Linear Algebra

CSCI 2820

Lecture 19

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ECES 122

Today

- Elementary row operations
- Row/column space
- Rank and nullspace

Exercise

- If either A or B (both square matrices) are not invertible, can AB be invertible?

case 1: B not invert

Assume $(AB)^{-1} = C$

$A^{-1} A B C C^{-1} = A^{-1} C^{-1}$

$B = A^{-1} C^{-1}$ invertible $\Rightarrow \Leftarrow$

$\exists \vec{x} : B\vec{x} = \vec{0} \Rightarrow (A \cdot B)\vec{x} = \vec{0}$

case 2: A not invert

(repeat similar proof)

Exercise

11.5 *Inverse of a block matrix.* Consider the $(n+1) \times (n+1)$ matrix

$$A = \begin{bmatrix} I & a \\ a^T & 0 \end{bmatrix},$$

where a is an n-vector.

(a) When is A invertible? Give your answer in terms of a. Justify your answer.

(b) Assuming the condition you found in part (a) holds, give an expression for the inverse matrix A^{-1}.

Solving systems of equations

$$\boxed{\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 - 5x_3 &= 10 \end{aligned}} \Rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{array}\right] \Rightarrow \text{augmented matrix}$$

b

coeff matrix

$$\begin{array}{l} -5\cdot[\text{equation } 1] \\ +\ [\text{equation } 3] \\ \hline [\text{new equation } 3] \end{array} \qquad \begin{array}{r} -5x_1 + 10x_2 - 5x_3 = 0 \\ 5x_1 \qquad\quad - 5x_3 = 10 \\ \hline 10x_2 - 10x_3 = 10 \end{array}$$

$$x_1 - 2x_2 + x_3 = 0$$

$$\rightarrow \left(\tfrac{1}{2}\,2x_2 - \tfrac{1}{2}\,8x_3 = \tfrac{1}{2}\,8\right)$$

$$x_2 - 4x_3 = 4$$

$$10x_2 - 10x_3 = 10$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 10 & -10 & 10 \end{bmatrix}$$

$$\begin{array}{l} -10\cdot[\text{equation } 2] \\ +\ [\text{equation } 3] \\ \hline [\text{new eq. } 3] \end{array}$$

$$\boxed{\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ \left(\tfrac{1}{30}\,30x_3 = -30\cdot\tfrac{1}{30}\right) \\ x_3 &= -1 \end{aligned}}$$

$\Rightarrow$ upper triangular

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Solving systems of equations

$$x_2 - 4x_3 = 8$$
$$2x_1 - 3x_2 + 2x_3 = 1$$
$$4x_1 - 8x_2 + 12x_3 = 1$$

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 4 & -8 & 12 & 1 \end{bmatrix} \rightarrow \begin{array}{c} \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 4 & -8 & 12 & 1 \end{bmatrix} \end{array} \begin{array}{l} -2\times \\ \\ + \end{array}$$

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -2 & 8 & -1 \end{bmatrix} \begin{array}{l} \\ \times 2 \\ + \end{array} \rightarrow \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 15 \end{bmatrix}$$

$$0 = 15$$

$$0x_1 + 0x_2 + 0x_3 = 15 \quad \times$$

~~Echelon form~~ $\boxed{\text{Elementary Row operations}}$

1) (Replacement) : Replace one row by the sum of itself and a multiple of another row

2) (Interchange) : Interchange two rows

3) (Scaling) Multiply all entries in a row by a non-zero scalar.

$A \xleftrightarrow[\text{of E.R.O}]{\text{any \#}} B$

A, B "row equivalent"

Q. . Is system consistent?

- is a solution unique?

Echelon form

Def: A rectangular matrix is in row echelon form

1. All non-zero rows are above any rows of all zeroes
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it
3. All entries in a column below a leading entry are zero.

Reduced echelon form

4. leading entry in each non-zero row is 1
5. Each leading 1 is the only nonzero entry in its column.

[each matrix is row equivalent to one and only one reduced echelon matrix]

e.g.
$$\begin{bmatrix} \boxed{2} & -3 & 2 & 1 \\ 0 & \boxed{1} & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix}$$
echelon form

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 29 \\ 0 & \boxed{1} & 0 & 16 \\ 0 & 0 & \boxed{1} & 3 \end{bmatrix}$$
reduced echelon form

Echelon form

Def: A pivot position in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A

A pivot column is a column of A that contains a pivot position.

Row reduction algorithm

EXAMPLE 3 Apply elementary row operations to transform the following matrix first into echelon form and then into reduced echelon form:

$$A = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

SOLUTION

STEP 1

Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.

pivot pos.

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Pivot column

Row reduction algorithm

STEP 2

Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.

