spaces
- Gram Schmidt
Refresher on $\text{std}$

The standard deviation is a measure of how much the entries of a vector differ from their mean value. Another measure of how much the entries of an $n$-vector $\vec{x} = (x_1, \ldots, x_n)$ differ from each other, called the mean square difference, is defined as

$$MSD(\vec{x}) = \frac{1}{n^2} \sum_{i,j} (x_i - \langle x \rangle)^2$$

Show that $MSD(\vec{x}) = 2 \text{std}(\vec{x})^2$

$$\text{std}(x) = \frac{\|x - (1^T x/n)1\|}{\sqrt{n}}.$$
\[
\text{MSD}(\tilde{\mathbf{a}}) = \frac{1}{n^2} \sum_{i,j=1}^{n} (\tilde{a}_i - \tilde{x}_j)^2 = \frac{1}{n^2} \sum_{i,j=1}^{n} (\tilde{a}_i^2 + \tilde{x}_j^2 - 2\tilde{a}_i \tilde{x}_j)
\]

\[
= \frac{1}{n^2} \left[ n \sum_{i=1}^{n} \tilde{a}_i^2 + n \sum_{j=1}^{n} \tilde{x}_j^2 - 2 \sum_{i,j=1}^{n} \tilde{a}_i \tilde{x}_j \right]
\]

\[
= \frac{2n}{n^2} \sum_{i=1}^{n} \tilde{a}_i^2 - \frac{2}{n^2} \sum_{j=1}^{n} \tilde{x}_j^2 - 2 \sum_{i,j=1}^{n} \tilde{a}_i \tilde{x}_j
\]

\[
= \frac{2 \sum_{i=1}^{n} \tilde{a}_i^2}{n} = \frac{2 \| \tilde{a} \|^2}{n}
\]

\[
\tilde{z} = (x_1 - \bar{x}, \ldots, x_n - \bar{x})
\]

\[
\sum \tilde{a}_i = \sum x_i - n \bar{x}
\]

\[
\sum \tilde{a}_i = 0 \implies \sum x_i = n \bar{x}
\]

\[
\sum \tilde{y}_j = 0 \implies \bar{y} = 0
\]
Refresher on linear independence/basis

Suppose \( \overrightarrow{x_1}, \ldots, \overrightarrow{x_k} \) are orthonormal \( n \)-vectors, and \( \overrightarrow{x} = \beta_1 \overrightarrow{x_1} + \cdots + \beta_k \overrightarrow{x_k} \), Where \( \beta_1, \ldots, \beta_k \) are scalars. Express \( \| \overrightarrow{x} \| \) in terms of \( \beta = (\beta_1, \ldots, \beta_k) \).

\[
\| \overrightarrow{x} \|^2 = \langle \overrightarrow{x}, \overrightarrow{x} \rangle = \langle \beta_1 \overrightarrow{x_1} + \cdots + \beta_k \overrightarrow{x_k}, \beta_1 \overrightarrow{x_1} + \cdots + \beta_k \overrightarrow{x_k} \rangle = \sum_{i=1}^{k} \beta_i^2 \langle \overrightarrow{x_i}, \overrightarrow{x_i} \rangle = \sum_{i=1}^{k} \beta_i^2
\]

\[
\langle \overrightarrow{x_i}, \overrightarrow{x_j} \rangle = \begin{cases} 0 & \text{if } i \neq j \text{ by statement} \\ \| \overrightarrow{x_i} \|^2 & \text{if } i = j 
\end{cases}
\]

\[
\| \overrightarrow{x} \| = \sqrt{\sum_{i=1}^{k} \beta_i^2} = \sqrt{\sum_{i=1}^{k} \| \overrightarrow{x_i} \|^2}
\]

\[
\overrightarrow{x} = \beta_1 \overrightarrow{x_1} + \cdots + \beta_k \overrightarrow{x_k}
\]

\[
\overrightarrow{x} = \overrightarrow{x_1} e_1 + \cdots + \overrightarrow{x_k} e_k
\]
Vector space

1.1 Definition A vector space (over \( \mathbb{R} \)) consists of a set \( V \) along with two operations ‘+’ and ‘.’ subject to the conditions that for all vectors \( \mathbf{v}, \mathbf{w}, \mathbf{u} \in V \) and all scalars \( r, s \in \mathbb{R} \):

