Linear Algebra

CSCI 2820

Lecture 11

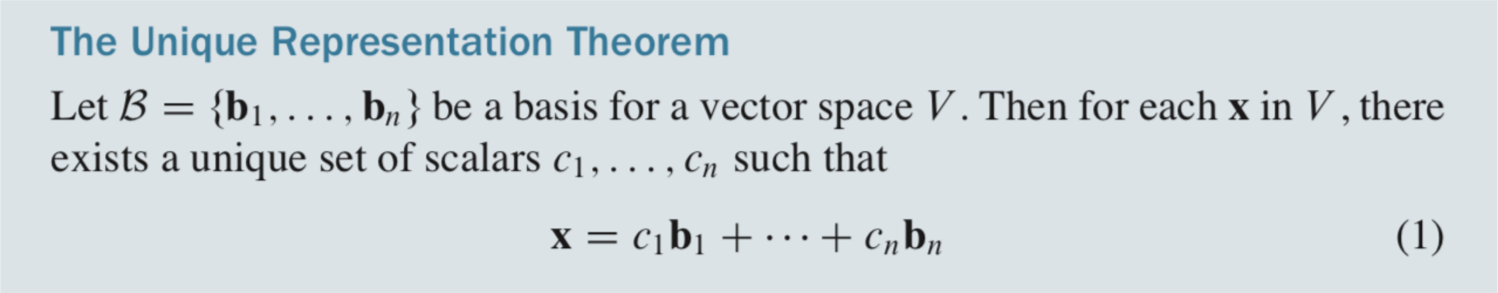
Prof. Alexandra Kolla

Alexandra.Kolla@Colorado.edu
ECES 122

Today

- Coordinate systems
- Projections
- Gram-Schmidt re-explained

Unique Representation



The Unique Representation Theorem

Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for a vector space V. Then for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, \dots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n \tag{1}$$

Coordinate Systems

DEFINITION

Suppose $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is a basis for V and $\mathbf{x}$ is in V. The **coordinates of $\mathbf{x}$ relative to the basis $\mathcal{B}$** (or the **$\mathcal{B}$-coordinates of $\mathbf{x}$**) are the weights $c_1, \dots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$.

Coordinate Systems

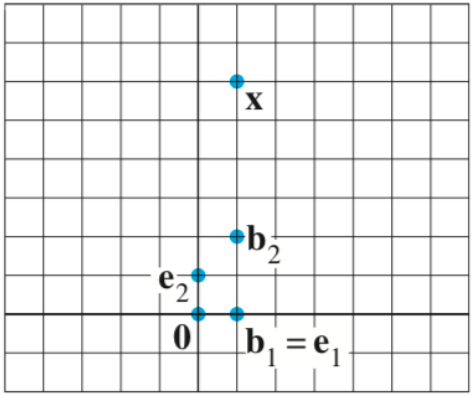


FIGURE 1 Standard graph paper.

