Intro to Quantum Computing
CSCI/PHYS 3090
CU Boulder Spring 2020
Problem Set 3 Solutions

Problem 1

We want to calculate the probabilities of getting $|0\rangle$ and $|1\rangle$ if we measure qubit one on the following states. We additionally would like to determine the post-measurement state on qubit 2 for each of the measurement outcomes.

The measurement probabilities and the post-measurement states can be determined by using the generalized Born-rule. We would like to get the states into the following form:

$$|\psi\rangle = \alpha_0 |0\rangle_1 \otimes |\psi_0\rangle_2 + \alpha_1 |0\rangle_1 \otimes |\psi_1\rangle_2 \tag{1}$$

where $|\psi_0\rangle_2$, $|\psi_1\rangle_2$ are normalized states. If our state is in the form shown above, then we get outcome "x" with probability $p_x = |\alpha_x|^2$ and the post-measurement state on qubit 2 is $|\psi_x\rangle_2$.

Part A

$$|\psi\rangle_{12} = \frac{1}{\sqrt{3}} |0\rangle_1 |1\rangle_2 + \sqrt{\frac{2}{3}} |1\rangle_1 |0\rangle_2$$

This state is already in the form given by Equation 1 and so we can get the following probabilities and post-measurement outcomes:

Outcome	Probability	Post-measurement state on qubit 2
0	$\frac{1}{3}$	$|1\rangle_2$
1	$\frac{2}{3}$	$|0\rangle_2$

Table 1: Outcomes for Part A

Part B

$$|\psi\rangle_{12} = \frac{1}{\sqrt{2}}|+\rangle_1|0\rangle_2 + \frac{1}{\sqrt{2}}|-\rangle_1|1\rangle_2$$
$$|\psi\rangle_{12} = \frac{1}{2}(|0\rangle_1 + |1\rangle_1)|0\rangle_2 + \frac{1}{2}(|0\rangle_1 - |1\rangle_1)|1\rangle_2$$
$$|\psi\rangle_{12} = \frac{1}{2}(|0\rangle_1|0\rangle_2 + |0\rangle_1|1\rangle_2 + |1\rangle_1|0\rangle_2 - |1\rangle_1|1\rangle_2)$$
$$|\psi\rangle_{12} = \frac{1}{2}(|0\rangle_1(|0\rangle_2 + |1\rangle_2) + |1\rangle_1(|0\rangle_2 - |1\rangle_2))$$
$$|\psi\rangle_{12} = \frac{1}{\sqrt{2}}(|0\rangle_1|+\rangle_2 + |1\rangle_1|-\rangle_2)$$

Therefore we get the following outcomes:

Outcome	Probability	Post-measurement state on qubit 2
0	$\frac{1}{2}$	$\|+\rangle_2$
1	$\frac{1}{2}$	$\|-\rangle_2$

Table 2: Outcomes for Part B

Part C

$$|\psi\rangle_{12} = \frac{1}{\sqrt{2+\sqrt{2}}}(|+\rangle_1|0\rangle_2 + |+\rangle_1|-\rangle_2)$$
$$|\psi\rangle_{12} = \frac{1}{\sqrt{2+\sqrt{2}}}|+\rangle_1(|0\rangle_2 + |-\rangle_2)$$
$$|\psi\rangle_{12} = \frac{1}{\sqrt{2+\sqrt{2}}}\frac{1}{\sqrt{2}}(|0\rangle_1 + |1\rangle_1)(|0\rangle_2 + |-\rangle_2)$$

We now want to normalize the second qubit state in order to properly apply the generalized Born rule.

$$|\psi\rangle_{12} = \frac{1}{\sqrt{2+\sqrt{2}}}\frac{1}{\sqrt{2}}(|0\rangle_1 + |1\rangle_1)(|0\rangle_2 + \frac{1}{\sqrt{2}}(|0\rangle_2 - |1\rangle_2))$$
$$|\psi\rangle_{12} = \frac{1}{\sqrt{2+\sqrt{2}}}\frac{1}{\sqrt{2}}(|0\rangle_1 + |1\rangle_1)((1 + \frac{1}{\sqrt{2}})|0\rangle_2 - \frac{1}{\sqrt{2}}|1\rangle_2)$$

$$\sqrt{(1+\frac{1}{\sqrt{2}})^2 + (\frac{1}{\sqrt{2}})^2} = \sqrt{1 + \frac{2}{\sqrt{2}} + \frac{1}{2} + \frac{1}{2}} = \sqrt{2+\sqrt{2}}$$

$$|\psi\rangle_{12} = \frac{1}{\sqrt{2}}(|0\rangle_1 + |1\rangle_1)(\frac{1+\frac{1}{\sqrt{2}}}{\sqrt{2+\sqrt{2}}}|0\rangle_2 - \frac{1}{\sqrt{2}\sqrt{2+\sqrt{2}}}|1\rangle_2)$$

$$|\psi\rangle_{12} = \frac{1}{\sqrt{2}}(|0\rangle_1 + |1\rangle_1)(\frac{1+\sqrt{2}}{\sqrt{4+2\sqrt{2}}}|0\rangle_2 - \frac{1}{\sqrt{4+2\sqrt{2}}}|1\rangle_2)$$

We now have the state in a form where we can simply use the Born rule to get the measurement outcomes.

