1. Grover, revisited (\(1/50\))

1.

Since each iteration of Grover’s algorithm rotates the state by an angle \(2\theta\) in the \(|y\rangle, |y\rangle_\perp\) plane, if after \(m\) iterations we have the state \(|y\rangle\) then after \(m - 1\) iterations we will have the state \(|\psi\rangle\) which is rotated \(2\theta\) away from \(|y\rangle\) towards \(|y\rangle_\perp\). We can then express the state \(|\psi\rangle\) as

\[
|\psi\rangle = a_y |y\rangle + a_x |y\rangle_\perp = \cos 2\theta |y\rangle + \sin 2\theta |y\rangle_\perp
\]

Since we have \(\sin \theta = \frac{1}{\sqrt{N}}\), our state \(|\psi\rangle\) is

\[
|\psi\rangle = \cos \left[ 2 \arcsin \left( \frac{1}{\sqrt{N}} \right) \right] |y\rangle + \sin \left[ 2 \arcsin \left( \frac{1}{\sqrt{N}} \right) \right] |y\rangle_\perp
\]

For large \(N\), to order \(\left( \frac{1}{\sqrt{N}} \right)^2\), we have

\[
|\psi\rangle \approx \left( 1 - \frac{2}{N} \right) |y\rangle + \frac{2}{\sqrt{N}} |y\rangle_\perp
\]

2.

Now instead of 1 marked element, we have \(\frac{N}{4}\) marked elements. We construct the state

\[
|\psi\rangle = \sqrt{\frac{4}{N}} \sum_{i=1}^{N/4} |a_i\rangle
\]

We begin as usual in the uniform superposition state

\[
|\phi\rangle = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} |x\rangle_n
\]

Then the angle of rotation \(\Theta\) is given by

\[
\sin \Theta = \langle \phi | \psi \rangle = \frac{N}{4} \sqrt{\frac{4}{N}} \frac{1}{\sqrt{N}} = \frac{1}{2}
\]

But this means that \(\Theta = \frac{\pi}{6}\). After one rotation by \(2\Theta\), the resulting state forms an angle of \(3\Theta = \frac{\pi}{2}\) with the state \(|\psi\rangle_\perp\). So the resulting state is just \(|\psi\rangle\) meaning we will find one of the marked items with certainty after one iteration.

There are a total $N = 2^n$ items with $k$ marked elements, $a_1, ..., a_k$. We construct the state

$$|\psi\rangle = \frac{1}{\sqrt{k}} \sum_{i=1}^{k} |a_i\rangle$$

We again start with the uniform superposition state

$$|\phi\rangle = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} |x_n\rangle$$

With the state $|\psi\rangle$ in the place of $|a\rangle$ which we had for a single marked element, the algorithm is the same except now our state is rotated by an angle $2\Theta$ with each iteration where

$$\sin \Theta = \langle \psi | \phi \rangle = \frac{k}{\sqrt{k\sqrt{N}}} = \sqrt{\frac{k}{N}}$$

If $N \gg m$, then $\Theta \approx \sqrt{\frac{k}{N}}$ and we see that in order to achieve success with high probability (finding one of the marked elements), the number of times we need to run the algorithm is approximately

$$\frac{\pi}{4} \sqrt{\frac{N}{k}}$$

If $N$ is large, then $\theta \approx \frac{1}{\sqrt{N}}$. Each iteration rotates the state by another $2\theta$. The initial uniform superposition state makes an angle $\theta$ with the state $|a\rangle_\perp$. After $\sqrt{2N}$ iterations of the algorithm, the resulting state will make an angle $\sqrt{2N} \cdot 2\theta + \theta = \sqrt{2N} \cdot \frac{2}{\sqrt{N}} + \frac{1}{\sqrt{N}} = 2\sqrt{2} + \frac{1}{\sqrt{N}} \approx 2\sqrt{2}$ with $|a\rangle_\perp$. The probability of success is then

$$P \approx \sin^2(2\sqrt{2}) = 0.095$$

There is roughly a 10% chance of success.