Linear Algebra

CSCI 2820

Lecture 14

Prof. Alexandra Kolla

Alexandra.Kolla@Colorado.edu
ECES 122

Today

- Matrices, contd.
- Examples and exercises

Matrix vector multiplication

A $m \times n$ matrix

$\vec{x}$ n-vector

def: Matrix-vector product $\underbrace{\vec{y}}_{m\text{-vector}} = A\underbrace{\vec{x}}_{n\text{-vector}}$

$$y_i = \sum_{k=1}^{n} A_{ik} x_k = A_{i1}x_1 + \cdots + A_{in}x_n, \quad i = 1, \ldots, m$$

$$= \langle b_i, x \rangle$$

eg:

$$\begin{bmatrix} 0 & 2 & -1 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0\cdot 2 + 2\cdot 1 + (-1)(-1) \\ (-2)\cdot 2 + (1)(1) + (1)(-1) \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

Matrix vector multiplication

$y_i = \langle b_i, x \rangle$, b_i^T is row i of A

$(b_i^T x)$

k-th column vector of A $\vec{a}_k$:

$$\boxed{\vec{y} = A\vec{x} = x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n}$$

verify:

$$\boxed{\begin{aligned} y_i &= x_1 a_1(i) + \dots + x_n a_n(i) \\ &= x_1 A_{i1} + \dots + x_n A_{in} \end{aligned}}$$

Matrix vector multiplication

examples A $m \times n$ matrix, $\vec{x}$ n-vector

(1) zero matrix. $A = 0$: $A\vec{x} = 0 \ \forall \vec{x}$

(2) Identity matrix $(n \times n)$. $I\vec{x} = \vec{x}$

(3) Picking out columns and rows : $A\vec{e}_j = \vec{a}_j$

$$\begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \\ A_{m1} & \cdots & A_{mn} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} A_{11} \\ \vdots \\ A_{m1} \end{bmatrix} = \vec{a}_1 \qquad (A^T e_i)^T = \vec{b}_i$$

(4) Averaging columns or rows: $A\frac{\vec{1}}{n} = \begin{bmatrix} \ \end{bmatrix} \to y_i = \frac{1}{n}\sum_{k=1}^{n} A_{ik}$

$$A^T\vec{1} = \begin{bmatrix} \ \end{bmatrix} \to y_i = \sum_{k=1}^{n} A_{ki} \qquad \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

Matrix vector multiplication

(5) Difference matrix

$(n-1) \times n$

$$D = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ & & \ddots & \ddots & & \\ 0 & 0 & 0 & \cdots & -1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}, \quad D\vec{x} = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ \vdots \\ x_n - x_{n-1} \end{bmatrix}$$

y_1

(6) Running sum matrix

$$S = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & 0 & \\ 1 & 1 & 1 & & \\ 1 & 1 & 1 & 1 & \\ \vdots & & & & \ddots \\ 1 & 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} x_1 \\ x_1 + x_2 \\ x_1 + x_2 + x_3 \\ \vdots \\ x_1 + \cdots + x_n \end{bmatrix}$$

Application examples:

Polynomial evaluation at multiple points

$$p(t) = c_1 + c_2 t + \cdots + c_{n-1} t^{n-2} + c_n t^{n-1}$$

value of $p(t)$ at $t_1, \ldots, t_m$

$$\vec{y} = \begin{bmatrix} p(t_1) \\ \vdots \\ p(t_m) \end{bmatrix}, \quad y_i = p(t_i)$$

$$A = \begin{bmatrix} 1 & t_1 & \cdots & t_1^{n-2} & t_1^{n-1} \\ 1 & t_2 & & t_2^{n-2} & t_2^{n-1} \\ \vdots & & & & \\ 1 & t_m & & t_m^{n-2} & t_m^{n-1} \end{bmatrix}, \quad \text{then} \quad \boxed{A\vec{c} = \vec{y}}$$

$$\vec{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

- Inner product : $\vec{a}, \vec{b}$, $\underbrace{a^T}_{\substack{1 \times n \\ \text{matrix}}} \underbrace{\vec{b}}_{n\text{-vector}} = \vec{y}$, $\vec{y}$ is 1-dim, scalar!

- Linear dependence of columns

$A = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}$, $\vec{a}_i$'s ar L.D if

$A\vec{x} = 0$ for some $\vec{x} \neq 0$

and L.I if $A\vec{x} = 0$ implies $x = 0$

Pf: $A\vec{x} = x_1\vec{a}_1 + \cdots + x_n\vec{a}_n = 0$ for some $\vec{x} \neq 0 \Rightarrow$ L.D

- If $\{\vec{a}_i\}$ basis ; for any n-vector $\vec{b}$, there is a unique n-vector $\vec{x}$: $A\vec{x} = \vec{b}$

(since $\{\vec{a}_i\}$ basis : $\vec{b} = \underbrace{x_1\vec{a}_1 + \cdots + x_n\vec{a}_n}_{A\vec{x}}$)

Properties of matrix-vector mult.

- $A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v}$ (distributes)
- $(aA)\vec{u} = a\cdot(A\vec{u}) = aA\vec{u}$
- $(A \pm B)\vec{u} = A\vec{u} \pm B\vec{u}$

Exercises

Suppose A is the adjacency matrix of a directed graph. The reversed graph is obtained by reversing the directions of all the edges of the original graph. What is the adjacency matrix of the reversed graph? (Express your answer in terms of A.) $\hookrightarrow B$

$$A_{ij} = \begin{cases} 1 & \text{if } (i,j) \text{ edge (or } (i,j) \in R) \\ 0 & \text{ow} \end{cases}$$

(square, $n \times n$)

answer: $(i,j) \in R$

$(j,i) \in R^{rev}$

$A_{ij} = 1 \Rightarrow B_{ji} = 1 \Rightarrow B = A^T$

$A_{ij} = 0 \Rightarrow B_{ji} = 0$

Exercises

Matrix-vector multiplication. For each of the following matrices, describe in words how x and $y = Ax$ are related. In each case x and y are n-vectors, with $n = 3k$.