Interchange rows 1 and 3. (We could have interchanged rows 1 and 2 instead.)

$$\begin{array}{l} \text{Pivot} \\ \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \end{array}$$

Row reduction algorithm

STEP 3

Use row replacement operations to create zeros in all positions below the pivot.

As a preliminary step, we could divide the top row by the pivot, 3. But with two 3's in column 1, it is just as easy to add -1 times row 1 to row 2.

$$\begin{array}{c}
\text{Pivot} \\
\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}
\end{array}$$

Row reduction algorithm

STEP 4

Cover (or ignore) the row containing the pivot position and cover all rows, if any, above it. Apply steps 1–3 to the submatrix that remains. Repeat the process until there are no more nonzero rows to modify.

With row 1 covered, step 1 shows that column 2 is the next pivot column; for step 2, select as a pivot the "top" entry in that column.

Pivot

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

New pivot column

For step 3, we could insert an optional step of dividing the "top" row of the submatrix by the pivot, 2. Instead, we add $-3/2$ times the "top" row to the row below. This produces

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

pivot

done!

Row reduction algorithm (reduced echelon form)

STEP 5

Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.

The rightmost pivot is in row 3. Create zeros above it, adding suitable multiples of row 3 to rows 2 and 1.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 2 & -4 & 4 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \begin{array}{l} \leftarrow \text{Row 1} + (-6) \cdot \text{row 3} \\ \leftarrow \text{Row 2} + (-2) \cdot \text{row 3} \\ \\ \end{array}$$

The next pivot is in row 2. Scale this row, dividing by the pivot.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \begin{array}{l} \\ \leftarrow \text{Row scaled by } \frac{1}{2} \\ \\ \end{array}$$

Create a zero in column 2 by adding 9 times row 2 to row 1.

$$1/3 \begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \begin{array}{l} \leftarrow \text{Row 1} + (9) \cdot \text{row 2} \\ \\ \\ \end{array}$$

Row reduction algorithm (reduced echelon form)

Finally, scale row 1, dividing by the pivot, 3.

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \quad \longleftarrow \text{Row scaled by } \tfrac{1}{3}$$

This is the $\boxed{\text{reduced echelon form of the original matrix.}}$

steps 1-4 : forward phase
step 5 is backward phase

$\boxed{\text{Thm:}}$ A linear system is consistent iff the rightmost column of the augmented matrix is not a pivot column.

if $[0 \ldots 0 \ \vec{b}]$, $\vec{b} \neq \vec{0}$ then inconsistent!

Row and column spaces

Def: The column space of a matrix A is the set $\text{Col}\,A$ of all linear combinations of columns of A.

$A = [\vec{a}_1 \ldots \vec{a}_n]$ $\vec{a}_i \in \mathbb{R}^m$, $\text{col}A = \text{span}\{\vec{a}_1, \ldots, \vec{a}_n\}$

eg: $A = \begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$

Q: is $\vec{b} \in \text{Col}A$?

rephrase: can $\vec{b}$ be written as $A\vec{x}$ for some $\vec{x}$?

$A\vec{x} = x_1\vec{a}_1 + \ldots + x_n\vec{a}_n$

rephrase': $A\vec{x} = \vec{b}$ consistent?

$\begin{bmatrix} 1 & -3 & -4 & 3 \\ -4 & 6 & -2 & 3 \\ -3 & 7 & 6 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & -2 & -6 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

this system is consistent

$\Rightarrow \vec{b} \in \text{col}A$

Column space, rank, nullspace

- Null space.

Def null space of A is the set Nul A of all solutions of the homogeneous equation $A\vec{x}=\vec{0}$

$A=[\vec{a}_1 \ldots \vec{a}_n]$, $A\vec{x}=\vec{0}$ $\vec{x}\in\mathbb{R}^n$, $\mathrm{Nul}\,A \subseteq \mathbb{R}^n$

In fact, Nul A is subspace of $\mathbb{R}^n$

(ex. verify)

- Rank of matrix A, rank A, is dimension of the column space of A.

pivot colums of A form a basis for col A $\Rightarrow$ rank A = # of pivot columns

(row equivalence $\Leftrightarrow$ same column space) ★★★

Thm. Rank thm if A has n columns, $\boxed{\mathrm{rank}\,A+\dim \mathrm{Nul}\,A = n}$

Column space, rank, nullspace

$$A = \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & 2 & 4 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & -6 & 4 & 14 & -20 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot columns

rank $A = 3$

dim nul $A = 2$

Invertibility and elementary operations

A $n \times n$ matrix. Tfae

1) $\text{col} A = \mathbb{R}^n$

2) $\dim \text{Col} A = n$

3) $\text{rank} A = n$

4) $\text{Nul} A = \{\vec{0}\}$

5) $\dim \text{Nul} A = 0$