Linear Algebra

CSCI 2820

Lecture 4

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ECES 122

Today

- Norm of vectors
- Distances
- Examples

Refresher on Linear functions

Which of the functions above is affine and why (or why not)?

counterexample $n=2$

Max $\Rightarrow$ last lecture

$\boxed{f(\alpha\vec{x}+\beta\vec{y}) \neq \alpha f(\vec{x})+\beta f(\vec{y})}$

- Minimum: The min element of an n-vector $\vec{x}$ $|x>$, $f(\vec{x})=\min\{x_1,x_2,\ldots,x_n\}$
- The average of the entries of the vectors with odd indices, minus the average of the entries of the vector with even indices (assume $n=2k$ is even)

$\langle\vec{w},\vec{x}\rangle$

$(x_1,x_2,x_3,x_4)=\vec{x}$

$\vec{w}=(1,-1,1,-1,\ldots)\cdot\frac{1}{2k}$ $\quad f(\vec{x})=\frac{x_1+x_3-x_2-x_4}{4}$

Norm

Euclidean norm of n-vector $\vec{x}$, $\|\vec{x}\|$

$$\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{\langle x, x \rangle}$$

Norm examples

$$\left\|\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}\right\| = ?\ \sqrt{2^2+(-1)^2+2^2} = \sqrt{4+1+4} = \sqrt{9} = 3$$

When vector is 1-dim? (scalar) absolute value

When vector is unit? $\Rightarrow$ 1 $[\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n]$

def: $\vec{v}$ is a unit vector iff $\|\vec{v}\|_2 = 1$

Norm Properties:

function $\mathbb{R}^n \to \mathbb{R}$

- Nonnegative homogeneity: $\|\beta\vec{x}\| = |\beta|\|\vec{x}\|$
- Triangle inequality: $\|\vec{x}+\vec{y}\| \le \|\vec{x}\| + \|\vec{y}\|$
- Non-negativity. $\|\vec{x}\| \ge 0$
- Definiteness. $\|\vec{x}\| = 0$ only if $\vec{x} = \vec{0}$

positive definiteness

verify: $\|\vec{x}\|^2 = 0 \Rightarrow x_1^2 + \ldots + x_n^2 = 0 \Rightarrow x_i^2 = 0 \Rightarrow \vec{x} = \vec{0}$

Norm

Root-mean-square value. (RMS)

$$\mathrm{rms}(\vec{x}) = \underbrace{\sqrt{\frac{x_1^2+\cdots+x_n^2}{n}}}_{ms(\vec{x})} = \frac{\|\vec{x}\|}{\sqrt{n}}$$

tells us what a "typical" value of $|x_i|$ is

$\vec{1}$, $\|\vec{1}\| = ?$ $\sqrt{1^2+\cdots+1^2} = \sqrt{n}$

$RMS(\vec{1}) = ?$ 1

if $\vec{v} = (a, a, \ldots, a)$, $RMS(\vec{v}) = |a|$

Norm of Sum : $\|\underbrace{\vec{x}+\vec{y}}\|^2 = (\vec{x}+\vec{y})^T(\vec{x}+\vec{y}) =$

$= \underbrace{\vec{x}^T\vec{x}} + \underbrace{\vec{x}^T\vec{y} + \vec{y}^T\vec{x}} + \underbrace{\vec{y}^T\vec{y}} =$

$= \|\vec{x}\|^2 + 2\vec{x}^T\vec{y} + \|\vec{y}\|^2$

Norm

Norm of block vectors $\vec{d} = \begin{pmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{pmatrix}$ or $\vec{a}$: 2 dim, $\vec{b}$: 3 dim, $\vec{c}$: 10 dim

$\vec{d} = (\vec{a}, \vec{b}, \vec{c})$

$\|\vec{d}\|^2 = \vec{d}^T\vec{d} = \vec{a}^T\vec{a} + \vec{b}^T\vec{b} + \vec{c}^T\vec{c} = \|\vec{a}\|^2 + \|\vec{b}\|^2 + \|\vec{c}\|^2$

$$\|\vec{d}\| = \|(\underbrace{\vec{a}, \vec{b}, \vec{c}}_{15\text{-dim}})\| = \underbrace{\sqrt{\|\vec{a}\|^2 + \|\vec{b}\|^2 + \|\vec{c}\|^2}} = \|(\|\vec{a}\|, \|\vec{b}\|, \|\vec{c}\|)\|$$

3-dim

$\vec{d} = (\vec{a}, \vec{b}, \vec{c})$ but: $\|\vec{d}\| = \|\vec{k}\|$

$\vec{k} = (\|\vec{a}\|, \|\vec{b}\|, \|\vec{c}\|)$

Norm

chebyshev inequality: $\vec{x}$ n-vector

assume k of its entries satisfy $x_i^2 \geq a^2$

$|x_i| \geq a,\ a > 0$

$\|\vec{x}\|^2 = x_1^2 + \cdots + x_n^2 \geq k a^2 \Rightarrow \boxed{k \leq \frac{\|\vec{x}\|^2}{a^2}}$ chebyshev

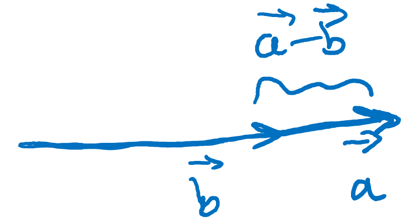
- $\|\vec{x}\|^2 / a^2 \geq n \Rightarrow$ nothing new ($k \leq n$)
- $\|\vec{x}\|^2 < a^2 \xRightarrow{\text{chebyshev}} k \leq \frac{\|\vec{x}\|^2}{a^2} < 1, \Rightarrow k = 0$

$$\boxed{\frac{k}{n} \leq \left(\frac{\mathrm{rms}(\vec{x})}{a}\right)^2}$$

Q: how many entries of $\vec{x}$ can exceed its RMS value by more than a factor of 5?

