Linear Algebra

CSCI 2820

Lecture 25

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ECES 122

Today

- More on spectrum

Determinant and eigenvalues

claim 2 : $\det A = \prod_{i=1}^{n} \lambda_i$

① Assume $A = PDP^{-1}$

$\det A = \det(PDP^{-1}) = \det P \cdot \det D \cdot \det P^{-1}$

$(\det P^{-1} = (\det P)^{-1})$ → $\underbrace{\det P \cdot (\det P)^{-1}}_{=1} \cdot \det D$

$= \det D = \prod_{i=1}^{n} \lambda_i$

② general A.

$f(\lambda) = \det(\lambda I - A) = \prod_{i=1}^{n} (\lambda - \lambda_i) = \lambda^n - \lambda^{n-1} \sum_{i=1}^{n} \lambda_i + \dots + \underbrace{(-1)^n \prod_{i=1}^{n} \lambda_i}$

• observe: if λ_i is root of char. poly, then λ_i is a root of $\det(\lambda I - A)$.

• by previous thms: $\boxed{\det A = (-1)^n \det(-A)}$

since $f(0) = \det(-A) = (-1)^n \prod_{i=1}^{n} \lambda_i$

$(-1)^n \det A = (-1)^n \prod_{i=1}^{n} \lambda_i$

$\Rightarrow \det A = \prod_{i=1}^{n} \lambda_i$

Determinant and eigenvalues

Trace and eigenvalues

Def: $\text{Trace } A = \sum_{i=1}^{n} a_{ii}$

$A \ n \times n$

Claim: $\text{Trace } A = \sum_{i=1}^{n} \lambda_i$

assume A is diagonalizable, $A = PDP^{-1}$

$D = \begin{bmatrix} \lambda_1 & & \emptyset \\ & \ddots & \\ \emptyset & & \lambda_n \end{bmatrix}$, $P = [\vec{v}_1 \ \dots \ \vec{v}_n]$

observe: $\text{Trace}(AB) = \text{Trace}(BA)$

$\text{Trace}(AB) = \sum_k \sum_j A_{kj} B_{jk} = \sum_k \sum_j B_{kj} A_{jk} = \text{Trace}(BA)$

- $\text{Trace } A = \text{Trace}(PDP^{-1}) = \text{Trace}(P^{-1}PD) = \text{Trace}(I \cdot D)$

$= \text{Trace}(D) = \sum_{i=1}^{n} \lambda_i$

Trace and eigenvalues

Complex Eigenvalues

We now allow $A_{n\times n}$ acting on $\mathbb{C}^n$

complex scalar λ is an eigenvalue

iff $\det(A-\lambda I)=0$, $\vec{x}$ eigenvector (non-zero)

iff $A\vec{x}=\lambda\vec{x}$

eg. $A=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$ $\quad \vec{x}\mapsto A\vec{x}$ on $\mathbb{R}^2$

$\det(A-\lambda I)=\lambda^2+1$. $\quad \lambda^2+1=0 \quad \begin{cases}\lambda_1 = i\\ \lambda_2=-i\end{cases}$

- $\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\underbrace{\begin{bmatrix}1\\ -i\end{bmatrix}}_{\vec{v_1}}=\begin{bmatrix}i\\ 1\end{bmatrix}=\underbrace{i}_{\lambda_1}\begin{bmatrix}1\\ -i\end{bmatrix}$

- $\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\underbrace{\begin{bmatrix}1\\ i\end{bmatrix}}_{\vec{v_2}}=\begin{bmatrix}-i\\ 1\end{bmatrix}=\underbrace{-i}_{\lambda_2}\begin{bmatrix}1\\ i\end{bmatrix}$

Complex Eigenvalues

$$A = \begin{bmatrix} 0.5 & -0.6 \\ 0.75 & 1.1 \end{bmatrix}$$

- Find evalues, evectors

$$\det(A - \lambda I) = 0 \Rightarrow \det \begin{bmatrix} 0.5-\lambda & -0.6 \\ 0.75 & 1.1-\lambda \end{bmatrix}$$

$$= \lambda^2 - 1.6\lambda + 1, \qquad \lambda = \frac{1}{2}\left[1.6 \pm \sqrt{(-1.6)^2 - 4}\right] = 0.8 \pm 0.6i$$

$$\left.\begin{aligned} \lambda_1 &= 0.8 - 0.6i \\ \lambda_2 &= 0.8 + 0.6i \end{aligned}\right\}$$