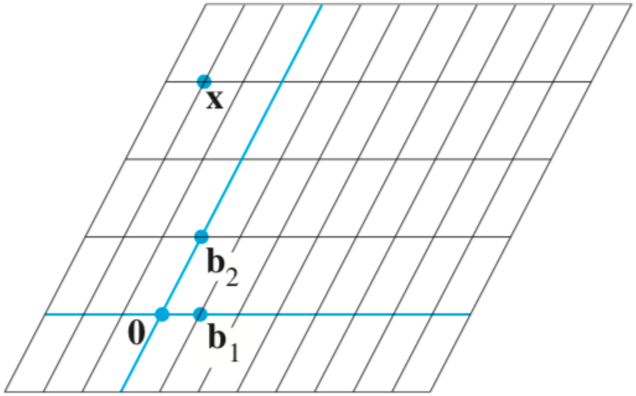
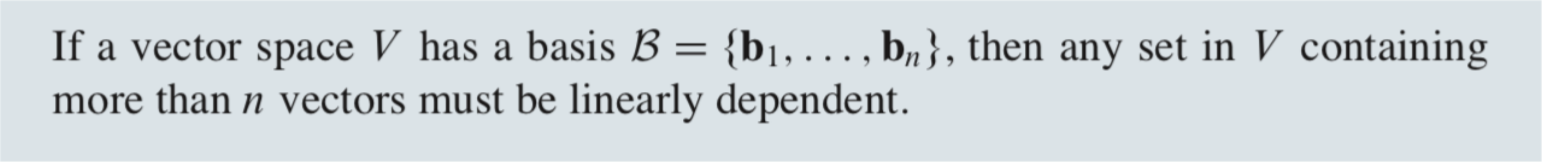
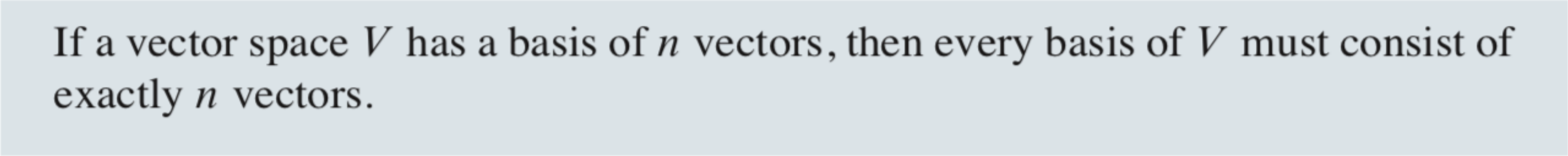


FIGURE 2 $\mathcal{B}$-graph paper.

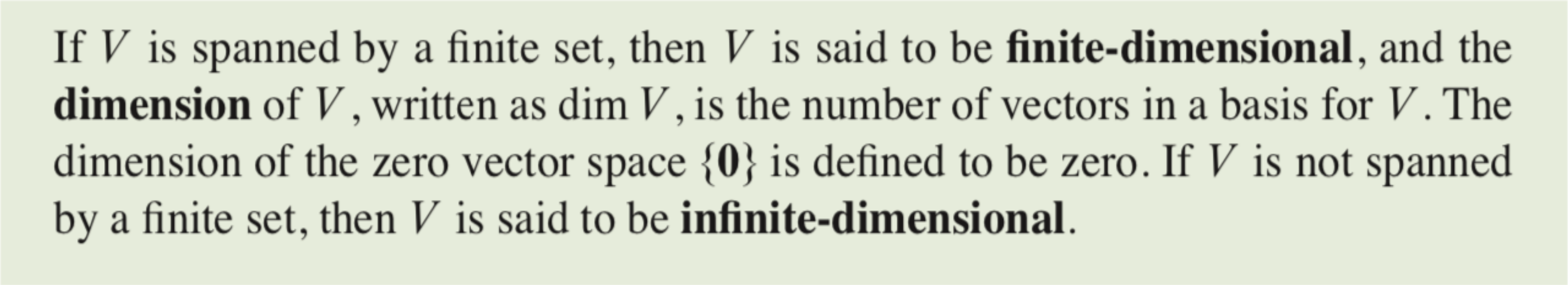
Dimension of Vector Space review



If a vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$, then any set in V containing more than n vectors must be linearly dependent.

