Problem 1:

Consider throwing \( m \) balls into \( n \) bins, and for convenience let the bins be numbered from 0 to \( n - 1 \). We say there is a \( k \)-gap starting at bin \( i \) if bins \( i, i+1, \cdots, i+k-1 \) are all empty.

- Determine the expected number of \( k \)-gaps.
- Prove a Chernoff-like bound for the number of \( k \)-gaps. (Hint: If you let \( X_i = 1 \) when there is a \( k \)-gap starting at bin \( i \), then there are dependencies between \( X_i \) and \( X_{i+1} \); to avoid these dependencies, you might consider \( X_i \) and \( X_{i+k} \).)

Problem 2:

Suppose that we vary the balls-and-bins process as follows. For convenience let the bins be numbered from 0 to \( n - 1 \). There are \( \log_2 n \) players. Each player randomly chooses a starting location \( l \) uniformly from \([0, n - 1]\) and then places one ball in each of the bins numbered \( l \mod n, l+1 \mod n, \cdots, l+n/\log_2 n - 1 \mod n \). Argue that the maximum load in this case is only \( O(\log \log n/\log \log \log n) \) with probability that approaches 1 as \( n \to \infty \).

Problem 3:

We consider another way to obtain Chernoff-like bounds in the setting of balls and bins without using the relationship between the real distribution and the Poisson distribution we saw in class. Consider \( n \) balls thrown randomly into \( n \) bins. Let \( X_i = 1 \) if the \( i \)-th bin is empty and 0 otherwise. Let \( X = \sum_{i=1}^{n} X_i \). Let \( Y_i, i = 1, \cdots, n \), be independent Bernoulli random variables that are 1 with probability \( p = (1 - 1/n)^n \). Let \( Y = \sum_{i=1}^{n} Y_i \).

- Show that \( E[X_1X_2 \cdots X_k] \leq E[Y_1Y_2 \cdots Y_k] \) for any \( k \geq 1 \).
- Show that \( E[e^{tX}] \leq E[e^{tY}] \) for all \( t \geq 0 \). (Hint: Use the expansion for \( e^x \) and compare \( E[X^k] \) to \( E[Y^k] \).)
- Derive a Chernoff bound for \( Pr(X \geq (1 + \delta)E[X]) \).