$$A\vec{x} = \lambda_{1,2}\vec{x} : \qquad A - \lambda_1 I =$$

$$\Rightarrow \boxed{(A - \lambda_1 I)\vec{x} = \vec{0}} \ *$$

$$\begin{bmatrix} -0.3+0.6i & -0.6 \\ 0.75 & 0.3+0.6i \end{bmatrix}$$

observe : the system

$$(-0.3+0.6i)x_1 - 0.6x_2 = 0 \ \checkmark$$

$$0.75x_1 + (0.3+0.6i)x_2 = 0 \ \checkmark$$

$$0.75x_1 = (-0.3-0.6i)x_2 \qquad \text{choose } x_2 = 5$$

$$x_1 = (-0.4-0.8i)x_2, \qquad x_1 = -2-4i$$

$$\boxed{\vec{v}_1 = \begin{bmatrix} -2-4i \\ 5 \end{bmatrix}}$$

Complex Eigenvalues

similarly, $\vec{v}_2 = \begin{bmatrix} -2+4i \\ 5 \end{bmatrix}$

- <u>Complex conjugates</u> of vectors

$\vec{x} \in \mathbb{C}^n$, $\overline{\vec{x}} \in \mathbb{C}^n$: entries are just complex conjugates of entries in $\vec{x}$

- <u>Real & imaginary parts of complex $\vec{v}$</u> :

$\mathrm{Re}\,\vec{x}$, $\mathrm{Im}\,\vec{x}$

e.g: $\vec{x} = \begin{bmatrix} 3-i \\ i \\ 2+5i \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} + i \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}$

$\mathrm{Re}\,\vec{x} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$, $\mathrm{Im}\,\vec{x} = \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}$, $\overline{\vec{x}} = \begin{bmatrix} 3+i \\ -i \\ 2-5i \end{bmatrix} = \mathrm{Re}\,\vec{x} - i \cdot \mathrm{Im}\,\vec{x}$

- $B_{m \times n}$, $\overline{B} = [\overline{b}_{ij}]$: $\overline{r\vec{x}} = \overline{r} \cdot \overline{\vec{x}}$, $\overline{B\vec{x}} = \overline{B} \cdot \overline{\vec{x}}$, $\overline{BC} = \overline{B} \cdot \overline{C}$, $\overline{rB} = \overline{r} \cdot \overline{B}$

Complex Eigenvalues

- let $A_{n\times n}$ real matrix.

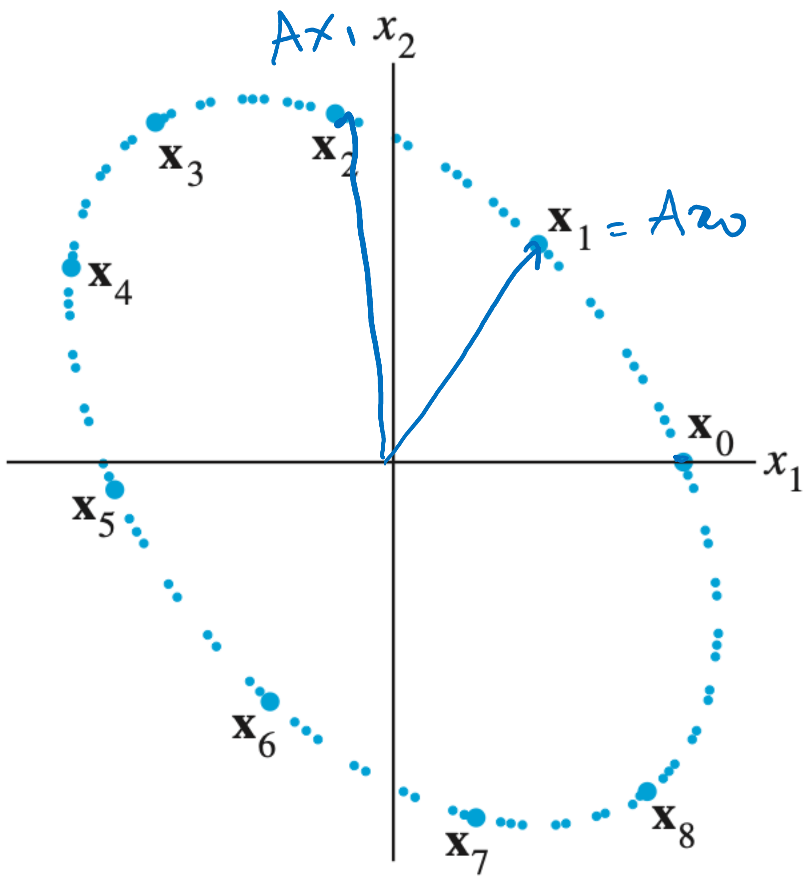
$\overline{A\vec{x}} = \overline{A}\,\overline{\vec{x}} = A\overline{\vec{x}}$.