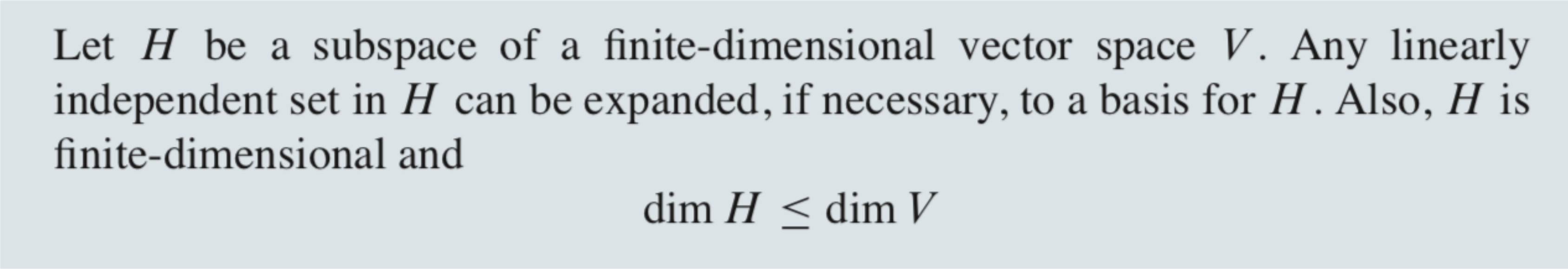


If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.



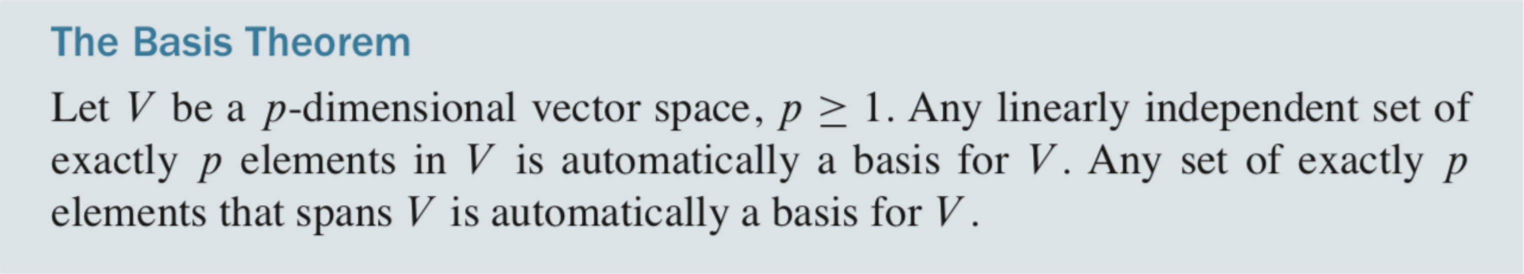
If V is spanned by a finite set, then V is said to be **finite-dimensional**, and the **dimension** of V, written as $\dim V$, is the number of vectors in a basis for V. The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be zero. If V is not spanned by a finite set, then V is said to be **infinite-dimensional**.

Dimension of Vector Space review



Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H. Also, H is finite-dimensional and

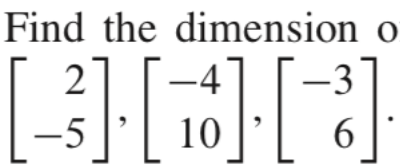
$$\dim H \leq \dim V$$



The Basis Theorem

Let V be a p-dimensional vector space, $p \geq 1$. Any linearly independent set of exactly p elements in V is automatically a basis for V. Any set of exactly p elements that spans V is automatically a basis for V.

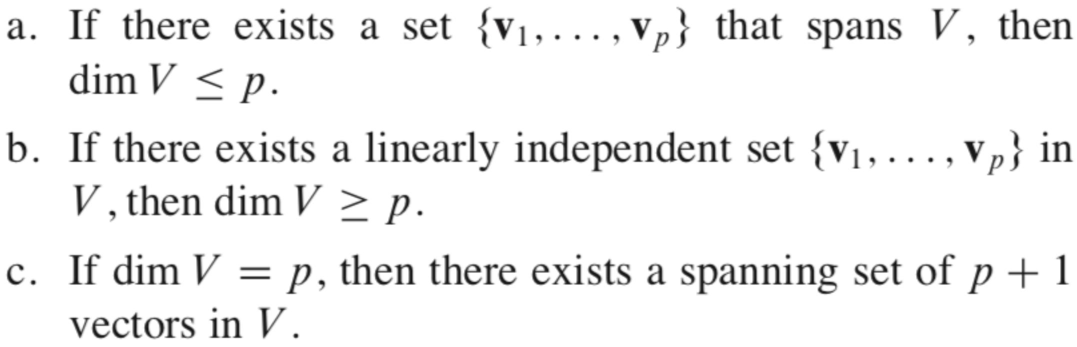
Dimension of Vector Space review



Find the dimension of the subspace H of $\mathbb{R}^2$ spanned by $\begin{bmatrix} 2 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 10 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \end{bmatrix}$.