Outcome	Probability	Post-measurement state on qubit 2
0	$\frac{1}{2}$	$\frac{1+\sqrt{2}}{\sqrt{4+2\sqrt{2}}} \|0\rangle_2 - \frac{1}{\sqrt{4+2\sqrt{2}}} \|1\rangle_2$
1	$\frac{1}{2}$	$\frac{1+\sqrt{2}}{\sqrt{4+2\sqrt{2}}} \|0\rangle_2 - \frac{1}{\sqrt{4+2\sqrt{2}}} \|1\rangle_2$

Table 3: Outcomes for Part C

Part D

$$|\psi\rangle_{12} = \frac{1}{\sqrt{2}}(|0\rangle_1 |+\rangle_2 + |+\rangle_1 |-\rangle_2)$$
$$|\psi\rangle_{12} = \frac{1}{\sqrt{2}}(|0\rangle_1 |+\rangle_2 + \frac{1}{\sqrt{2}}(|0\rangle_1 + |1\rangle_1) |-\rangle_2)$$
$$|\psi\rangle_{12} = |0\rangle_1 (\frac{1}{\sqrt{2}} |+\rangle_2 + \frac{1}{2} |-\rangle_2) + \frac{1}{2} |1\rangle_1) |-\rangle_2$$
$$|\psi\rangle_{12} = \sqrt{\frac{3}{4}} |0\rangle_1 (\sqrt{\frac{2}{3}} |+\rangle_2 + \frac{1}{\sqrt{3}} |-\rangle_2) + \frac{1}{2} |1\rangle_1) |-\rangle_2$$

Outcome	Probability	Post-measurement state on qubit 2
0	$\frac{3}{4}$	$\sqrt{\frac{2}{3}} \|+\rangle_2 + \frac{1}{\sqrt{3}} \|-\rangle_2$
1	$\frac{1}{4}$	$\|-\rangle_2$

Table 4: Outcomes for Part D

Problem 2

Given that $|u\rangle$,$|v\rangle$,$|w\rangle$, and $|z\rangle$ are real vectors, then we want to show that the inner product of $(|u \otimes v\rangle, |w \otimes z\rangle)$ is equal to $\langle u|w\rangle \cdot \langle v|z\rangle$. We will have to assume that $|u\rangle$ and $|w\rangle$ have the same dimension and that $|v\rangle$ and $|z\rangle$ have the same dimension for the inner product to be well defined.

We can define the tensor product in terms of how it operates on vectors in some basis:

$$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}, \quad v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}, w = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{pmatrix}, \quad z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}$$

$$|u\rangle \otimes |v\rangle = \begin{pmatrix} u_1 v_1 \\ u_1 v_2 \\ \vdots \\ u_1 v_n \\ u_2 v_1 \\ \vdots \\ u_m v_n \end{pmatrix}, \quad |w\rangle \otimes |z\rangle = \begin{pmatrix} w_1 z_1 \\ w_1 z_2 \\ \vdots \\ w_1 z_n \\ w_2 z_1 \\ \vdots \\ w_m z_n \end{pmatrix}$$

The inner product is the following:

$$\begin{gathered}
(|u\rangle \otimes |v\rangle, |w\rangle \otimes |z\rangle) = u_1 v_1 w_1 z_1 + u_1 v_2 w_1 z_2 + \cdots + u_1 v_n w_1 z_n + u_2 v_1 w_2 z_1 + \cdots + u_m v_n w_m z_n \\
(|u\rangle \otimes |v\rangle, |w\rangle \otimes |z\rangle) = (u_1 w_1)(v_1 z_1 + v_2 z_2 + \ldots v_n z_n) + \ldots (u_m w_m)(v_1 z_1 + v_2 z_2 + \ldots v_n z_n) \\
(|u\rangle \otimes |v\rangle, |w\rangle \otimes |z\rangle) = (u_1 w_1 + u_2 w_2 + \cdots + u_m z_m) \langle v|z\rangle \\
(|u\rangle \otimes |v\rangle, |w\rangle \otimes |z\rangle) = \langle u|w\rangle \cdot \langle v|z\rangle
\end{gathered}$$

