Intro to Quantum Computing
CSCI/PHYS 3090
CU Boulder Spring 2020
Problem Set 5 Solutions

Problem 1

We would like to determine whether the following states are entangled and express them in the Schmidt basis: $|\psi\rangle = \sum_k \sqrt{\sigma_k}\, |\tilde{k}\rangle\, |\tilde{\tilde{k}}\rangle$.

Part A

We can express the state in Schmidt form by just factoring the state in the following way:

$$|\psi_a\rangle = \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle)$$
$$|\psi_a\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$
$$|\psi_a\rangle = |+\rangle\,|-\rangle$$

This state is clearly not entangled as we just showed that it was separable into a single tensor product and only has one non-zero Schmidt coefficient.

Part B

We are interested in the state $|\psi_b\rangle = \frac{1}{\sqrt{6}}|00\rangle + \frac{1}{\sqrt{3}}|01\rangle + \frac{1}{\sqrt{6}}|10\rangle - \frac{1}{\sqrt{3}}|11\rangle$. We can construct the following coefficient matrix:

$$A = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{pmatrix}$$

Since this matrix is not symmetric, then we want to perform a singular value decomposition in order to express $|\psi_b\rangle$ in Schmidt form. This is a decomposition of the form $A = U\Sigma V^\dagger$ where Σ is a diagonal matrix, and U and V are unitaries.

$$A^\dagger A = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{pmatrix}$$

The singular values are therefore $\sqrt{\sigma_0} = \frac{1}{\sqrt{3}}$ and $\sqrt{\sigma_1} = \sqrt{\frac{2}{3}}$. The columns of V are the eigenvectors of $A^\dagger A$, which we can now calculate.

$\sigma_0 = \frac{1}{3}$ **Case:**

$$(A^\dagger A - \sigma_0 I)v_0 = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{3} \end{pmatrix} v_0 = 0 \implies v_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$\sigma_1 = \frac{2}{3}$ **Case:**

$$(A^\dagger A - \sigma_1 I)v_1 = \begin{pmatrix} -\frac{1}{3} & 0 \\ 0 & 0 \end{pmatrix} v_1 = 0 \implies v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Therefore we have that $V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$. We can calculate U by finding the eigenvectors of $AA^\dagger$.

$\sigma_0 = \frac{1}{3}$ **Case:**

$$(AA^\dagger - \sigma_0 I)v_0 = \begin{pmatrix} \frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{1}{6} \end{pmatrix} v_0 = 0 \implies v_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\sigma_1 = \frac{2}{3}$ **Case:**

$$(AA^\dagger - \sigma_1 I)v_1 = \begin{pmatrix} -\frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{6} \end{pmatrix} v_1 = 0 \implies v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Therefore we have that $U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = H$. This gives us the singular value decomposition:

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & \sqrt{\frac{2}{3}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This is useful because we had the state in the form $|\psi\rangle = \sum_{ij} a_{ij} |i\rangle |j\rangle$ and we found the SVD of the A matrix, allowing us to express $a_{ij} = \sum_k u_{ik} \sqrt{\sigma_k} v^\dagger_{kj}$. Therefore,

$$|\psi\rangle = \sum_{ijk} u_{ik} \sqrt{\sigma_k} v^\dagger_{kj} |i\rangle |j\rangle = \sum_k \sqrt{\sigma_k} |\tilde{k}\rangle |\tilde{\tilde{k}}\rangle$$

where $|\tilde{k}\rangle = \sum_i u_{ik} |i\rangle$ and $|\tilde{\tilde{k}}\rangle = \sum_j v^\dagger_{kj} |j\rangle$. We have the following Schmidt basis states:

$$|\tilde{0}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle$$
$$|\tilde{1}\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |-\rangle$$
$$|\tilde{\tilde{0}}\rangle = |0\rangle$$
$$|\tilde{\tilde{1}}\rangle = |1\rangle$$

Therefore we can finally express the state in the following way:

$$|\psi_b\rangle = \frac{1}{\sqrt{3}}\,|+\rangle\,|0\rangle + \sqrt{\frac{2}{3}}\,|-\rangle\,|1\rangle$$

This state is clearly entangled as there are two non-zero Schmidt coefficients and therefore the state cannot be written as a single tensor product.

Part C

The state that we are interested in is $|\psi_c\rangle = \frac{1}{2}(|00\rangle - i\,|01\rangle + |10\rangle + i\,|11\rangle)$. We can construct the coefficient matrix as follows:

$$A = \begin{pmatrix} \frac{1}{2} & -\frac{i}{2} \\ \frac{1}{2} & \frac{i}{2} \end{pmatrix}$$

We can perform a SVD in a similar fashion as was demonstrated in Part B with the difference that we now have degenerate singular values. We get the following:

$$A = \begin{pmatrix} \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Therefore we can write the Schmidt basis states as follows:

$$\begin{aligned} |\tilde{0}\rangle &= -\frac{i}{\sqrt{2}}\,|0\rangle + \frac{i}{\sqrt{2}}\,|1\rangle = -i\,|-\rangle \\ |\tilde{1}\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle \\ |\tilde{\tilde{0}}\rangle &= |1\rangle \\ |\tilde{\tilde{1}}\rangle &= |0\rangle \end{aligned}$$

The state is therefore the following in Schmidt form:

$$|\psi_c\rangle = \frac{1}{\sqrt{2}}(-i\,|-\rangle\,|1\rangle + |+\rangle\,|0\rangle)$$

This state is clearly entangled as there are two non-zero Schmidt coefficients.