Dimension of Vector Space review

True or False?



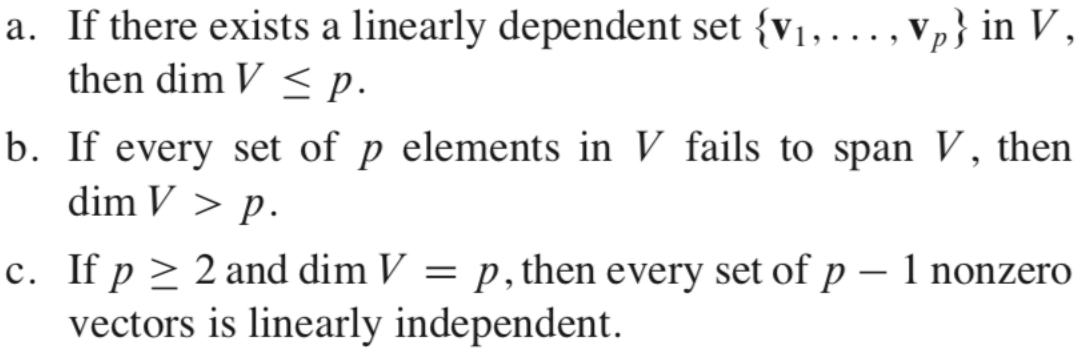
a. If there exists a set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ that spans V, then $\dim V \leq p$.

b. If there exists a linearly independent set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in V, then $\dim V \geq p$.

c. If $\dim V = p$, then there exists a spanning set of $p + 1$ vectors in V.

Dimension of Vector Space review

True or False?



a. If there exists a linearly dependent set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in V, then $\dim V \leq p$.

b. If every set of p elements in V fails to span V, then $\dim V > p$.

c. If $p \geq 2$ and $\dim V = p$, then every set of $p - 1$ nonzero vectors is linearly independent.

Orthogonality, Inner Product, norm, and distance review

Orthogonal Complements

$\vec{z}$ is orthogonal to every vector in a subspace W of $\mathbb{R}^n$, we say $\vec{z}$ is orthogonal to W.

Def: set of all vectors $\vec{z}$ that are orthogonal to W is called orthogonal complement of W

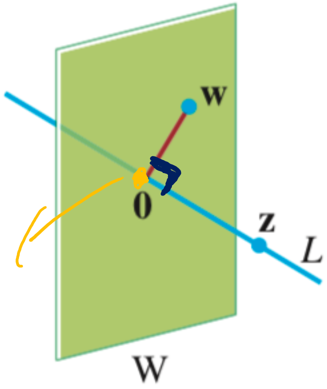
$(W^{\perp})$

eg.

Orthogonal Complements

W = plane through the origin (subspace of $\mathbb{R}^3$)

L: line throug the origin (subspace of $\mathbb{R}^3$)



$\begin{pmatrix}0\\0\\0\end{pmatrix} = \vec{0}$

$\forall \vec{z} \in L : \quad \vec{z} \cdot \vec{w} = 0$

$\vec{w} \in W$

$L = W^{\perp}$, $W = L^{\perp}$

1. A vector **x** is in $W^{\perp}$ if and only if **x** is orthogonal to every vector in a set that spans W.
2. $W^{\perp}$ is a subspace of $\mathbb{R}^n$.

Orthogonality

Show that if **x** is in both W and $W^{\perp}$, then $\mathbf{x} = \mathbf{0}$.