Problem 3

We begin by defining the bitwise inner product between two bit strings $x = (x_1, \ldots, x_n)$ and $a = (a_1, \ldots, a_n)$ as the following:

$$x \cdot a = x_1 a_1 + \cdots + x_n a_n$$

where addition is assumed to mean addition modulo 2. The addition between two bit strings in this problem is bitwise addition modulo 2: $x + a = (a_1 + x_1, \ldots, a_n + x_n)$. We would like to prove the following relation:

$$\sum_{x \in \{0,1\}^n} (-1)^{(a+y) \cdot x} = \prod_{j=1}^{n} \sum_{x_j=0}^{1} (-1)^{(a_j + y_j) x_j} \tag{2}$$

Part A

We will begin by considering the $n = 1$ case. Let's begin by looking at the RHS.

$$\prod_{j=1}^{1} \sum_{x_j=0}^{1} (-1)^{(a_j + y_j) x_j} = \sum_{x_1=0}^{1} (-1)^{(a_1 + y_1) x_1} = (-1)^{(a_1 + y_1) 0} + (-1)^{(a_1 + y_1) 1}$$

We can now consider the LHS:

$$\begin{gathered}
\sum_{x \in \{0,1\}^1} (-1)^{(a+y) \cdot x} = (-1)^{(a+y) \cdot 0} + (-1)^{(a+y) \cdot 1} = (-1)^{(a_1 + y_1) \cdot 0} + (-1)^{(a_1 + y_1) \cdot 1} \\
\sum_{x \in \{0,1\}^1} (-1)^{(a+y) \cdot x} = (-1)^{(a_1 + y_1) 0} + (-1)^{(a_1 + y_1) 1}
\end{gathered}$$

We therefore see that the Equation 2 holds for the case of n = 1.

Part B

Now let's consider the case of $n = 2$. We will try to work the RHS of the equation into the LHS.

$$\prod_{j=1}^{2}\sum_{x_j=0}^{1}(-1)^{(a_j+y_j)x_j} = \prod_{j=1}^{2}((-1)^{(a_j+y_j)0} + (-1)^{(a_j+y_j)1})$$

$$\prod_{j=1}^{2}\sum_{x_j=0}^{1}(-1)^{(a_j+y_j)x_j} = ((-1)^{(a_1+y_1)0} + (-1)^{(a_1+y_1)1})((-1)^{(a_2+y_2)0} + (-1)^{(a_2+y_2)1})$$

$$\prod_{j=1}^{2}\sum_{x_j=0}^{1}(-1)^{(a_j+y_j)x_j} = (-1)^{(a_1+y_1)0}(-1)^{(a_2+y_2)0} + (-1)^{(a_1+y_1)1}(-1)^{(a_2+y_2)0}$$
$$+(-1)^{(a_1+y_1)0}(-1)^{(a_2+y_2)0} + (-1)^{(a_1+y_1)1}(-1)^{(a_2+y_2)1}$$

$$\prod_{j=1}^{2}\sum_{x_j=0}^{1}(-1)^{(a_j+y_j)x_j} = (-1)^{(a_1+y_1)0+(a_2+y_2)0} + (-1)^{(a_1+y_1)1+(a_2+y_2)0}$$
$$+(-1)^{(a_1+y_1)0+(a_2+y_2)1} + (-1)^{(a_1+y_1)1+(a_2+y_2)1}$$

$$\prod_{j=1}^{2}\sum_{x_j=0}^{1}(-1)^{(a_j+y_j)x_j} = (-1)^{(a+y)\cdot(0,0)} + (-1)^{(a+y)\cdot(1,0)} + (-1)^{(a+y)\cdot(0,1)} + (-1)^{(a+y)\cdot(1,1)}$$

$$\prod_{j=1}^{2}\sum_{x_j=0}^{1}(-1)^{(a_j+y_j)x_j} = \sum_{x\in\{0,1\}^2}(-1)^{(a+y)\cdot x}$$

Therefore, we have shown that Equation 2 holds for the $n = 2$ case.

