Linear Algebra

CSCI 2820

Lecture 10

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ECES 122

Today

- Vector Spaces review
- Linear independence review
- Basis review

Vector space

1.1 Definition A *vector space* (over $\mathbb{R}$) consists of a set V along with two operations '$+$' and '$\cdot$' subject to the conditions that for all vectors $\vec{v}, \vec{w}, \vec{u} \in V$ and all *scalars* $r, s \in \mathbb{R}$:

(1) the set V is closed under vector addition, that is, $\vec{v} + \vec{w} \in V$

(2) vector addition is commutative, $\vec{v} + \vec{w} = \vec{w} + \vec{v}$

(3) vector addition is associative, $(\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{w} + \vec{u})$

(4) there is a *zero vector* $\vec{0} \in V$ such that $\vec{v} + \vec{0} = \vec{v}$ for all $\vec{v} \in V$

(5) each $\vec{v} \in V$ has an *additive inverse* $\vec{w} \in V$ such that $\vec{w} + \vec{v} = \vec{0}$

(6) the set V is closed under scalar multiplication, that is, $r \cdot \vec{v} \in V$

(7) scalar multiplication distributes over scalar addition, $(r+s) \cdot \vec{v} = r \cdot \vec{v} + s \cdot \vec{v}$

(8) scalar multiplication distributes over vector addition, $r \cdot (\vec{v} + \vec{w}) = r \cdot \vec{v} + r \cdot \vec{w}$

(9) ordinary multiplication of scalars associates with scalar multiplication, $(rs) \cdot \vec{v} = r \cdot (s \cdot \vec{v})$

(10) multiplication by the scalar 1 is the identity operation, $1 \cdot \vec{v} = \vec{v}$.

Vector space, contd.

eg. $n \geq 0$, $\mathbb{P}_n$ the set of polynomials of degree at most n:

- "vectors": $p(t) = a_0 + a_1 t + \cdots + a_n t^n$ $\left(\begin{matrix} a_i \in \mathbb{R} \\ t \in \mathbb{R} \end{matrix}\right)$

degree of $p(t)$ is the highest power of t whose coefficient is not zero. (If $p(t) = a_0$ what is the degree?)

deg $= 0$

① closed under addition: $q(t) = b_0 + b_1 t + \cdots + b_n t^n$

$(p+q)(t) = p(t) + q(t) = (a_0 + b_0) + (a_1 + b_1)t + \cdots + (a_n + b_n)t^n \in \mathbb{P}_n$ ✓

② closed under scalar mult: $(cp)(t) = cp(t) = ca_0 + ca_1 t + \cdots + ca_n t^n$

$\in \mathbb{P}_n$

$\Rightarrow \vec{p} = \begin{pmatrix} a_0 \\ \vdots \\ a_n \end{pmatrix}$

Vector space, contd.

Subspaces

Vector space, contd.

DEFINITION

A **subspace** of a vector space V is a subset H of V that has three properties:

a. The zero vector of V is in H.[2]

b. H is closed under vector addition. That is, for each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u}+\mathbf{v}$ is in H.

c. H is closed under multiplication by scalars. That is, for each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

- $\{0\}$ subspace of any vector space
- $\mathbb{R}^2$. Is this a subspace of $\mathbb{R}^3$?

$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ $\boxed{\mathbb{R}^2 \not\subset \mathbb{R}^3}$ $\rightarrow \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

$$H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} : s,t \in \mathbb{R} \right\} \subseteq \mathbb{R}^3$$

✓ subspace of $\mathbb{R}^3$.

Vector space, contd.

Subspace spanned by a set

$S = \text{span}\{\vec{v}_1, \dots, \vec{v}_k\}$ = set of all vectors that can be written as linear combinations of $\vec{v}_1 \dots \vec{v}_k$

$$\vec{v} = a_1\vec{v}_1 + \dots + a_k\vec{v}_k \in S$$

eg 1

- eg. $\vec{v}_1, \vec{v}_2 \in V$ (V is vector space)

  $H = \text{span}\{\vec{v}_1, \vec{v}_2\}$. Show: H is a subspace of V

  - zero element: $\vec{0} = 0\vec{v}_1 + 0\vec{v}_2 \in H$
  - closed under "+": $\left.\begin{array}{l}\vec{u} = a_1\vec{v}_1 + a_2\vec{v}_2 \\ \vec{v} = b_1\vec{v}_1 + b_2\vec{v}_2\end{array}\right\} \vec{u}+\vec{v} = (a_1+b_1)\vec{v}_1 + (a_2+b_2)\vec{v}_2 \in H$
  - closed under "∘": similar.

Vector space, contd.