$x \in W^{\perp}$ implies that $\forall \vec{w} \in W$ $\langle w, x \rangle = 0$

$x \in W$ $\hookrightarrow \langle x, x \rangle = 0 \Rightarrow \|x\|^2 = 0 \Rightarrow \vec{x} = \vec{0}$

Orthogonality

True or False?

a. $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$. T

b. For any scalar c, $\mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$. T

c. If the distance from **u** to **v** equals the distance from **u** to $-\mathbf{v}$, then **u** and **v** are orthogonal. T

$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\langle u,v\rangle = \|u\|^2 + \|v\|^2 + 2\langle u,v\rangle$$

$$\Rightarrow \langle u,v\rangle = 0$$

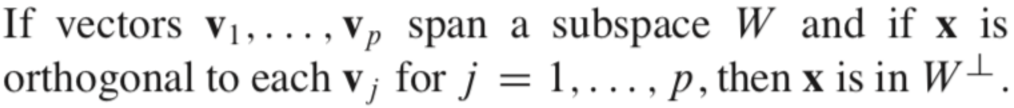
If vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ span a subspace W and if $\mathbf{x}$ is orthogonal to each $\mathbf{v}_j$ for $j = 1, \ldots, p$, then $\mathbf{x}$ is in $W^{\perp}$. T

$$\vec{w} = c_1\vec{v}_1 + \cdots + c_p\vec{v}_p$$

$$\langle x, w\rangle = c_1\underbrace{\langle x, v_1\rangle}_{0} + \cdots + c_p\underbrace{\langle x, v_p\rangle}_{0} = 0$$

Orthogonality

True or False?



If vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ span a subspace W and if $\mathbf{x}$ is orthogonal to each $\mathbf{v}_j$ for $j = 1, \ldots, p$, then $\mathbf{x}$ is in $W^{\perp}$.

Orthogonal sets

$\{\vec{v}_1, \dots, \vec{v}_k\}$ is orthogonal set if

$\langle v_i, v_j \rangle = 0 \quad \forall\ i \neq j$

Orthogonal sets

If $S = \{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is linearly independent and hence is a basis for the subspace spanned by S.

Assume $c_1\vec{u}_1 + \dots + c_p\vec{u}_p = \vec{0}$

$$\langle c_1\vec{u}_1 + \dots + c_p\vec{u}_p, \vec{u}_1\rangle = \vec{0} \Rightarrow$$

$$\underbrace{c_1\langle u_1, u_1\rangle} + c_2\underset{0}{\cancel{\langle u_2, u_1\rangle}} + \dots + c_p\underset{0}{\cancel{\langle u_p, u_1\rangle}} = 0 \Rightarrow$$

