Linear Algebra

CSCI 2820

Lecture 13

Prof. Alexandra Kolla

Alexandra.Kolla@Colorado.edu
ECES 122

Today

- Matrices

Matrices

$$A = \begin{bmatrix} 0 & 1 & -2.3 & 0.1 \\ 1.3 & 4 & -0.1 & 0 \\ 4.1 & -1 & 0 & 1.7 \end{bmatrix}$$

size or dim

3×4

$m \times n$ matrix

has m rows, n columns

elements = values in this array

i,j element A_{ij} ($A_{i,j}$ or $A[i,j]$)

↳ row and column indices

$A_{11} = 0$

set of real $m \times n$ matrices is denoted $\mathbb{R}^{m \times n}$

Matrices

Matrices

- square matrices : equal rows & columns ($n \times n$)
- tall : $m \times n$, $m > n$
- wide : $m \times n$, $n > m$
- vectors as matrices : n-vector is $n \times 1$ matrix (column vector)

  row vector = $1 \times n$ matrix
- 1×1 matrix is just a scalar.
- Columns and rows of matrix ($m \times n$ matrix A)

  j-th column is given by vector $a_j = \begin{bmatrix} A_{1j} \\ A_{2j} \\ \vdots \\ A_{mj} \end{bmatrix}$ $j = 1, \ldots, n$

  "column vector"

  "row vector" $b_i = [A_{i1}, \ldots A_{in}]$ $i = 1, \ldots, m$

Matrices

2×3 matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

$b_1 = [1 \quad 2 \quad 3]$

$a_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

Block matrices $A = \begin{bmatrix} B & C \\ D & E \end{bmatrix}$ B, C, D, E blocks or submatrices.

C is the 1,2 block of A.

for block vectors : $\vec{v} = \begin{pmatrix} \vec{v}_1 \\ \vdots \\ \vec{v}_n \end{pmatrix}$ any dim of the $\vec{v}_i$ is ok

but for block matrices : - Matrices in the same (block) row must have same number of rows ($\overset{D,E}{B}$, C same # rows)

- Matrices in the same column must have same # of columns (C, E same # colums, B, D same # colums)

Examples

$$A = \begin{bmatrix} B & C \\ D & E \end{bmatrix} \qquad B = [0\ 2\ 3] \qquad D = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 5 \end{bmatrix}$$

$$C = [-1] \qquad E = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 2 & 3 & -1 \\ 2 & 2 & 1 & 4 \\ 1 & 3 & 5 & 4 \end{bmatrix} \rightsquigarrow A_{2:3,1:2} = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$$

$A_{m \times n}$. p, q, r, s : $1 \le p \le q \le m$, $1 \le r \le s \le n$

a_i are column vectors

b_j are row vectors

$$A_{p:q,r:s} = \begin{bmatrix} A_{pr} & A_{p,r+1} & \cdots & A_{ps} \\ A_{p+1,r} & A_{p+1,r+1} & \cdots & A_{p+1,s} \\ \vdots & \vdots & & \\ A_{qr} & A_{q,r+1} & \cdots & A_{qs} \end{bmatrix}$$

$A = [a_1 \ \ldots \ a_n]$

or $A = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$

- why matrices? Images, Rainfall data: A_{42} : rain at location 4 day 2

examples

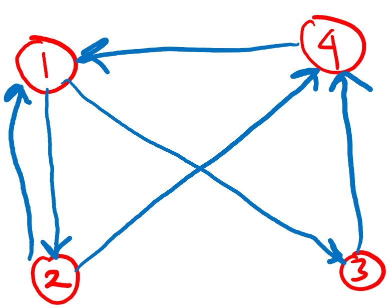
- Matrix representation of a graph.

suppose we have n objects $\{1, \ldots, n\}$

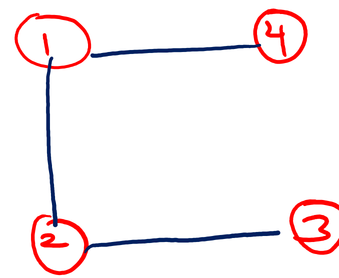
relation R, $(i,j) \in R$ choice i is preferred to choice j.

