CSCI 6214— Randomized Algorithms— FALL 2021
Take-Home Final

Problem 1:

Let $Z_1, \cdots, Z_k$ be i.i.d. random variables distributed according to the Binomial distribution with parameters n and p. Let $M = \max_i Z_i$. Show that $E[M] = np + O(\sqrt{np}\log k)$.

Problem 2:

Let G be an arbitrary k-regular directed graph (i.e., every vertex has in and out degree k). In this problem, we will show, using the Lovasz Local Lemma (LLL), that G contains at least $\lfloor \frac{k}{3\ln k} \rfloor$ vertex-disjoint cycles.

1. Suppose the vertices of G are partitioned into $c = \lfloor \frac{k}{3\ln k} \rfloor$ components by assigning each vertex to a component chosen independently and u.a.r. For each vertex v, let A_v be the event that v does not have an edge to a vertex in all c of the components (including its own component). (Note that A_v includes the event that no vertex is assigned to some component.) Show that $Pr[A_v] \leq \frac{1}{3k^2 \ln k}$.

2. Let D_v denote the "dependency set" of event A_v (i.e., A_v is independent of all events A_u except for those in D_v. Show that $|D_v| \leq (k+1)^2$.

3. Deduce from parts (a) and (b) and the LLL that G contains at least c vertex disjoint cycles.

Problem 3:

This question involves looking into Poisson approximations.
Theorem 5.7 (Chapter 5, Probability and Computing book) shows that any event that occurs with small probability in the balls-in-bins setting where the number of balls in each bin is an independent poisson random variable also occurs with small probability in the standard ball-in-bins model. We proved in class a similar statement for random graphs: Every event that happens with small probability in the $G_{n,p}$ model also happens with small probability in the $G_{n,N}$ model for $N = \binom{n}{2}p$. Here $G_{n,N}$ is the set of graphs with n vertices and exactly N edges. An undirected graph on n vertices is disconnected if there exists a set of $k < n$ vertices such that there is no edge between this set and the rest of the graph. Otherwise, the graph is connected. Show that there exists a constant c such that if $N \geq cn \log n$ then, with probability $O(e^{-n})$, a graph chosen randomly from $G_{n,N}$ is connected.

Problem 4:
Let G = (V,E) be an arbitrary connected undirected graph with n vertices and m edges. Consider the following random process on G, which is a very simple model for the spread of

rumors or infections. Initially, each vertex of G is colored either black or white (with no constraints). Then, at each step, all vertices simultaneously update their colors, independently of all other vertices, as follows:

- with probability 1/2 do nothing
- else pick a neighboring vertex u.a.r. and adopt the color of that vertex (Note that all vertices make these decisions before any vertex changes color.)

It should be clear that this process will eventually terminate with all vertices black or all white, after which no further change is possible.

1. Let the random variable X_t denote the sum of the degrees of all the white vertices at time t. Show that X_t is a martingale with respect to the obvious filter.
2. Use the optional stopping theorem to compute the probability that the process terminates in the all-white configuration, as a function of the initial configuration.
3. Use the optional stopping theorem again to show that the expected duration of the process is at most $O(m^2)$ steps. [NOTE: Justify carefully any claims you make about the conditional variance of X_t.]

Problem 5:

Let $C_n = (V, E)$ denote the cycle of length n; that is, the vertices are $V = \{v_1, ..., v_n\}$, and there is an edge between v_i and v_{i+1}, for $i = 1, ..., n-1$, and v_n is connected to v_1. You are starting a random walk from $X_0 = v_1$. At the i-th step, you do one of the following with probability 1/3

- Stay put (i.e., $X_i = X_{i-1}$).
- Go left (i.e., if $X_{i-1} = v_j$, then $X_i = v_{j-1}$, where $v_0 \equiv v_n$).
- Go right (i.e., $X_{i-1} = v_j$, then $X_i = v_{j+1}$, where $v_{n+1} \equiv v_1$).

Let δ be a parameter, $\delta \in (0, 1)$, and let $v_t \in V$ be an arbitrary vertex. Let $m = m(\delta, n)$ be the minimum number of steps, so that for all $i \geq m$, we have that

$$|Pr[X_i = v_t] - 1/n| \leq \delta/n$$

Provide an upper and lower bound, as tight as possible, on the value of $m(\delta, n)$.