<u>Theorem</u>

if $\vec{v}_1, \dots, \vec{v}_k \in V$, $H = \text{span}\{\vec{v}_1, \dots, \vec{v}_k\}$ is a subspace of V.

subspace spanned of generated by $\vec{v}_1 \dots \vec{v}_k$

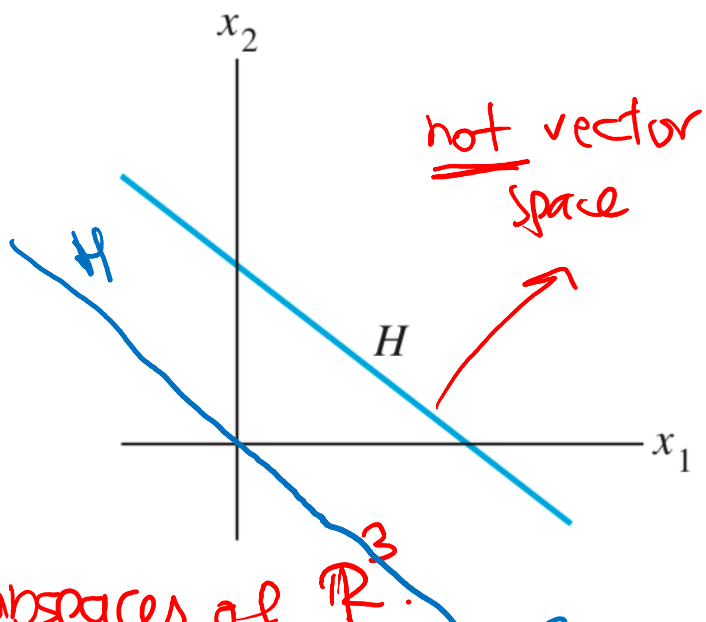
<u>ex</u>: $H = \left\{ \begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} : a, b \in \mathbb{R} \right\}$. Show H is subspace of $\mathbb{R}^4$.

$$H \ni \vec{v} = \begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} = a \underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{\vec{v}_1} + b \underbrace{\begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}_2}$$

<u>$\vec{v}_1, \vec{v}_2$ l. indep</u>

$H = \text{Span}\{\vec{v}_1, \vec{v}_2\}$, thus subspace of $\mathbb{R}^4$.

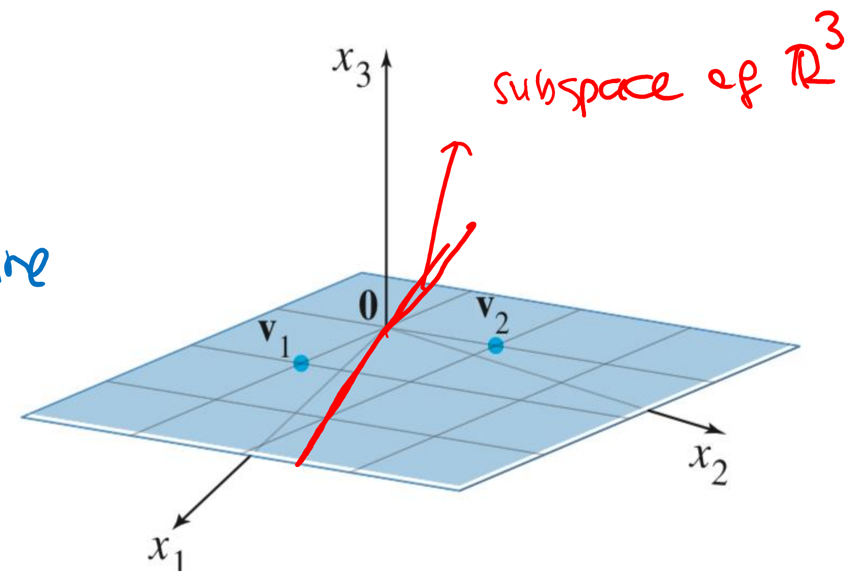
Vector space, contd.



Subspaces of $\mathbb{R}^3$.

every subspace of $\mathbb{R}^3$ is either span $\{\vec{v}_1, \vec{v}_2\}$ → plane

or span $\{\vec{v}\}$ → line



Vector space, contd.

ex: $H \subseteq \mathbb{R}^2$, $H = \left\{ \begin{pmatrix} 3s \\ 2+5s \end{pmatrix} : s \in \mathbb{R} \right\}$

is this a vector space? NO

- $\vec{0} \notin H$ $\quad \begin{pmatrix} 3s \\ 2+5s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} s = 0 \\ s = -5/2 \end{matrix}$

Vector space, contd.

Linear Independence/Basis

DEFINITION Let H be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ in V is a **basis** for H if

(i) $\mathcal{B}$ is a linearly independent set, and

(ii) the subspace spanned by $\mathcal{B}$ coincides with H; that is,

$$H = \text{Span}\,\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$$

$\{\vec{v}_1, \ldots, \vec{v}_k\}$ L.I iff $a_1\vec{v}_1 + \cdots + a_k\vec{v}_k = \vec{0} \iff a_i = 0 \ \forall i$

Observation: If $H \subsetneq V$, (ii) implies that all $\vec{b}_i$'s belong to H. [$\vec{b}_i \in \text{span}\{\vec{b}_1, \ldots, \vec{b}_p\}$

$\vec{b}_i = 0 \cdot \vec{b}_1 + \cdots + 1 \cdot \vec{b}_i + \cdots + 0\,\vec{b}_p$ $\in$ span.