Problem 2

4. Bob and Claire share many identically prepared copies of the two qubit state:

$|\psi\rangle = \sqrt{1-2x}\,|00\rangle + \sqrt{x}\,|01\rangle + \sqrt{x}\,|10\rangle$, $0 \le x \le \frac{1}{2}$

If Bob measures $|0\rangle$ in $\{|0\rangle, |1\rangle\}$ basis, then Claire always measures $|\phi\rangle$ in the $\{|\phi\rangle, |\phi_\perp\rangle\}$ basis. Bob always obtains $|\phi\rangle$ if Claire obtains $|0\rangle$.

a. Bob obtains outcome 0: $|\psi\rangle \to |\psi'\rangle = \dfrac{(|0\rangle_B {}_B\langle 0| \otimes I)|\psi\rangle}{\sqrt{\langle\psi|(|0\rangle\langle 0|\otimes I)^\dagger(|0\rangle\langle 0|\otimes I)|\psi\rangle}}$

$$(|0\rangle\langle 0| \otimes I)|\psi\rangle = \sqrt{1-2x}\,|00\rangle + \sqrt{x}\,|01\rangle$$
$$= |0\rangle \otimes (\sqrt{1-2x}\,|0\rangle + \sqrt{x}\,|1\rangle)$$

$$\langle\psi|(|0\rangle\langle 0|\otimes I)^\dagger(|0\rangle\langle 0|\otimes I)|\psi\rangle = 1-2x+x = 1-x$$

$$|\psi'\rangle = |0\rangle \otimes \underbrace{\frac{1}{\sqrt{1-x}}(\sqrt{1-2x}\,|0\rangle + \sqrt{x}\,|1\rangle)}_{|\phi\rangle}$$

For Claire to always measure $|\phi\rangle$, then the state in her subsystem must be $|\phi\rangle$ after Bob measures $|0\rangle$ and $|\phi_\perp\rangle$ must be orthogonal to $|\phi\rangle$.

$$|\phi_\perp\rangle = a|0\rangle + b|1\rangle$$

$$\langle\phi|\phi_\perp\rangle = \frac{1}{\sqrt{1-x}}(a\sqrt{1-2x} + b\sqrt{x}) = 0$$

$$\Rightarrow \frac{a}{b} = -\frac{\sqrt{x}}{\sqrt{1-2x}}$$ $a^2+b^2=1$ for normalization

$$a = \frac{\sqrt{x}}{\sqrt{1-x}},\quad b = -\frac{\sqrt{1-2x}}{\sqrt{1-x}}$$

$$\Rightarrow \boxed{|\phi\rangle = \frac{1}{\sqrt{1-x}}(\sqrt{1-2x}\,|0\rangle + \sqrt{x}\,|1\rangle)} \quad \boxed{|\phi_\perp\rangle = \frac{1}{\sqrt{1-x}}(\sqrt{x}\,|0\rangle - \sqrt{1-2x}\,|1\rangle)}$$

4b. Bob and Claire both measure in the $\{|\phi\rangle, |\phi_\perp\rangle\}$ basis

$$P(x) = |(\langle\phi_\perp| \otimes \langle\phi_\perp|)|\psi\rangle|^2$$

$$P(x) = \left|\frac{1}{1-x}\left(x\langle 00| + (1-2x)\langle 11| - \sqrt{x}\sqrt{1-2x}\,(\langle 01| + \langle 10|)\right)\left(\sqrt{1-2x}\,|00\rangle + \sqrt{x}\,(|01\rangle + |10\rangle)\right)\right|^2$$

$$P(x) = \frac{1}{(1-x)^2}\left|x\sqrt{1-2x} - 2x\sqrt{1-x}\right|^2$$

$$P(x) = \frac{1}{(1-x)^2}\left|-x\sqrt{1-2x}\right|^2$$

$$P(x) = \frac{x^2(1-2x)}{(1-x)^2} \Rightarrow \boxed{P(x) = \frac{x^2 - 2x^3}{(1-x)^2}}$$

c. $$\frac{dP}{dx} = \frac{2x - 6x^2}{(1-x)^2} - 2\,\frac{(x^2-2x^3)(-1)}{(1-x)^3} = 0$$

$$x - 3x^2 + \frac{x^2 - 2x^3}{1-x} = 0$$

$$1 - 3x + \frac{x - 2x^2}{1-x} = 0$$

$$1 - 4x + 3x^2 + x - 2x^2 = 0$$

$$x^2 - 3x + 1 = 0 \Rightarrow x = \frac{3 \pm \sqrt{9-4}}{2}$$

$$x = \frac{3 \pm \sqrt{5}}{2}$$

Since $x \leq \frac{1}{2}$, then only the x_- solution is valid.

$$\Rightarrow \boxed{x = \frac{3-\sqrt{5}}{2}} \qquad P\left(\frac{3-\sqrt{5}}{2}\right) = \frac{5\sqrt{5} - 11}{2}$$ ← Mathematica

4d. Albert's reasoning was wrong because he used local realism to make his prediction. His flaw was in trying to consider what might have happened if Bob or Claire had chosen to use a basis other than the one that they actually did. Since the two basis sets are non-commuting, then it is not possible to simultaneously describe the results of Alice' and Bob's measurements in the separate basis sets.