Linear Algebra

CSCI 2820

Lecture 24

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ECES 122

Today

- Characteristic Equations
- Determinant properties reminder
- Eigenvalue calculation
- Diagonalization

Characteristic equations

$A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$ find evalues?

Find all λ s.t: $\boxed{(A-\lambda I)\vec{x} = \vec{0}}$

$\Leftrightarrow$ find all λ s.t $A-\lambda I$ not invertible.

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix}$$

$A-\lambda I$ not invert $\Leftrightarrow$ $\det(A-\lambda I) = 0$

$$\left\{ \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc \right\}$$

$$\det(A-\lambda I) = (2-\lambda)(-6-\lambda) - 3\cdot 3 = \boxed{\lambda^2 + 4\lambda - 21}$$

$$= \boxed{(\lambda-3)(\lambda+7) = 0}$$

When is $\det = 0$? $\lambda = 3$ and $\lambda = -7$ evalues of A

Characteristic equations

Determinants : A $n\times n$, let U be any echelon obtained from A by row replacements + interchanges let r = # of interchanges. Then

$$\det(A) = \begin{cases} (-1)^r \cdot \begin{pmatrix}\text{product of} \\ \text{pivots}\end{pmatrix} & \text{when } A \text{ invert.} \\ 0 & \text{when } A \text{ not invert} \end{cases}$$

$$\boxed{\begin{array}{l} \underline{\text{Invertible Matrix thm.}} \\ A \ n\times n \text{ matrix is invert. iff} \\ \bullet \text{ The number } 0 \text{ is } \underline{\text{not}} \text{ an eigenvalue of } A \\ \bullet \ \det(A) \neq 0 . \end{array}}$$

Characteristic equations

Properties of det

A, B n×n matrices

- A invert. iff $\det A \neq 0$
- $\det AB = (\det A)\cdot(\det B)$
- $\det A^T = \det A$
- A triangular $\Rightarrow$ $\det A$ is product of diag.
- row replacement does not change det, while row interchange changes the sign.

Def: (characteristic equation) : $\boxed{\det(A-\lambda I)=0}$

A scalar λ is evalue of A iff λ satisfies characteristic equation.

eg: find the char. equation $\begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$(\lambda-5)(\lambda-3)(\lambda-5)(\lambda-1)$

$(\lambda-5)^2(\lambda-3)(\lambda-1) = \lambda^4 - 14\lambda^3 + 68\lambda^2 - 130\lambda + 75 = 0$

Characteristic equations

A 6×6, $\lambda^6 - 4\lambda^5 - 12\lambda^4 = \lambda^4(\lambda-6)(\lambda+2)$

Similarity

A, B $n \times n$ matrices. A similar to B if there is an invertible matrix P s.t

$P^{-1}AP = B$ or equiv. $A = PBP^{-1}$

(A similar to B $\Leftrightarrow$ B similar to A, $Q = P^{-1}$

$B = QAQ^{-1}$)

Thm: if A, B similar then they have the same characteristic poly. hence, the same evalues (with same mult.)

Proof: $B = P^{-1}AP$. $B - \lambda I = P^{-1}AP - \lambda P^{-1}P = P^{-1}(AP - \lambda P)$

$= P^{-1}(A - \lambda I)P$

$\det(B - \lambda I) = \det(P^{-1}(A - \lambda I)P) = \det(P^{-1}) \cdot \det(A - \lambda I) \cdot \det(P)$

$= \det(A - \lambda I) \cdot \underbrace{\det(P^{-1})\det(P)}_{\det(P^{-1}P) = \det I = 1} = \det(A - \lambda I)$

Similarity

Warning: $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

same evalues but not similar

Diagonalization

$A = PDP^{-1}$, D is diagonal

allows us to compute matrix powers quickly.

eg. $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$, $D^2 = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$

$= \begin{bmatrix} 5^2 & 0 \\ 0 & 3^2 \end{bmatrix}$

$\vdots$

$D^k = \begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix}$

<u>claim</u>: if $A = PDP^{-1}$ then, A^k is also easy to compute.

Diagonalization

eg 2 : $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$. $P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$

$D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$

$A = PDP^{-1}$

$P^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$

$$A^2 = (PD\underbrace{P^{-1}).(P}_{I}DP^{-1}) = PD^2P^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5^2 & 0 \\ 0 & 3^2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$\vdots$

$$A^k = PD^kP^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2.5^k - 3^k & 5^k - 3^k \\ 2.3^k - 2.5^k & 2.3^k - 5^k \end{bmatrix}$$

Diagonalization

Def: $A_{n\times n}$ is diagonalizable if similar to a diag. matrix.

