Linear Algebra

CSCI 2820

Lecture 11

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ECES 122

Today

- Coordinate systems
- Projections
- Gram-Schmidt re-explained

Unique Representation

The Unique Representation Theorem

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, \ldots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n \tag{1}$$

Coordinate Systems

DEFINITION

Suppose $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is a basis for V and $\mathbf{x}$ is in V. The **coordinates of x relative to the basis $\mathcal{B}$** (or the **$\mathcal{B}$-coordinates of x**) are the weights $c_1, \dots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$.

eg 1. $\vec{x} = \begin{bmatrix} 1 \\ 6 \end{bmatrix} = 1 \cdot \vec{e}_1 + 6 \cdot \vec{e}_2$ "standard basis"

eg 2. $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ $\vec{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix} = -2\vec{b}_1 + 3 \cdot \vec{b}_2 = -2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

Coordinate Systems

$\vec{x} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$ $[\vec{x}]_B = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

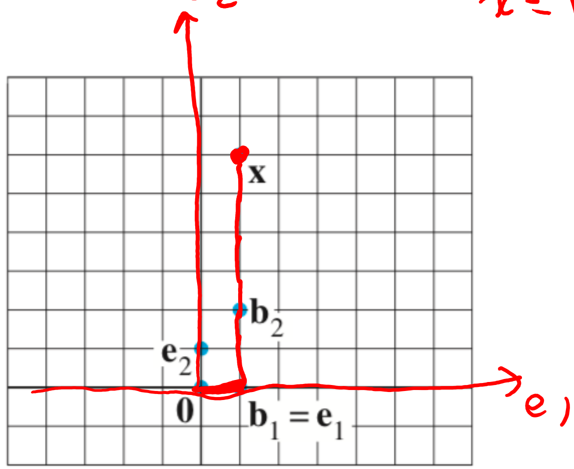

FIGURE 1 Standard graph paper.

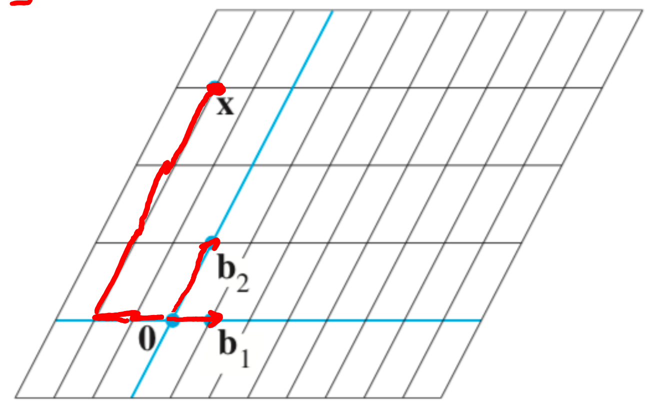

FIGURE 2 $\mathcal{B}$-graph paper.

Dimension of Vector Space review

If a vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then any set in V containing more than n vectors must be linearly dependent.

If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.

If V is spanned by a finite set, then V is said to be **finite-dimensional**, and the **dimension** of V, written as dim V, is the number of vectors in a basis for V. The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be zero. If V is not spanned by a finite set, then V is said to be **infinite-dimensional**.

Dimension of Vector Space review

Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H. Also, H is finite-dimensional and

$$\dim H \leq \dim V$$

The Basis Theorem

Let V be a p-dimensional vector space, $p \geq 1$. Any linearly independent set of exactly p elements in V is automatically a basis for V. Any set of exactly p elements that spans V is automatically a basis for V.

Dimension of Vector Space review

Find the dimension of the subspace H of $\mathbb{R}^2$ spanned by

$$\begin{bmatrix} 2 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 10 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \end{bmatrix}.$$

$\vec{x}_1 \quad \vec{x}_2 \quad \vec{x}_3$

$\vec{x}_2 = -2 \cdot \vec{x}_1$

$\{\vec{x}_1, \vec{x}_3\}$ basis for H

linearly indep

$\dim H = 2$ $(H = \mathbb{R}^2)$

Dimension of Vector Space review

True or False?

a. If there exists a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ that spans V, then $\dim V \leq p$. T

b. If there exists a linearly independent set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in V, then $\dim V \geq p$. T

does not imply L.I.

eg $V = \mathbb{R}^3$

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \subseteq \mathbb{R}^3$ ($\vec{v}_1$, $\vec{v}_2$)

c. If $\dim V = p$, then there exists a spanning set of $p + 1$ vectors in V. T

$V = \mathbb{R}^3$ $\quad \text{span}\{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_1 + 5\vec{e}_2 + 3\vec{e}_3\} = \mathbb{R}^3$

$\dim(V) = 3$ $\quad \text{span}\{\vec{e}_1, \vec{e}_2, \vec{e}_3\} = \mathbb{R}^3$

if $\dim V = p+2$ then "c" would be F

Dimension of Vector Space review

True or False?

F a. If there exists a linearly dependent set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in V, then $\dim V \leq p$. $V = \mathbb{R}^3$. $|\{\vec{e}_1, 3\vec{e}_1\}| = 2$ but $\dim V = 3$

T b. If every set of p elements in V fails to span V, then $\dim V > p$.

F c. If $p \geq 2$ and $\dim V = p$, then every set of $p - 1$ nonzero vectors is linearly independent.

Orthogonality, Inner Product Review