CSCI3090: Intro to Quantum Computing, Spring 2020
HW 0 (due at noon on Monday, January 20, 2020)

This homework contains two problems. **Read the instructions for submitting homework on the course webpage**.

Collaboration Policy: For this homework, each student should work independently and write up their own solutions and submit them.

Read the course policies before starting the homework.

- Homework 0 tests your familiarity with prerequisite material: linear algebra, matrices, induction. It is meant to help you identify gaps in your background knowledge. You are responsible for filling those gaps. The course web page has pointers to several excellent online resources for prerequisite material. If you need help, please ask the instructors.
- Each student must submit individual solutions for these homework problems.
- Please carefully read the course policies on the course web site. If you have any questions, please ask in lecture, or by email. In particular:
  - Submit separately stapled solutions, one for each numbered problem, with your name and ID clearly printed on each page.
  - You may use any source at your disposal: paper, electronic, human, or other, but you must write your solutions in your own words, and you must cite every source that you use (except for official course materials). Please see the academic integrity policy for more details.
  - No late homework will be accepted for any reason.
  - Answering "I don't know" to any (non-extra-credit) problem or subproblem, on any homework or exam, is worth 25% partial credit.
  - Proofs containing phrases like "and so on" or "repeat this process for all n", instead of an explicit loop, recursion, or induction, will receive a score of 0.
  - Unless explicitly stated otherwise, every homework problem requires a proof.

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1 Required problems

1. (30 PTS.) Lucas Numbers

The Lucas numbers L_n are defined recursively as follows:

$$L_n = \begin{cases} 2 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ L_{n-2} + L_{n-1} & \text{otherwise.} \end{cases}$$

You may recognize this as the Fibonacci recurrence, but with a different base case ($L_0 = 2$ instead of $F_0 = 0$). Similarly, the anti-Lucas numbers Γ_n are defined recursively as follows:

$$\Gamma_n = \begin{cases} 1 & \text{if } n = 0 \\ 2 & \text{if } n = 1 \\ \Gamma_{n-2} - \Gamma_{n-1} & \text{otherwise.} \end{cases}$$

The first few Luca numbers are as follows:

n	0	1	2	3	4	5	6	7
L_n	2	1	3	4	7	11	18	29

(a) Prove that $\Gamma_n = (-1)^{n-1} L_{n-1}$ for every positive integer n.

(b) Prove that any non-negative integer can be written as the sum of distinct non-consecutive Lucas numbers; that is, if L_i appears in the sum, then L_{i-1} and L_{i+1} cannot.
For example,

$$4 = L_3$$
$$8 = 7 + 1 = L_4 + L_1$$
$$15 = 11 + 4 = L_5 + L_3$$
$$16 = 11 + 4 + 1 = L_5 + L_3 + L_1$$
$$23 = 18 + 4 + 1 = L_6 + L_3 + L_1$$

2. (30 PTS.) Tournament ranking.
A set of n tennis players $P_1, \ldots, P_n$ play in a tournament in which every player plays a match with every other player (a total of $\binom{n}{2}$ matches are played). There are no ties so for each pair of players (P_i, P_j), $i \neq j$ either P_i wins over P_j or vice-versa. We write $P_i \prec P_j$ if P_j wins against P_i in their match. We wish to rank the players from 1 to n with 1 being the best and n being the worst and justify the ranking. Let $P_{i_1}, P_{i_2}, \ldots, P_{i_n}$ be a ranking of the players from 1 to n; here $i_1, i_2, \ldots, i_n$ is a permutation of $\{1, 2, \ldots, n\}$. A ranking is justified if $P_{i_n} \prec P_{i_{n-1}} \prec \ldots \prec P_{i_1}$. Prove the following via induction: for any integer n and *any* given outcomes of the $\binom{n}{2}$ matches, there is a ranking that is justified.

(20 PTS.) The Hadamard Matrix.

The *Hadamard Matrix* is defined recursively as below:

$$\mathbf{H}_1 \triangleq \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\mathbf{H}_n = \mathbf{H}_1 \otimes \mathbf{H}_{n-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{H}_{n-1} & \mathbf{H}_{n-1} \\ \mathbf{H}_{n-1} & -\mathbf{H}_{n-1} \end{bmatrix}$$

Obviously $\mathbf{H}^T = \mathbf{H}$ is real and symmetric. Here are two examples for $n = 2$ and $n = 3$:

$$\mathbf{H}_2 = \mathbf{H}_1 \otimes \mathbf{H}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{H}_1 & \mathbf{H}_1 \\ \mathbf{H}_1 & -\mathbf{H}_1 \end{bmatrix} = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$\mathbf{H}_3 = \mathbf{H}_1 \otimes \mathbf{H}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{H}_2 & \mathbf{H}_2 \\ \mathbf{H}_2 & -\mathbf{H}_2 \end{bmatrix} = \frac{1}{\sqrt{8}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}$$

Note that $\mathbf{H}_n$ is $N = 2^n$ by $N = 2^n$ matrix.

1. Find the inverse of H_n.
2. Show that for all n, H_n is Unitary.

(20 PTS.) Diagonalization.

Determine whether the given matrix A is diagonalizable. If so, find a matrix S and a diagonal matrix D, such that $A = SDS^{-1}$.

(1) $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 3 & 4 \end{bmatrix}$ (2) $\mathbf{A} = \begin{bmatrix} 0 & 5 \\ 4 & 1 \end{bmatrix}$

(3) $\mathbf{A}=\begin{bmatrix} -5 & 3 \\ -3 & 1 \end{bmatrix}$

(4) $\mathbf{A}=\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$