Q: does a diagonalizable matrix need to be invertible? No

Thm: $A_{n\times n}$ is diagonalizable if and only if A has n linearly ind. evectors. In fact, $A = PDP^{-1}$ with D diagonal, iff the colums of P are the n linearly indep. eigenvectors of A. In this case, diag. entries of D are eigenvalues of A that correspond respectively to the eigenvectors in P.

Diagonalization

ex 1. let $A = \begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$A\vec{v}_1 = \lambda_1 \vec{v}_1 \Rightarrow \lambda_1 = 1$

& $A\vec{v}_2 = \lambda_2 \vec{v}_2 \Rightarrow \lambda_2 = 3$. Diagonalize A.

(give me P, D: $A = PDP^{-1}$)

$P = [\vec{v}_1 \; \vec{v}_2] = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

ex 2. A 4x4, evalues 5, 3, -2. Suppose I know that the eigenspace for $\lambda = 3$ is 2 dim. Can I say that A is diagonalizable?

Yes. by thm.

Diagonalization

<u>Proof</u> : $P = [\vec{v_1} \ \cdots \ \vec{v_n}]$, $D = \begin{bmatrix} \lambda_1 & & \emptyset \\ & \ddots & \\ \emptyset & & \lambda_n \end{bmatrix}$

$AP = [A\vec{v_1} \ \cdots \ A\vec{v_n}]$ (1)

$PD = P \begin{bmatrix} \lambda_1 & & \emptyset \\ & \ddots & \\ \emptyset & & \lambda_n \end{bmatrix} = [\lambda_1\vec{v_1} \ \lambda_2\vec{v_2} \cdots \ \lambda_n\vec{v_n}]$ (2)

- suppose A diagonalizable, $A = PDP^{-1}$

$A = PDP^{-1} \Leftrightarrow AP = PD \overset{(1)}{\underset{(2)}{\Longleftrightarrow}} [A\vec{v_1} \ \cdots \ A\vec{v_n}] = [\lambda_1\vec{v_1} \ \cdots \ \lambda_n\vec{v_n}]$

$A\vec{v_i} = \lambda_i\vec{v_i} \quad \forall i \quad (\Rightarrow)$

assume $A\vec{v_i} = \lambda_i\vec{v_i}$

$(\Leftarrow)$ $P = [\vec{v_1} \ \cdots \ \vec{v_n}]$, $D = \begin{bmatrix} \lambda_1 & & \emptyset \\ & \ddots & \\ \emptyset & & \lambda_n \end{bmatrix}$.

$\Rightarrow AP = PD \Rightarrow A = PDP^{-1}$

Diagonalization

$$A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

$$\det(A-\lambda I) = -(\lambda-1)(\lambda+2)^2$$

$\lambda = 1$ mult 1

$\lambda = -2$ mult 2

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad \lambda = 1$$

$$\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \lambda = -2$$

impossible to diagonalize

<u>not</u> diagonalizable

Diagonalization

ex. diagonalize

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}, \quad D, P?$$

s.t $A = PDP^{-1}$

step 1: find evalues

$\det(A - \lambda I) = 0 \Leftrightarrow -\lambda^3 - 3\lambda^2 + 4 = 0 \Leftrightarrow$

$-(\lambda-1)(\lambda+2)^2$: $\lambda = 1$ mult. 1

$\lambda = -2$ mult. 2

$$D = \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{-2} & 0 \\ 0 & 0 & \boxed{-2} \end{bmatrix}$$

step 2: Find 3 linearly ind evectors of A.

$\lambda = 1$, $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$\lambda = -2$, $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ $\quad \vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$P = [\vec{v}_1 \; \vec{v}_2 \; \vec{v}_3]$

$$\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$P = [\vec{v}_2 \; \vec{v}_1 \; \vec{v}_3]$

Diagonalization

<u>Thm</u> : A $n \times n$ with n distinct evalues is diagonalizable

<u>eg</u> : $A = \begin{bmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{bmatrix}$

(1) is A diagonalizable?

(2) is A invertible?

Diagonalization

if not distinct evalues?

A has n distinct eval. and $\vec{v}_1 \dots \vec{v}_n$ evectors

$P = [\vec{v}_1 \dots \vec{v}_n]$ $\{\vec{v}_i\}$ LI so P invertible

how do I find P invertible?

Thm. $A_{n \times n}$ with distinct eval $\lambda_1 \dots \lambda_p$

(a) For $1 \le k \le p$, dimension of the eigenspace of λ_k is less than or equal to multiplicity of evalue λ_k

(b) A is diagonalizable iff sum of the dimensions of the eigenspace equals n. This happens only if: (i) characteristic poly factors completely into linear factors. (ii) dimension of espace for each λ_k equals multiplicity of λ_k.

(c) if A diagonalizable, B_k is basis for espace λ_k, $\{B_1, \dots, B_p\}$ is basis for $\mathbb{R}^n$.