Part C

We will now show the inductive step by showing that if we assume that the identity holds for the n-th case, then it holds for the $(n+1)$-th case. We will start with the RHS and prove that it is equal to the LHS. Define the following strings: $a' = (a_1, \ldots, a_n)$ and $y' = (y_1, \ldots, y_n)$ where the bit strings a and y that we are given are (a', a_{n+1}) and (y', y_{n+1}) .

$$\prod_{j=1}^{n+1}\sum_{x_j=0}^{1}(-1)^{(a_j+y_j)x_j} = (\sum_{x_{n+1}=0}^{1}(-1)^{(a_{n+1}+y_{n+1})x_{n+1}})\prod_{j=1}^{n}\sum_{x_j=0}^{1}(-1)^{(a_j+y_j)x_j}$$

$$\prod_{j=1}^{n+1}\sum_{x_j=0}^{1}(-1)^{(a_j+y_j)x_j} = ((-1)^{(a_{n+1}+y_{n+1})0} + (-1)^{(a_{n+1}+y_{n+1})1})\sum_{x\in\{0,1\}^n}(-1)^{(a'+y')\cdot x}$$

$$\prod_{j=1}^{n+1}\sum_{x_j=0}^{1}(-1)^{(a_j+y_j)x_j} = \sum_{x\in\{0,1\}^n}(-1)^{(a_{n+1}+y_{n+1})0}(-1)^{(a'+y')\cdot x} + \sum_{x\in\{0,1\}^n}(-1)^{(a_{n+1}+y_{n+1})1}(-1)^{(a'+y')\cdot x}$$

$$\prod_{j=1}^{n+1}\sum_{x_j=0}^{1}(-1)^{(a_j+y_j)x_j} = \sum_{x\in\{0,1\}^n}(-1)^{(a'+y')\cdot x+(a_{n+1}+y_{n+1})0} + \sum_{x\in\{0,1\}^n}(-1)^{(a'+y')\cdot x+(a_{n+1}+y_{n+1})1}$$

$$\prod_{j=1}^{n+1}\sum_{x_j=0}^{1}(-1)^{(a_j+y_j)x_j} = \sum_{x\in\{0,1\}^n}(-1)^{((a',a_{n+1})+(y',y_{n+1}))\cdot(x,0)} + \sum_{x\in\{0,1\}^n}(-1)^{((a',a_{n+1})+(y',y_{n+1}))\cdot(x,1)}$$

$$\prod_{j=1}^{n+1}\sum_{x_j=0}^{1}(-1)^{(a_j+y_j)x_j} = \sum_{x\in\{0,1\}^n}(-1)^{(a+y)\cdot(x,0)} + \sum_{x\in\{0,1\}^n}(-1)^{(a+y)\cdot(x,1)}$$

We now observe that we are really summing over all possible bit strings of length $n+1$ since we are adding together the case where the $(n+1)$-th bit is 0 and the case when it is 1 for each possible string of n least significant bits.

$$\prod_{j=1}^{n+1} \sum_{x_j=0}^{1} (-1)^{(a_j+y_j)x_j} = \sum_{x\in\{0,1\}^{n+1}} (-1)^{(a+y)\cdot x}$$

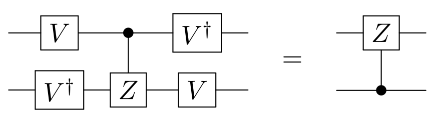
Therefore we have shown the inductive step and proven by induction that Equation 2 holds.

Problem 4

In this problem we are trying to find unitaries that wil inetrchange the control and target qubits for ControlZ and ControlY gates.

Part A

We want to find a unitary V such that the follow is true:

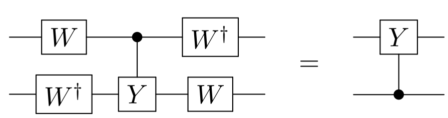


We can start by just expressing CZ_{12} and CZ_{21} as 4x4 matrices.

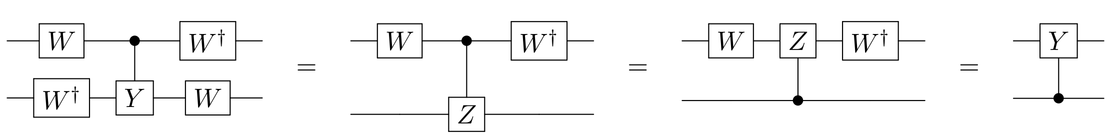
$$CZ_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$
$$CZ_{21} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The CZ gate is symmetric under exchange of the control and target qubits as the only computational basis state that the gate acts on non-trivially is $|1\rangle_1 |1\rangle_2$. Therefore the unitary that satisfies the above circuit equation is $V = I$ (the identity).

Part B



We can determine the unitary W by noting that in the previous problem we found that the CZ gate is symmetric in its target and control qubits. This means that if we can find a gate W such that $W^\dagger Y W = Z$, then we can turn CY_{12} into CZ_{12}. This is equivalent to CZ_{21}, so then when we act the inverse gates that take us back from Z to Y, then we can take CZ_{21} to CY_{21}. Diagrammatically, this is the following argument:



The gate W that performs the required operation is

$$W = e^{i\frac{\pi}{4}X} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

It can be verified that $W^{\dagger}YW = Z$ and therefore the gate satisfies the above diagrammatic equation.