Distance

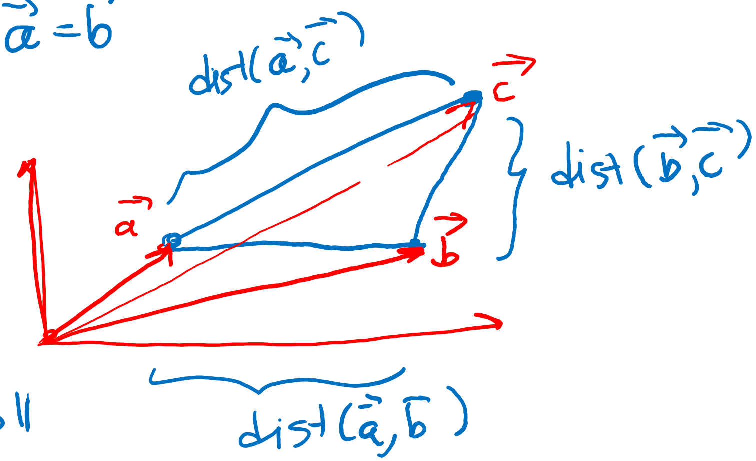
$\text{dist}(\vec{a},\vec{b}) = \|\vec{a}-\vec{b}\|$



$\text{dist}(\vec{a},\vec{b}) = 0 \; ? \quad \|\vec{a}-\vec{b}\| = 0$

if $\vec{a}-\vec{b} = \vec{0} \Rightarrow \vec{a} = \vec{b}$

- Triangle



$\text{dist}(\vec{a},\vec{b}) = \|\vec{a}-\vec{b}\|$

$\text{dist}(\vec{b},\vec{c}) = \|\vec{b}-\vec{c}\|$

$\text{dist}(\vec{a},\vec{c}) = \|\vec{a}-\vec{c}\|$

$\|\vec{a}-\vec{c}\| \le \|\vec{a}-\vec{b}\| + \|\vec{b}-\vec{c}\|$ (triangle ineq. for the actual triangle)

follows from (3): $\|\vec{a}-\vec{c}\| = \|(\vec{a}-\vec{b})+(\vec{b}-\vec{c})\| \le \|\vec{a}-\vec{b}\| + \|\vec{b}-\vec{c}\|$

Distance

Distance

$$u = \begin{bmatrix} 1.8 \\ 2.0 \\ -3.7 \\ 4.7 \end{bmatrix}, \qquad v = \begin{bmatrix} 0.6 \\ 2.1 \\ 1.9 \\ -1.4 \end{bmatrix}, \qquad w = \begin{bmatrix} 2.0 \\ 1.9 \\ -4.0 \\ 4.6 \end{bmatrix}.$$

Distance

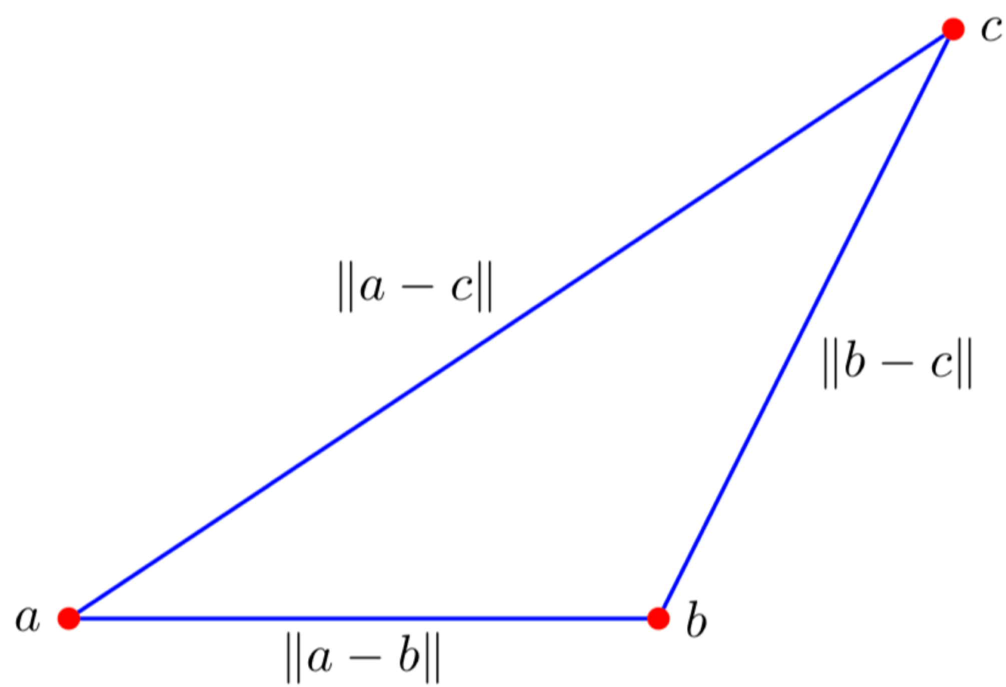


Figure 3.2 Triangle inequality.