- $S = \{1, t, t^2, \ldots, t^n\}$. Claim: S is basis for $\mathbb{P}_n$

("standard" basis of $\mathbb{P}_n$) (ii) immediate from definitions.

(i) L.I., Suppose: $p(t) = c_0 \cdot 1 + c_1 t + c_2 t^2 + \cdots + c_n t^n = 0(t) \Rightarrow c_0 = c_1 = \cdots = c_n = 0$

$\Rightarrow \forall t \in \mathbb{R} \quad p(t) = 0$.

Fundamental theorem in algebra: how many values t can I have such that $p(t) = 0 \rightsquigarrow$ at most n roots!

Linear Independence/Basis

let $\vec{v}_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 6 \\ 16 \\ -5 \end{bmatrix}$

$H = \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. $\vec{v}_3 = 5\vec{v}_1 + 3\vec{v}_2$

show that $\text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \text{span}\{\vec{v}_1, \vec{v}_2\}$

$H = \text{span}\{\vec{v}_1, \vec{v}_2\}$

① $\text{span}\{\vec{v}_1, \vec{v}_2\} \subseteq H$: $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 = c_1\vec{v}_1 + c_2\vec{v}_2 + 0\cdot\vec{v}_3 \in H$

② $H \subseteq \text{span}\{\vec{v}_1, \vec{v}_2\}$ $\vec{x} \in H$: $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$

$(\vec{v}_3 = 5\vec{v}_1 + 3\vec{v}_2)$

$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3(5\vec{v}_1 + 3\vec{v}_2)$

$= (c_1 + 5c_3)\vec{v}_1 + (c_2 + 3c_3)\vec{v}_2 \in \text{span}\{\vec{v}_1, \vec{v}_2\}$

$\{\vec{v}_1, \vec{v}_2\}$ basis for H.

Linear Independence/Basis

Linear Independence/Basis

The Spanning Set Theorem

Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ be a set in V, and let $H = \text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$.

a. If one of the vectors in S—say, $\mathbf{v}_k$—is a linear combination of the remaining vectors in S, then the set formed from S by removing $\mathbf{v}_k$ still spans H.

b. If $H \neq \{\mathbf{0}\}$, some subset of S is a basis for H.

ex: $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $H = \left\{ \begin{bmatrix} s \\ s \\ 0 \end{bmatrix} : s \in \mathbb{R} \right\}$ then

q_2: what is a basis for H?

every vector in H is a linear combination of $\vec{v}_1$ and $\vec{v}_2$: $\begin{bmatrix} s \\ s \\ 0 \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \boxed{s \cdot \vec{v}_1 + s\vec{v}_2}$

Q: Is $\{\vec{v}_1, \vec{v}_2\}$ a basis for H? (is span$\{\vec{v}_1, \vec{v}_2\} = H$? does $\vec{v}_1 \in H$? $\vec{v}_2 \in H$?)

$\vec{v}_1 \notin H$, $\vec{v}_2 \notin H$ $\Rightarrow \{\vec{v}_1, \vec{v}_2\}$ not a basis.

$\boxed{a\vec{v}_1 + b\vec{v}_2}$

Linear Independence/Basis

ex: V, W vector spaces

$T: V \to W$ linear functions

$U: V \to W$ $\quad [f(a\vec{x}+b\vec{y}) = af(\vec{x}) + bf(\vec{y})]$

let $\{\vec{v}_1, \dots, \vec{v}_p\}$ basis for V. Assume $\boxed{T(\vec{v}_j) = U(\vec{v}_j) \ \forall i}$

Show: $T(\vec{x}) = U(\vec{x}) \quad \forall \vec{x} \in V$

: $\{\vec{v}_1, \dots, \vec{v}_p\}$ basis for V. $\vec{x} \in V$, can be written as linear comb. of $\vec{v}_i$'s ($V = \text{span}\{\vec{v}_1, \dots, \vec{v}_p\}$) $\vec{x} = c_1\vec{v}_1 + \dots + c_p\vec{v}_p$, $c_i \in \mathbb{R}$

Since T is linear, U is linear:

$$T(\vec{x}) = T(c_1\vec{v}_1 + \dots + c_p\vec{v}_p) = c_1T(\vec{v}_1) + \dots + c_pT(\vec{v}_p)$$
$$= c_1U(\vec{v}_1) + \dots + c_pU(\vec{v}_p) = U(c_1\vec{v}_1 + \dots + c_p\vec{v}_p)$$
$$= U(\vec{x})$$

Linear Independence/Basis

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