- typically drawn as graph. (directed)

$$R = \{(1,2), (1,3), (2,1), (2,4), (3,4), (4,1)\}$$



(*)



$R_s = \{(1,2), (2,1)$
$(2,3)(3,2)$
$(4,1)(1,4)\}$

Relations, or graphs are represented as matrices (n×n dim)
"adjacency matrices"

$$A_{ij} = \begin{cases} 1 & \text{if } (i,j) \in R \\ 0 & \text{o/w} \end{cases}$$

eg for (*)

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

examples

Zero and Identity

zero : $A_{ij} = 0 \ \forall i,j$ $\quad O_{m \times n}$

identity : $I_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$ $\quad I_{m \times n}$

$$I_{2\times2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_{4\times4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$I = [e_1 \ e_2 \ \dots e_n]$$

- sparse matrices $\quad nnz(A) = \#$ of non-zero entries.
- diagonal matrices : $\begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1.2 \end{bmatrix}$

  $A_{ij} = 0 \quad i \neq j$

  $\mathrm{diag}(a_1, \dots, a_n) \qquad A_{11} = a_1, \dots, A_{nn} = a_n$

Other matrices

Triangular matrices

upper triangular : $A_{ij} = 0$ for $i > j$

lower -"- : $A_{ij} = 0$ for $i < j$

eg. $\begin{bmatrix} 1 & -1 & 0.7 \\ 0 & 1.2 & -1.1 \\ 0 & 0 & 3.2 \end{bmatrix}$ $\begin{bmatrix} -0.6 & 0 \\ -0.3 & 3.5 \end{bmatrix}$

Transpose, addition, norm

- A is $m \times n$, transpose A^T : $(A^T)_{ij} = A_{ji}$

$$\begin{bmatrix} 0 & 4 \\ 7 & 0 \\ 3 & 1 \end{bmatrix}^T = \begin{bmatrix} 0 & 7 & 3 \\ 4 & 0 & 1 \end{bmatrix}, \quad (A^T)^T = A$$

- Transpose of block matrix

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^T = \begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix} \quad \text{(ex.)}$$

- symmetric matrix : ① square $n \times n$
  ② $A = A^T$, $A_{ij} = A_{ji}$

Transpose, addition, norm

- matrix addition

$$\begin{bmatrix} 0 & 4 \\ 7 & 0 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 9 & 3 \\ 3 & 5 \end{bmatrix}$$

- commutativity : $A+B = B+A$
- associativity : $(A+B)+C = A+(B+C) = A+B+C$
- zero : $A+0 = 0+A = A$
- transpose of sum $(A+B)^T = A^T + B^T$ (ex)
- scalar-matrix mult

$$(-2)\begin{bmatrix} 1 & 6 \\ 9 & 3 \\ 6 & 0 \end{bmatrix} = \begin{bmatrix} -2 & -12 \\ -18 & -6 \\ -12 & 0 \end{bmatrix}$$

$(\beta A)^T = \beta A^T$

$(\beta+\gamma)A = \beta A + \gamma A$

$(\beta\gamma)A = \beta(\gamma A)$

Transpose, addition, norm

Norm: $m \times n$ matrices can be seen as $m \cdot n$ vectors, $\|A\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}^2}$ (agrees when A is just row or column vector)

- satisfies all norm properties.
  - $\|A\| \geq 0$
  - $\|A\| = 0$ iff $A = 0$
  - $\|\gamma A\| = |\gamma| \|A\|$
  - $\|A+B\| \leq \|A\| + \|B\|$
- distance between A, B: $\|A-B\|$
- $\|A\| = \|A^T\|$
- $\|A\|^2 = \|a_1\|^2 + \dots + \|a_n\|^2$

Transpose, addition, norm