(a) $A = \begin{bmatrix} 0 & 0 & I_k \\ 0 & I_k & 0 \\ I_k & 0 & 0 \end{bmatrix}$.

$\vec{y}$ is a $3k$-dim column vector $\begin{bmatrix} \; \end{bmatrix}_{3k \times 1}$

(b) $A = \begin{bmatrix} E & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & E \end{bmatrix}$, where E is the $k \times k$ matrix with all entries $1/k$.

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_{3k} \end{bmatrix}, = \begin{bmatrix} x_1 \\ \vdots \\ x_k \\ x_{k+1} \\ \vdots \\ x_{2k} \\ x_{2k+1} \\ \vdots \\ x_{3k} \end{bmatrix} = \begin{bmatrix} \vec{x}_{(1)} \\ \vec{x}_{(2)} \\ \vec{x}_{(3)} \end{bmatrix}$$

$3k \times 1$ matrix

$$\vec{x}_{(1)} = \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix}, \quad \vec{x}_{(2)} = \begin{bmatrix} x_{k+1} \\ \vdots \\ x_{2k} \end{bmatrix}, \quad \vec{x}_{(3)} = \begin{bmatrix} x_{2k+1} \\ \vdots \\ x_{3k} \end{bmatrix}$$

Exercises

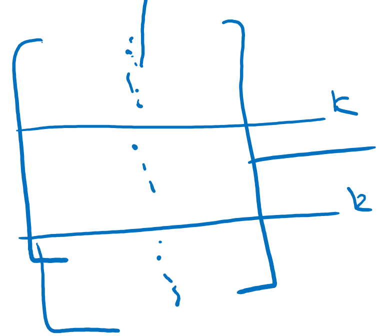
$$\underbrace{\begin{bmatrix} 0 & 0 & I_k \\ 0 & I_k & 0 \\ I_k & 0 & 0\end{bmatrix}}_{A} \begin{bmatrix}\vec{x}(1)\\ \vec{x}(2)\\ \vec{x}(3)\end{bmatrix} = \begin{bmatrix} 0\cdot\vec{x}(1) + 0\cdot\vec{x}(2) + I_k \vec{x}(3) \\ 0\cdot\vec{x}(1)+I_k\vec{x}(2)+0\cdot\vec{x}(3) \\ I_k\vec{x}(1)+0\cdot\vec{x}(2)+0\cdot\vec{x}(3)\end{bmatrix} =$$

$$= \begin{bmatrix}\vec{x}(3)\\ \vec{x}(2)\\ \vec{x}(1)\end{bmatrix} = \begin{bmatrix} x_{2k+1}\\ \vdots \\ x_{3k} \\ x_{k+1} \\ \vdots \\ x_{2k} \\ x_1 \\ \vdots \\ x_k\end{bmatrix}$$

(b)

$$\begin{bmatrix} E&0&0\\0&E&0\\0&0&E\end{bmatrix}\begin{bmatrix}\vec x(1)\\ \vec x(2)\\ \vec x(3)\end{bmatrix}$$

$$= \begin{bmatrix}E\vec x(1)\\E\vec x(2)\\E\vec x(3)\end{bmatrix} =$$



$$\frac{x_1+\cdots+x_k}{k}, \quad \frac{x_{k+1}\cdots x_{2k}}{k}, \quad \frac{x_{2k+1}+\cdots+x_{3k}}{k}$$

Exercises

Let A and B be two $m \times n$ matrices. Under each of the assumptions below, determine whether $A = B$ must always hold, or whether $A = B$ holds only sometimes.

(a) Suppose $Ax = Bx$ holds for all n-vectors x.

(b) Suppose $Ax = Bx$ for some nonzero n-vector x.

(a) $\underline{\vec{x} = \vec{e}_j}$ always

$A\vec{e}_j = B\vec{e}_j \Rightarrow \underline{\vec{a}_j = \vec{b}_j} \Rightarrow A_{ij} = B_{ij} \ \forall i$

do this for every $\vec{e}_j$: $A_{ij} = B_{ij} \ \forall i, j$

(b) only sometimes

$\vec{x} = \vec{e}_1$: $\vec{a}_1 = \vec{b}_1 = \vec{v}$

$A = [\vec{v} \ \vec{a}_2 \ldots \vec{a}_n]$ $B = [\vec{v}, \vec{b}_2, \ldots, \vec{b}_n]$

$\vec{a}_i \neq \vec{b}_i$

Exercises

Norm of matrix-vector product. Suppose A is an $m \times n$ matrix and x is an n-vector. A famous inequality relates $\|x\|$, $\|A\|$, and $\|Ax\|$:

$$\|Ax\| \leq \|A\|\|x\|.$$

The left-hand side is the (vector) norm of the matrix-vector product; the right-hand side is the (scalar) product of the matrix and vector norms. Show this inequality. *Hints.* Let a_i^T be the ith row of A. Use the Cauchy–Schwarz inequality to get $(a_i^T x)^2 \leq \|a_i\|^2\|x\|^2$. Then add the resulting m inequalities.

Exercises