CSCI 6214— Randomized Algorithms— FALL 2021
Problem Set #5

Problem 1:

Consider a graph in $G_{n,p}$, with $p = c \ln n/n$. Use the second moment method or the conditional expectation inequality to prove that if $c < 1$ then, for any constant $\epsilon > 0$ and for n sufficiently large, the graph has isolated vertices with probability at least $1 - \epsilon$.

Problem 2:

Show that if

$$4\binom{k}{2}\binom{n}{k-2}\frac{2}{2^{\binom{k}{2}}} \leq 1$$

then it is possible to color the edges of K_n with two colors so that it has no monochromatic K_k subgraph.

Problem 3:

Consider the set-balancing problem. We claim that there is an $n \times n$ matrix A for which $\|Ab\|_\infty$ is $\Omega(\sqrt{n})$ for any choice of vector b. For convenience here we assume that n is even. Here, for a matrix M, and a vector b, $\|Mb\|_\infty$ is the largest entry in absolute value of the vector Mb.

- We have used the fact that

  $$n! \leq e\sqrt{n}(\frac{n}{e})^n$$

  Now show that

  $$n! \geq a\sqrt{n}(\frac{n}{e})^n$$

  for some positive constant a.

- Let $b_1, b_2, \cdots, b_{m/2}$ all equal 1, and let $b_{m/2+1}, b_{m/2+2}, \cdots, b_m$ all equal -1. Let $Y_1, Y_2, \cdots, Y_m$ each be chosen independently and uniformly at random from $\{0, 1\}$. Show that there exists a positive constant c such that, for sufficiently large m,

  $$Pr[|\sum_{i=1}^{m} b_i Y_i| > c\sqrt{m}] > 1/2$$

  (Hint: Condition on the number of Y_i that are equal to 1.)

- Let $b_1, b_2, \cdots, b_m$ each be equal to either 1 or -1. Let $Y_1, Y_2, \cdots, Y_m$ each be chosen independently and uniformly at random from $\{0, 1\}$. Show that there exists a positive constant c such that, for sufficiently large m,

  $$Pr[|\sum_{i=1}^{m} b_i Y_i| > c\sqrt{m}] > 1/2$$

- Prove that there exists a matrix A for which $\|Ab\|_\infty$ is $\Omega(\sqrt{n})$ for any choice of vector b.

Problem 4:

Consider the problem of whether graphs in $G_{n,p}$ have cliques of constant size k. Suggest an appropriate threshold function for this property. Generalize the argument used for cliques of size 4, using the second moment method.