PHYS-CSCI 3090 Spring 2020
Final Exam /100

May 1, 2020

This final exam is similar in flavor to a problem set, and the rules are similar too: You may work together, use your book and notes and any other resources you find helpful. However, what you submit must be your own work written in your own words and reflecting your own understanding. We are also asking you to identify the people you have worked with.

Collaborators:

1. An Error detecting code.

Consider the 4 qubit quantum code with stabilizers given by

$$XXXX \tag{1}$$
$$ZZZZ. \tag{2}$$

As usual, the code is the space of $+1$ eigenvectors of these two stabilizers. This code encodes two qubits.

a) Please write down a basis for the 4 dimensional code space.

b) This code is not a good error correcting code: different single qubit errors lead to the same syndrome. Find some distinct one qubit error that give the same syndrome.

c) This code is still useful as an error *detecting* code: show that any single-qubit error will lead to a syndrome that is not 00. Thus, if you measure the syndromes and get 00, you can be sure that there wasn't a single-qubit error.

d) Find logical $\bar{X}_1$, $\bar{Z}_1$, $\bar{X}_2$, $\bar{Z}_2$ for the two encoded qubits. That is, find strings of Paulis that satisfy:

$$\bar{X}_1\bar{Z}_1 = -\bar{Z}_1\bar{X}_1 \tag{3}$$
$$\bar{X}_2\bar{Z}_2 = -\bar{Z}_2\bar{X}_2 \tag{4}$$
$$\bar{X}_1\bar{Z}_2 = \bar{Z}_2\bar{X}_1 \tag{5}$$
$$\bar{X}_2\bar{Z}_1 = \bar{Z}_1\bar{X}_2 \tag{6}$$
$$\bar{X}_1\bar{X}_2 = \bar{X}_2\bar{X}_1 \tag{7}$$
$$\bar{Z}_1\bar{Z}_2 = \bar{Z}_2\bar{Z}_1. \tag{8}$$

In words: the logical X's and Z's should anticommute if they're on the same qubit, and commute if they're on different qubits.

e) Suppose we have the usual error model: on each qubut, with probability $1-3p$ there is no error, and with probability p there is an X error, with probability p there is a Y error and with probability p there is a Z error. What is the probability of measuring a syndrome 00?

f) Suppose you do measure 00. What is the probability of an undetected error? I.e., what the probability there's *was* a logical error but you didn't notice it?

2. GHZ revisited.

Recall the GHZ experiment: Alice, Bob, and Charlie are all spatially separated, and have these weird boxes. The boxes has doors labled X and Y on them, and each party can choose to either open X or Y, but not both. Inside, they find a 0 or a 1. Furthermore, if exactly one of them opens the X door, and two of them open the Y door, they always find an odd number of 1's. If reality can be described by a local hidden variable theory, then if they all open the X doors, they'll also always get an odd number of 1's.

Quantum mechanics gives a different prediction, which we'll see here. Let

$$|\psi\rangle = \frac{1}{\sqrt{2}}[|000\rangle + |111\rangle]. \tag{9}$$

a) Show that

$$ZZI|\psi\rangle = |\psi\rangle \tag{10}$$
$$IZZ|\psi\rangle = |\psi\rangle \tag{11}$$
$$XXX|\psi\rangle = |\psi\rangle. \tag{12}$$

b) Use the fact that $ZX = iY$ to show that

$$XYY|\psi\rangle = -|\psi\rangle \tag{13}$$
$$YXY|\psi\rangle = -|\psi\rangle \tag{14}$$
$$YYX|\psi\rangle = -|\psi\rangle. \tag{15}$$

c) If "opening X" and "opening Y" correspond to measuring X and measuring Y on $|\psi\rangle$, argue that 1) when exactly one person measures X and the other two measure Y, they always find an odd number of 1's. But, if they all measure X they will not find that they always get an odd number of ones. What do they find?

3. Impatient Grover's search.

Consider a function f where there are K values such that $f(x) = 1$ for $x \in \{0,1\}^n$. We will investigate what happens if someone impatiently runs Grover's search by measuring the state of the quantum algorithm between each iteration of Grover's search to see if it has found a solution yet? Formally, between each Grover iteration, they measure according to the following two projection operators (also called a POVM):

$$\{M_0 = \sum_{x:f(x)=0} |x\rangle\langle x|, \quad M_1 = \sum_{x:f(x)=1} |x\rangle\langle x|\}$$

.

Note that, if the state $|\phi\rangle$ is measured according to the POVM above, the state after measurement collapses to $|\phi_0\rangle = \frac{M_0|\phi\rangle}{\sqrt{\langle\phi|M_0|\phi\rangle}}$ (outcome "0") with probability $\langle\phi|M_0|\phi\rangle$ and to $|\phi_1\rangle = \frac{M_1|\phi\rangle}{\sqrt{\langle\phi|M_1|\phi\rangle}}$ (outcome "1") with probability $\langle\phi|M_1|\phi\rangle$.

- How long does Grover take if $K = 1$, i.e. there is one "marked" element?
- How long does Grover take for arbitrary K (as a function of K)

4. Algorithmic endeavors.

1. Suppose you are given a 2^k -to-1 function $f : \{0,1\}^n \to \{0,1\}^n$ such that there exist n-bit strings $a_1, ..., a_k$, such that for all $x \in \{0,1\}^n$ and for $1 \le i \le k$, $f(x + a_i) = f(x)$. What information about the a_i's can we hope to reconstruct from f? Work out the details of the quantum algorithm, including the classical reconstruction.

2. Assume a divides q and b divides q. What is the Quantum Fourier transform mod q of the uniform superposition on all $0 \le x \le q$ such that a divides x or b divides x?