1. the set \( V \) is closed under vector addition, that is, \( \mathbf{v} + \mathbf{w} \in V \)
2. vector addition is commutative, \( \mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v} \)
3. vector addition is associative, \( (\mathbf{v} + \mathbf{w}) + \mathbf{u} = \mathbf{v} + (\mathbf{w} + \mathbf{u}) \)
4. there is a zero vector \( \mathbf{0} \in V \) such that \( \mathbf{v} + \mathbf{0} = \mathbf{v} \) for all \( \mathbf{v} \in V \)
5. each \( \mathbf{v} \in V \) has an additive inverse \( \mathbf{w} \in V \) such that \( \mathbf{w} + \mathbf{v} = \mathbf{0} \)
6. the set \( V \) is closed under scalar multiplication, that is, \( r \cdot \mathbf{v} \in V \)
7. scalar multiplication distributes over scalar addition, \( (r + s) \cdot \mathbf{v} = r \cdot \mathbf{v} + s \cdot \mathbf{v} \)
8. scalar multiplication distributes over vector addition, \( r \cdot (\mathbf{v} + \mathbf{w}) = r \cdot \mathbf{v} + r \cdot \mathbf{w} \)
9. ordinary multiplication of scalars associates with scalar multiplication, \( (rs) \cdot \mathbf{v} = r \cdot (s \cdot \mathbf{v}) \)
10. multiplication by the scalar 1 is the identity operation, \( 1 \cdot \mathbf{v} = \mathbf{v} \).

For \( \mathbb{R}^n \Rightarrow \mathbf{0} = \left( \begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \end{array} \right) \)

\( r = 3 \)

\( s = 2 \)

\( r \cdot (s \mathbf{v}) = 3 \cdot \left( 2 \left( \begin{array}{c} -1 \\ 2 \\ -5 \\ 3 \end{array} \right) \right) = \left( \begin{array}{c} -6 \\ 6 \\ -15 \\ 9 \end{array} \right) \)

\( (rs) \cdot \mathbf{v} = \left( \begin{array}{c} -6 \\ 6 \\ -15 \\ 9 \end{array} \right) \)

\( \mathbf{v} = \left( \begin{array}{c} -1 \\ 2 \\ -5 \\ 3 \end{array} \right) \)
Vector space, contd.

- Example:

\[ V = \{ (0), (1) \} \]

not a vector space

\[ (1) + (1) = (0) \notin V \]

\[ L = \{ (y) \mid y = 3x \} \]

line through origin

\[ +, \cdot \quad \text{"regular" vector addition, mult of } \mathbb{R}^2 \]

\[ (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \]

\[ r (y) = (ry) \]

L "inherits" these operations

- ex: \[ K = \{ (x, y) \} \]

\[ x, y \text{ are integers} \]

is this a vector space with +, \cdot \text{ over reals?}\

\[ 0.5 \cdot (\frac{1}{2}) = (0.5) \notin K \quad \text{NOT a vector space!} \]
Vector space, contd.

(1) closure under +
\[ \vec{v}_1 = (x_1, y_1), \quad \vec{v}_2 = (x_2, y_2), \quad \vec{v}_1 + \vec{v}_2 = (x_1 + x_2, y_1 + y_2) \]

\[ y_1 = 3x_1 \quad y_2 = 3x_2 \]

(2), (3) straightforward from \( \mathbb{R}^2 \).

(4) \[ (x, y) + (0, 0) = (x, y) \]

(5) inverse under +:
\[ (-x, -y) + (x, y) = (0, 0) \]
if \( y = 3x \) then \( -y = 3(-x) \)
Vector space, contd.

\[
\{(0)\} \quad \text{this is the trivial vector space}
\]
Vector space, contd.
Vector space, contd.
Gram Schmidt Algo

given: a list of n-vectors \( \vec{a}_1, \ldots, \vec{a}_k \) for \( i = 1 \ldots k \)

1. Orthogonalization \( \vec{q}_i = \vec{a}_i - \langle \vec{q}_1, \vec{a}_i \rangle \vec{q}_1 - \ldots - \langle \vec{q}_{i-1}, \vec{a}_i \rangle \vec{q}_{i-1} \)

2. Test for linear dependence. If \( \vec{q}_i = 0 \), then quit

3. Normalization, \( \vec{z}_i = \vec{q}_i / ||\vec{q}_i|| \)

For step 2: \( i = 1 \):

\( \vec{q}_1 = \vec{a}_1 \), \( \vec{z}_1 = \frac{\vec{a}_1}{||\vec{a}_1||} \)

Claim 1: If algo doesn't quit at any iteration then \( \vec{a}_i \) are indeed l.o.s.

Claim 2: If algo quits before the end \( (say \vec{q}_j = 0) \) then \( \vec{a}_i \) is linearly dep. and \( \vec{a}_j = c_1 \vec{a}_1 + \ldots + c_k \vec{a}_k \).
The following hold for Gram-Schmidt:

for \( i = 1, \ldots, k \), assuming \( \mathbf{a}_1, \ldots, \mathbf{a}_k \) are linearly independent.

1) \( \tilde{\mathbf{a}}_i \neq \mathbf{0} \) so the linear dependence test in step 2 fails.

2) \( \tilde{\mathbf{a}}_1, \ldots, \tilde{\mathbf{a}}_i \) are orthonormal.

3) \( \mathbf{a}_i \) is a linear combination of \( \tilde{\mathbf{a}}_1, \ldots, \tilde{\mathbf{a}}_i \).

4) \( \tilde{\mathbf{a}}_i \) is a linear combination of \( \mathbf{a}_1, \ldots, \mathbf{a}_i \).