If λ is evalue of A, $\vec{x}$ evector

$A\overline{\vec{x}} = \overline{A\vec{x}} = \overline{\lambda\vec{x}} = \overline{\lambda}\cdot\overline{\vec{x}}$

($\overline{\lambda}$: evalue; $\overline{\vec{x}}$: evector)

Complex Eigenvalues



Real and Imaginary parts of eigenvectors

Real and Imaginary parts of eigenvectors

Real and Imaginary parts of vectors

Real and Imaginary parts of vectors

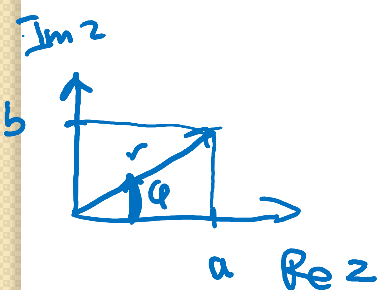
Hidden Rotations

Eg: $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ $a, b \neq 0 \in \mathbb{R}$

easy to calculate: $\lambda_{1,2} = a \pm bi$

define $r = |\lambda| = \sqrt{a^2+b^2}$

$$C = r \begin{bmatrix} a/r & -b/r \\ b/r & a/r \end{bmatrix} = \underbrace{\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}}_{\text{scaling}} \underbrace{\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}}_{\text{rotation}}$$



φ is "argument" of $\lambda = a + bi$

Hidden Rotations

$$A = \begin{bmatrix} 0.5 & -0.6 \\ 0.75 & 1.1 \end{bmatrix} \qquad \lambda_1 = 0.8 - 0.6i$$

$$\vec{v}_1 = \begin{bmatrix} -2-4i \\ 5 \end{bmatrix}$$

define: $P = \begin{bmatrix} \mathrm{Re}\,\vec{v}_1 & \mathrm{Im}\,\vec{v}_1 \end{bmatrix} = \begin{bmatrix} -2 & -4 \\ 5 & 0 \end{bmatrix}$

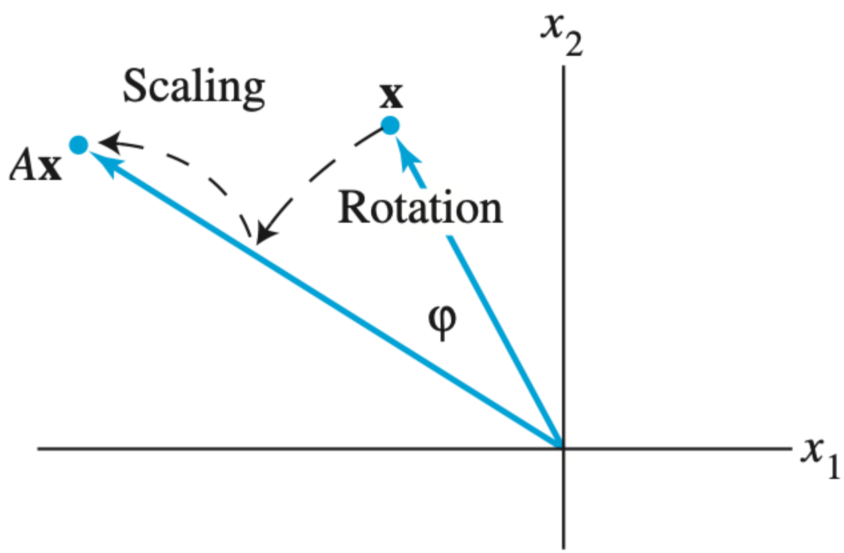
$$\underline{C = P^{-1} A P} = \underbrace{\frac{1}{20}\begin{bmatrix} 0 & 4 \\ -5 & -2 \end{bmatrix}}_{P^{-1}} \underbrace{\begin{bmatrix} 0.5 & -0.6 \\ 0.75 & 1.1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} -2 & -4 \\ 5 & 0 \end{bmatrix}}_{P}$$

$$= \begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{bmatrix} \quad \left(\text{of the form } \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \right)$$

$r = \sqrt{a^2+b^2} = \sqrt{0.8^2+0.6^2} = 1$, C is pure rotation!

$$A = PCP^{-1} = \underline{P} \underbrace{\begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{bmatrix}}_{C} \underline{P^{-1}}$$

Hidden Rotations



Hidden Rotations

<u>Thm</u> A 2×2, real matrix with a complex eigenvalue $\lambda = a - bi$ $(b \neq 0)$, and corresponding eigenvector $\vec{v}$ in $\mathbb{C}^2$, then

$$A = PCP^{-1}, \quad P = [\operatorname{Re}\vec{v} \;\; \operatorname{Im}\vec{v}], \quad C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Hidden Rotations

FCQs!

- Visit colorado.campuslabs.com/courseeval