$$c_1\|\vec{u}_1\|^2 = 0 \Rightarrow c_1 = 0$$

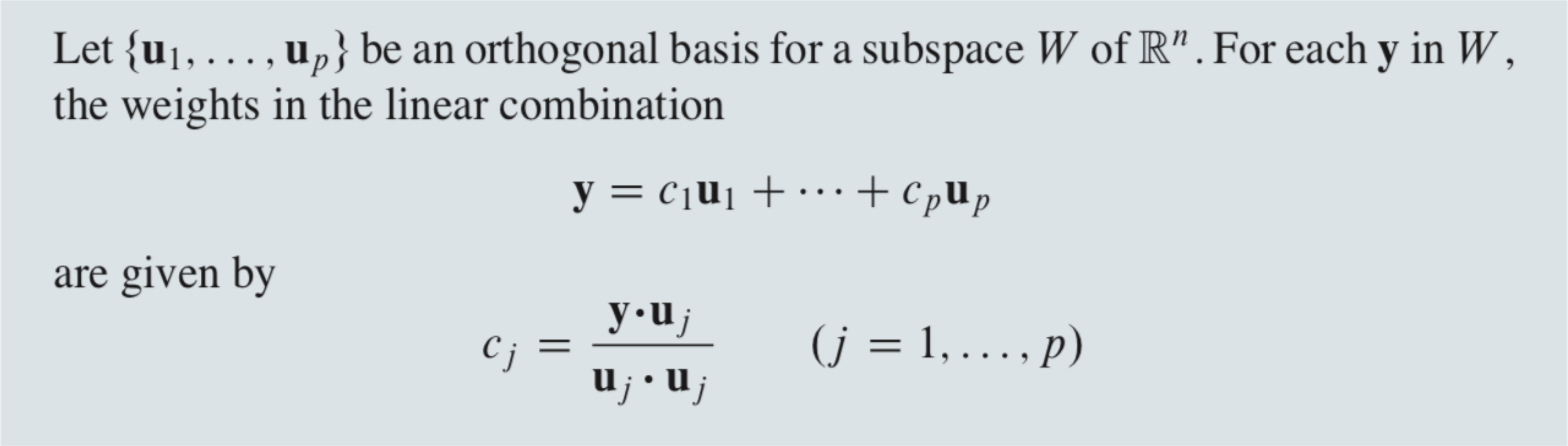
repeat same argument for

$$\vec{u}_2, \vec{u}_3, \dots, \vec{u}_p$$

$$\Downarrow \qquad \Downarrow \qquad \Downarrow$$

$$c_2 = 0 \qquad c_3 = 0 \qquad c_p = 0$$

Orthogonal sets



Let $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. For each $\mathbf{y}$ in W, the weights in the linear combination

$$\mathbf{y} = c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p$$

are given by

$$c_j = \frac{\mathbf{y}\cdot\mathbf{u}_j}{\mathbf{u}_j\cdot\mathbf{u}_j} \qquad (j = 1, \dots, p)$$

Orthogonal Projections

$\vec{u} \in \mathbb{R}^n$

we want to decompose $\vec{y} \in \mathbb{R}^n$ into sum of two vectors, one a multiple of $\vec{u}$, the other orthogonal to $\vec{u}$.

$$\vec{y} = a\vec{u} + \vec{z}, \qquad \vec{z} \perp \vec{u}$$

$$\vec{z} = \vec{y} - a\vec{u} \perp \vec{u}$$

$$\text{if } \langle \vec{y} - a\vec{u}, \vec{u} \rangle = 0$$

$$\Rightarrow \langle \vec{y}, \vec{u} \rangle - a \langle \vec{u}, \vec{u} \rangle = 0$$

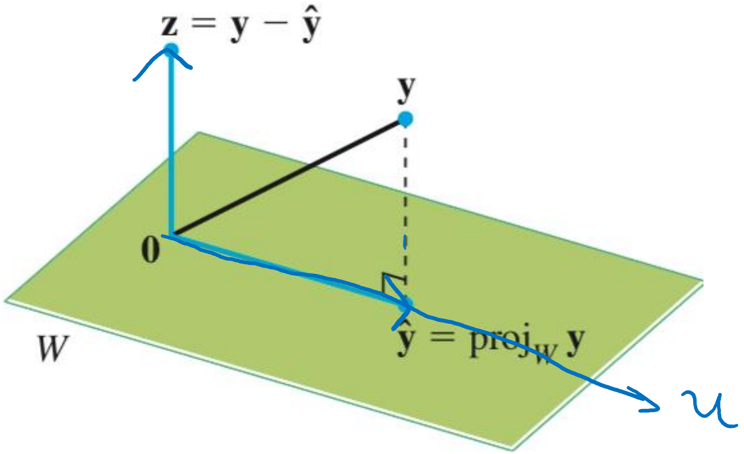
$$\boxed{a = \frac{\langle \vec{y}, \vec{u} \rangle}{\langle \vec{u}, \vec{u} \rangle}}$$

$$\vec{y} = \underbrace{\boxed{\frac{\langle \vec{y}, \vec{u} \rangle}{\langle \vec{u}, \vec{u} \rangle} \vec{u}}}_{\text{orthogonal proj of } \vec{y} \text{ on } \vec{u}} + \underbrace{\boxed{\vec{z}}}_{\text{component orthogonal to } \vec{u}}$$

$$\boxed{\text{proj}_L \vec{y} = \frac{\langle y, u \rangle}{\langle u, u \rangle} \vec{u}}$$

$L =$ line through $\vec{0}$ and $\vec{u}$

Orthogonal Projections



Orthogonal Projections